Coexistence in competing species models, I: the Neuhauser-Pacala model

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Abstract

This is a concise version of the lecture note for the first part of the learning session "Competing species models" held on 3rd August 2017 at the National University of Singapore[1]. This part is centered around the Neuhauser-Pacala model [3] in terms of methods of "Bernoulli subsystems" in [1, 4]. The second part of the session is led by Matthias Hammer and discusses related models of interacting diffusions[2].

Contents

1 The model

Recall that the competitive Lotka-Volterra ordinary differential equations for two species are based on the logistic equations. There are self-interactions within species and interactions between the two species. These intraspecific and interspecific interactions have species-wise impacts. If we denote by N_0 and N_1 the number of individuals of species 0 and 1, respectively, then the equations are written as

$$\begin{cases} \dot{N}_0 = & r_0 N_0 \left(1 - \dfrac{N_0 + \alpha_{01} N_1}{K_0}\right) = N_0 \left(r_0 - \dfrac{r_0}{K_0} N_0 - \dfrac{r_0 \alpha_{01}}{K_0} N_1\right) = N_0(a - bN_0 - cN_1), \\ \dot{N}_1 = & r_1 N_1 \left(1 - \dfrac{N_1 + \alpha_{10} N_0}{K_1}\right) = N_1 \left(r_1 - \dfrac{r_1}{K_1} N_1 - \dfrac{r_1 \alpha_{10}}{K_1} N_0\right) = N_1(a' - b'N_1 - c'N_0), \end{cases}$$

where

- r_i = intrinsic growth rate of species i,

[1] See http://www2.ims.nus.edu.sg/Programs/017gene/wk.php for details.
[2] The slides can be found in http://ims.nus.edu.sg/events/2017/gene/files/competing_species2.pdf.

- K_i = carrying capacity of species i,
- α_{ij} = strength of interspecific competition.

The Lotka-Volterra model is simple enough to allow a very detailed analysis. But its evolution incorporates neither the spatial dispersal of species nor stochasticity. To incorporate these two features, we view the equations in the following way first. Set the density of species-0 and the density of species-1 to be

$$p_0 = \frac{N_0}{N_0 + N_1} \quad \text{and} \quad p_1 = \frac{N_1}{N_0 + N_1},$$

respectively. Then by the differential equations satisfied by N_0 and N_1,

$$\begin{aligned}\dot{p_0} &= \frac{(N_0 + N_1)\dot{N_0} - N_0(\dot{N_0} + \dot{N_1})}{(N_0 + N_1)^2} \\ &= \frac{N_0 N_1}{(N_0 + N_1)^2}\left[(a - bN_0 - cN_1) - \left(a' - b'N_1 - c'N_0\right)\right] \\ &= -\frac{N_0 N_1}{(N_0 + N_1)^2}\left(a' + bN_0 + cN_1\right) + \frac{N_0 N_1}{(N_0 + N_1)^2}\left(a + b'N_1 + c'N_0\right).\end{aligned}$$

If we assume

$$a = a',$$

then the above equation for $\dot{p_0}$ reduces to

$$\dot{p_0} = -\frac{N_0 N_1}{(N_0 + N_1)^2}(bN_0 + cN_1) + \frac{N_0 N_1}{(N_0 + N_1)^2}\left(b'N_1 + c'N_0\right).$$

Now divide both sides by $N_0 + N_1$, by b' and finally by $\lambda p_1 + p_0$. Then up suitable time changes, the differential equation can be written as

$$\dot{p_0} = -p_0 \frac{\lambda p_1}{\lambda p_1 + p_0}(p_0 + \alpha_{01} p_1) + p_1 \frac{p_0}{\lambda p_1 + p_0}(p_1 + \alpha_{10} p_0), \tag{1}$$

since, for $a = a'$,

$$\lambda \stackrel{\text{def}}{=} b/b' = K_1/K_0, \quad c/b = \alpha_{01} \quad \text{and} \quad c'/b' = \alpha_{10}.$$

Equation (1) has an obvious interpretation in terms of a death-birth process.

A model by Neuhauser and Pacala introduced in [3] is a stochastic spatial generalization of the last differential equation. Put in a general framework, we can describe the model in the following way. Let q be an irreducible transition probability on a nonempty set E and have zero trace:

$$\sum_{x \in E} q(x, x) = 0.$$

For $x \in E$, $\xi \in \{0, 1\}^E$ and $\sigma \in \{0, 1\}$, we define the local frequencies of σ's by

$$f_\sigma(x, \xi) = \sum_{y \in E} q(x, y)\mathbb{1}_{\{\sigma\}}\big(\xi(y)\big).$$

Then the Lotka-Volterra model is generalized from a density-dependent model to a frequency dependent model and from a differential equation to a spin system in the sense of [2, page 122–123]. The flip rates are given by

$$\begin{aligned} 0 \to 1 \text{ with rate } &\underbrace{(f_0 + \alpha_{01} f_1)}_{\text{death rate}} \underbrace{\left(\frac{\lambda f_1}{\lambda f_1 + f_0}\right)}_{\text{birth probability}}, \\ 1 \to 0 \text{ with rate } &(f_1 + \alpha_{10} f_0)\left(\frac{f_0}{\lambda f_1 + f_0}\right). \end{aligned}$$

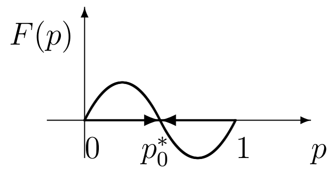


Figure 1

It's not difficult to show that the density of 1's in this particle system on a complete graph over N vertices converges to a solution of (1) as $N \to \infty$. See also the slides of the second part on stochastic spatial generalizations of the Lotka-Volterra model.

The main interests for the Neuhauser-Pacala model are in characterizing its equilibria. If we reconsider the density version (1) of the Lotka-Volterra model, then the fact that $p_0 + p_1 = 1$ gives

$$\dot{p_0} = \frac{p_0(1-p_0)}{\lambda(1-p_0)+p_0}\left\{(1-\lambda\alpha_{01}) - p_0\big[(1-\lambda\alpha_{01}) + (\lambda - \alpha_{10})\big]\right\} = \frac{F(p_0)}{\lambda(1-p_0)+p_0}.$$

By setting the right-hand side to zero, the foregoing equation has a unique nontrivial equilibrium given by

$$p_0^* = \frac{(1-\lambda\alpha_{01})}{(1-\lambda\alpha_{01}) + (\lambda - \alpha_{10})}$$

if $0 \le \alpha_{10} < \lambda$ and $0 \le \alpha_{01} < 1/\lambda$; see the graph of the polynomial F given in Figure 1.

The rest is aimed to methods which prove such a unique characterization of equilibria for the stochastic spatial version by Neuhauser and Pacala.

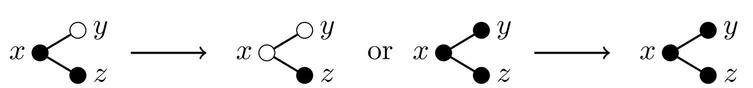
2 Constructions and duality

We focus on the symmetric Neuhauser-Pacala model, that is $\lambda = 1$ and $\alpha_{01} = \alpha_{10} = \alpha \in [0, 1)$. This is the most well studied case. Now the flip rates can be written as

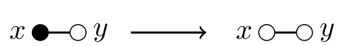
$$\begin{cases} 0 \to 1 & \text{with rate } (f_0 + \alpha f_1) f_1 = (1-\alpha) f_0 f_1 + \alpha f_1, \\ 1 \to 0 & \text{with rate } (f_1 + \alpha f_0) f_0 = (1-\alpha) f_1 f_0 + \alpha f_0 \end{cases}$$

(using $f_0 + f_1 \equiv 1$). These flip rates are probabilistic mixtures of two separate mechanisms:

(1) By $f_1 f_0 = f_0 f_1$, the children of two neighbors invade the focal site subject to pairwise annihilation. For example, we have transitions



(2) By f_0 and f_1, we have voting. For example,



Here, a filled vertex is occupied by a 1-individual and by a 0-individual otherwise. By the cancellative additivity that $1+1 \equiv 0$ in $\mathbb{Z}_2$, the process can be constructed by the following Poisson events: in $\mathbb{Z}_2^E$,

$$\eta \to \eta + \{x\} \times \{y, z\}\eta = \underbrace{[\eta - \eta(x)\mathbf{1}_x]}_{\text{types outside } x} + \underbrace{[\eta(x) + \eta(y) + \eta(z)]\mathbf{1}_x}_{\text{type at } x},$$

$$\text{with rate } r(\{x\}\times\{y,z\}) = (1-\alpha)q(x,y)q(x,z),$$

$$\eta \to \eta + \{x\}\times\{x,y\}\eta = [\eta - \eta(x)\mathbf{1}_x] + [\eta(x)+\eta(x)+\eta(y)]\mathbf{1}_x \underbrace{\equiv}_{\substack{\text{mod }(2)\\ \text{at every site}}} [\eta - \eta(x)\mathbf{1}_x] + \eta(y)\mathbf{1}_x$$

$$\text{with rate } r(\{x\}\times\{x,y\}) = \alpha q(x,y),$$

where

$$\{x\}\times\{y,z\} = \begin{array}{c} \\ x \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & y & z & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad \{x\}\times\{x,y\} = \begin{array}{c} \\ x \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccc} & x & y & \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and the zeros denote block matrices with zero entries of obvious dimensions.

On a finite set E, this construction can be reduced to a more explicit construction of the process: with initial condition $\mathbf{1}_A$,

$$\eta_t^A \equiv (\mathrm{Id} + J_{N_t})\cdots(\mathrm{Id}+J_1)\mathbf{1}_A,$$

where $J_1, J_2, \cdots$ are i.i.d. random matrices with law

$$\mathbb{P}(J_1 = J) = \frac{r(J)}{\sum_{J'} r(J')}$$

and (N_t) is an independent Poisson process with rate

$$\mathbb{E}N_1 = \sum_{J'} r(J').$$

In this case, the duality is easy to see:

$$\begin{aligned} \langle \mathbf{1}_B, \eta_t^A\rangle &\equiv \mathbf{1}_B^\top(\mathrm{Id}+J_{N_t})\cdots(\mathrm{Id}+J_1)\mathbf{1}_A && (2)\\ &= [(\mathrm{Id}+J_1^\top)\cdots(\mathrm{Id}+J_{N_t}^\top)\mathbf{1}_B]^\top\mathbf{1}_A \\ &= \langle \widehat{\eta}_t^{t,B}, \mathbf{1}_A\rangle \overset{(\mathrm{d})}{\equiv} \langle \widehat{\eta}_t^B, \mathbf{1}_A\rangle. && (3) \end{aligned}$$

Here,

$$\widehat{\eta}_t^B \overset{\text{def}}{=} (\mathrm{Id}+J_{N_t}^\top)\cdots(\mathrm{Id}+J_1^\top)\mathbf{1}_B.$$

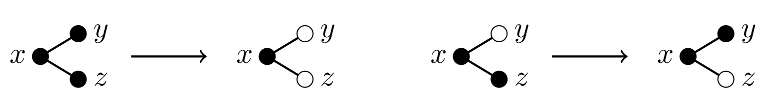
We can reverse this reduction of the constructions of (η_t^A) and define $(\widehat{\eta}_t^B)$ by

(1) Poisson events of branching with annihilation:

$$\xi \to \xi + \{y,z\}\times\{x\}\xi \equiv [\xi - \xi(y)\mathbf{1}_y - \xi(z)\mathbf{1}_z] + [\eta(x)+\eta(y)]\mathbf{1}_y + [\xi(x)+\xi(z)]\mathbf{1}_z,$$

$$\text{with rate } \widehat{r}(\{y,z\}\times\{x\}) \overset{\text{def}}{=} r(\{x\}\times\{y,z\})$$

such as



(2) and Poisson events of random walks with annihilation: in $\mathbb{Z}_2^E$,

$$\xi \to \xi + \{x,y\} \times \{x\}\xi \equiv [\xi - \xi(x)\mathbf{1}_x - \xi(y)\mathbf{1}_y] + [\xi(x) + \xi(x)]\mathbf{1}_x + [\xi(x) + \xi(y)]\mathbf{1}_y$$
$$\text{with rate } \widehat{r}(\{x,y\} \times \{x\}) \stackrel{\text{def}}{=} r(\{x\} \times \{x,y\})$$

such as

$$x \bullet\!\!-\!\!\bullet\, y \longrightarrow x \circ\!\!-\!\!\circ\, y \qquad x \circ\!\!-\!\!\bullet\, y \longrightarrow x \circ\!\!-\!\!\bullet\, y\,.$$

Notice that in contrast to the Neuhauser-Pacala model where death events are followed by birth events at each updating step, the dual process *reverses* the orders of birth events and death events and is an invasion process.

Taking expectation of both sides of (3), we have the following theorem when E is a finite set. It still holds on infinite sets.

Theorem 2.1 (Parity dual equation). *For any irreducible kernel (E, q) with zero trace, finite subsets A, B of E, and $t \geq 0$,*

$$\mathbb{P}(\langle \mathbf{1}_B, \eta_t^A \rangle \equiv 1) = \mathbb{P}(\langle \widehat{\eta}_t^B, \mathbf{1}_A \rangle \equiv 1).$$

The validity of the theorem on infinite sets can be obtained by various methods:

(1) Check the Feynman-Kac duality equation by generator calculations: for

$$f(t, A, B) \stackrel{\text{def}}{=} \mathbb{P}(\langle \mathbf{1}_B, \eta_t^A \rangle \equiv 1),$$

we consider the forward equation:

$$\frac{d}{dt} f(t, A, B) = \mathbb{E}\big[\mathsf{L}^{\text{Neuhauser-Pacala}} f(t, \eta_t^A, B)\big] = [\mathsf{L}^{\text{dual}} f(t, A, \cdot)](B).$$

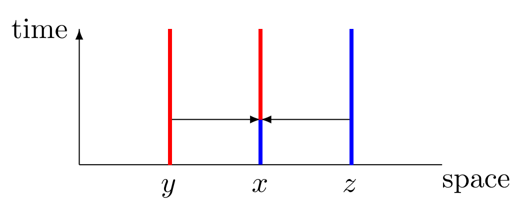
The algebra to check the second equality is a bit long but straightforward.

(2) Use graphical representations of the Poisson events defining (η_t^A) at the beginning of this section. This uses the interpretation that the right-hand side of (2) is a number of oriented paths in space and time. These paths travel along arrows defined by 1's in the random matrices $\mathrm{Id} + J_m$ so that $x \to y$ if and only if $(\mathrm{Id} + J_m)(y, x) = 1$ (*not* $(\mathrm{Id} + J_m)(x, y) = 1$). But self-loops are ignored. Then

$$\eta_t^A(x) \equiv \#\{\text{oriented paths from } A \times \{0\} \text{ to } (x, t)\}, \quad x \in E.$$

Note that $\langle \widehat{\eta}_t^{t,B}, \mathbf{1}_A \rangle$ counts the numbers of reversely oriented paths between $A \times \{0\}$ and $B \times \{t\}$ in $\mathbb{Z}_2$ and so is equal to $\langle \mathbf{1}_B, \eta_t^A \rangle$ in $\mathbb{Z}_2$.

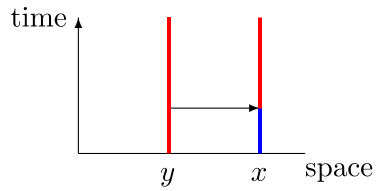
Example 2.2. Write $\color{red}{\bullet} = 1$ and $\color{blue}{\bullet} = 0$. Then for the Neuhauser-Pacala model, if $E = \{x, y, z\}$, the initial condition $\mathbf{1}_y$ after the update $\{x\} \times \{y, z\}$ can be visualized by



The two arrows are those defined by the following matrix which are not self-loops:

$$\mathrm{Id} + \begin{matrix} & x & y & z \\ x & 0 & 1 & 1 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{matrix} = \mathrm{Id} + \begin{matrix} & x & y & z \\ x & 1 & 1 & 1 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{matrix}.$$

Similarly, for $E = \{x, y\}$, the initial condition $\eta_0 = \mathbf{1}_y$ after the voting update of $\eta \mapsto \eta + \{x\} \times \{x, y\}\eta$ can be visualized by



One can check that the arrow is the one defined by the following matrix which is not a self-loop:

$$\mathrm{Id} + \{x\} \times \{x, y\} = \mathrm{Id} + \begin{matrix} & x & y \\ x & 1 & 1 \\ y & 0 & 0 \end{matrix} = \begin{matrix} & x & y \\ x & 0 & 1 \\ y & 0 & 1 \end{matrix}.$$

□

The usefulness of this parity dual equation follows from the fact that parity events uniquely determine a measure.

Proposition 2.3. *For finite measures ν_1 and ν_2, $\nu_1\{\eta; |\eta \cap B| \equiv 1\} = \nu_2\{\eta; |\eta \cap B| \equiv 1\}$ for all finite B implies that $\nu_1 = \nu_2$.*

Proof. The proof follows upon rewriting the following test functions in two different way:

$$\prod_{x \in A} [2\eta(x) - 1] = (-1)^{|\eta^{\complement} \cap A|} = 1 - 2\mathbb{1}_{\{\eta; |\eta^{\complement} \cap A| \equiv 1\}},$$
$$\prod_{x \in A} [2\eta(x) - 1] = 2^{|A|} \prod_{x \in A} \eta(x) + \text{a lower-order polynomial in } \eta(x) \text{ for } x \in A,$$

and then use an induction on $|A|$. □

3 Invariance of equiparity coexistence

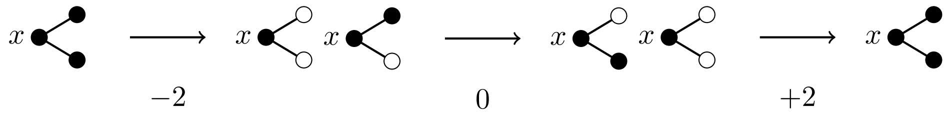
If we start the process with the Bernoulli product measure $\beta_{1/2}$ with density 1/2, then the corresponding parity dual equation can be written as

$$\mathbb{P}_{\beta_{1/2}}(\langle \mathbf{1}_B, \eta_t \rangle \equiv 1) = \mathbb{P}\left(\sum_{m=1}^{\langle \widehat{\eta}_t^B, \mathbf{1} \rangle} X_m \equiv 1, \widehat{\eta}_t^B \neq \mathbf{0} \right). \tag{4}$$

Here, $X_1, X_2, \cdots$ are i.i.d. $\{0, 1\}$-valued Bernoulli variables with mean 1/2 and independent of $\widehat{\eta}_t^B$. It is clear that

$$\text{the right-hand side of (4)} = \frac{1}{2}\mathbb{P}(\widehat{\eta}_t^B \neq \mathbf{0}).$$

If $\alpha = 0$, the process $(\widehat{\eta}_t^B)$ cannot die out unless $B = \varnothing$. This is because any 0-site has no effect on any other sites since 0 is the additive identity in $\mathbb{Z}_2$ and any 1-site x can only change its neighbors in three ways:



(5)

Here, the integer below each diagram is the difference of the numbers of filled sites between the right-hand graph and the left-hand graph.

Theorem 3.1. $\alpha = 0 \Longrightarrow \beta_{1/2}$ *invariant.*

There are two natural questions stemming from this theorem.

Question 3.2. Is $\beta_{1/2}$ also the unique limiting distribution whenever we start with mixing initial conditions?

Question 3.3. What if $\alpha \in (0,1)$?

These questions are closely related to the following conjecture due to Neuhauser and Pacala [3, Conjecture 1].

Conjecture 3.4. *There is coexistence in the symmetric Neuhauser-Pacala model on* $\mathbb{Z}^d$ *for any* $d \geq 2$.

We give a tangential discussion of the role of $\beta_{1/2}$ in the model.

Theorem 3.5. *For all* $u \in (0,1)$,

$$\begin{aligned}
&\langle \widehat{\eta}^B_{t_k}, \mathbf{1}\rangle \xrightarrow[t_k\to\infty]{\text{(d)}} \langle \widehat{\eta}^B_{\infty}, \mathbf{1}\rangle \quad \textit{in} \quad \mathbb{Z}_+ \cup \{+\infty\} \\
\Longrightarrow &\mathbb{P}_{\beta_u}\big(\langle \mathbf{1}_B, \eta_{t_k}\rangle \equiv 1\big) \xrightarrow[t_k\to\infty]{} \frac{1}{2}\mathbb{E}\left[1-(1-2u)^{\langle \widehat{\eta}^B_{\infty},\mathbf{1}\rangle}; \langle \widehat{\eta}^B_{\infty}, \mathbf{1}\rangle \geq 1\right].
\end{aligned}$$

Remark 3.6. (1) On a finite set, $(\beta_u)_{0<u<1}$ is quite large since it can generate uniform distributions over sets of configurations with fixed sizes.

(2) The proof of the theorem can be modified in an obvious way to show a similar formula if we use Bernoulli product measures with site-dependent densities, where the densities $\neq 0, 1$ are bounded away from 0 and 1. □

Corollary 3.7. $\mathbb{P}_{\beta_u}\big(\langle \mathbf{1}_B, \eta_{t_k}\rangle \equiv 1\big) \to \frac{1}{2}$ *if and only if* (1) $u = \frac{1}{2}$ *and the dual process survives:*

$$\lim_{t_k\to\infty} \mathbb{P}(\langle \widehat{\eta}^B_{t_k}, \mathbf{1}\rangle \geq 1) = 1,$$

or (2) $u \neq \frac{1}{2}$, *the dual process survives:*

$$\lim_{t_k\to\infty} \mathbb{P}(\langle \widehat{\eta}^B_{t_k}, \mathbf{1}\rangle \geq 1) = 1$$

and there is extinction versus unbounded growth in the dual process [4]:

$$\lim_{t_k\to\infty} \mathbb{P}(1 \leq \langle \widehat{\eta}^B_{t_k}, \mathbf{1}\rangle \leq L) = \mathbb{P}(1 \leq \langle \widehat{\eta}^B_{\infty}, \mathbf{1}\rangle < \infty) = 0.$$

The proof of the last theorem is an application of the parity dual equation and the following lemma.

Lemma 3.8 (Parity deviations). *Let $X_1, X_2, \cdots, X_N$ be independent $\mathbb{Z}_+$-valued random variables with $\mathbb{P}(X_m \equiv 1) = u_m$. Then it holds that*

$$\mathbb{P}\left(\sum_{m=1}^N X_m \equiv 0\right) - \mathbb{P}\left(\sum_{m=1}^N X_m \equiv 1\right) = \prod_{m=1}^N (1-2u_m).$$

Proof. Write the left-hand side as $\mathbb{E}\big[(-1)^{\sum_{m=1}^N X_m}\big]$. □

Proof of Theorem 3.5. With a slight abuse of notation, we write β_u as a random configuration independent of $(\widehat{\eta}_t^B)$. By the parity dual equation, we have

$$\begin{aligned}\lim_{k\to\infty} \mathbb{P}_{\beta_u}\big(\langle \mathbf{1}_B, \eta_{t_k}\rangle \equiv 1\big) &= \lim_{k\to\infty} \mathbb{P}\big(\langle \widehat{\eta}_{t_k}^B, \beta_u\rangle \equiv 1\big)\\ &= \lim_{k\to\infty}\Big[\mathbb{P}\big(\langle \widehat{\eta}_{t_k}^B, \beta_u\rangle \equiv 1, \langle \widehat{\eta}_\infty^B, \mathbf{1}\rangle = \infty\big) + \mathbb{P}\big(\langle \widehat{\eta}_{t_k}^B, \beta_u\rangle \equiv 1, \langle \widehat{\eta}_\infty^B, \mathbf{1}\rangle < \infty\big)\Big]\\ &= \frac{1}{2}\mathbb{P}\big(\langle \widehat{\eta}_\infty^B, \mathbf{1}\rangle = \infty\big) + \frac{1}{2}\mathbb{E}\left[1-(1-2u)^{\langle \widehat{\eta}_\infty^B, \mathbf{1}\rangle}; 1 \le \langle \widehat{\eta}_\infty^B, \mathbf{1}\rangle < \infty\right],\end{aligned}$$

where the last equality follows from Lemma 6. □

These lemma and theorem should provide the basic idea behind the related methods in [1, 3, 4]. There, the main technical issue for not starting with Bernoulli measures is to identify "Bernoulli subsystems" which have sizes growing to infinity as $t \to \infty$. The frameworks of those methods can be roughly fit into the following (trivial but presumably instructive) complications of the above context.

Let $X_m(t)$, $m \in \mathcal{I}_t$, be a collection of random variables such that $X_m(t)$, $m \in \mathcal{B}_t$, are conditionally independent given $\mathscr{F}_t$ and $\sum_{m\in\mathcal{I}_t\setminus\mathcal{B}_t} X_m(t) \in \mathscr{F}_t$, and so define a Bernoulli subsystem. Since $\sum_{m=1}^N X_m \equiv 1$ and $\sum_{m=1}^N X_m \equiv 0$ are complements to each other, the lemma shows that

$$\mathbb{P}\left(\sum_{m\in\mathcal{I}_t} X_m(t) \equiv 1 \,\middle|\, \mathscr{F}_t\right) = \mathbb{P}\left(\sum_{m\in\mathcal{B}_t} X_m(t) \equiv 1 - \sum_{m\in\mathcal{I}_t\setminus\mathcal{B}_t} X_m(t) \,\middle|\, \mathscr{F}_t\right) \simeq \frac{1}{2}, \tag{6}$$

where the error bound for the approximate equality depends only on how $u_m(t)$'s are close to 0 and 1 and the size of $\mathcal{B}_t$.

Domination by oriented percolation. In more detail, the method in [3], extended from [1], uses the following parity dual equation (which is a simple consequence of the Markov property of (η_t^A) and the old parity dual equation):

$$\mathbb{P}(\langle \mathbf{1}_B, \eta_{2t}^A\rangle \equiv 1) = \mathbb{P}(\langle \widehat{\eta}_t^B, \eta_t^A\rangle \equiv 1).$$

Then one writes $\langle \widehat{\eta}_t^B, \eta_t^A\rangle = |\widehat{\eta}_t^B \cap \eta_t^A|$ (configurations are identified as sets of vertices with 1's) as a sum of the $\{0,1\}$-valued random variables

$$X_m(t) \equiv |\widehat{\eta}_t^B \cap \eta_t^A \cap E_m|, \quad m \in \mathcal{I}_t = \mathbb{Z},$$

where $\{E_m\}$ is a partition of the space. The proof proceeds by picking out the space-time boxes where both of the processes (η_t^A) and $(\widehat{\eta}_t^B)$ are "isolated" locally. Namely, the Poisson clocks in the constructions of these processes, which govern interactions between sites inside the spatial regions and sites outside, do not ring within the time intervals. These isolated processes can be further refined to the set

$$X_m(t), \quad m \in \mathcal{B}_t,$$

of independent Bernoulli variables when suitably conditioned, if E_m and $E_{m'}$ are far away from each for every $m \neq m'$. The remaining task is to show that $\mathcal{B}_t$ can grow to infinity with probability one. This question is reduced to a question about domination by oriented percolation.

Extinction versus unbounded growth. The method in [4, Section 3] takes a different route, and the discussion above is more similar to it. Now we consider the dual equation in the form: for fixed $s > 0$,

$$\mathbb{P}(\langle \mathbf{1}_B, \eta^A_{s+t}\rangle \equiv 1) = \mathbb{P}(\langle \widehat{\eta}^B_t, \eta^A_s\rangle \equiv 1) = \mathbb{E}\big[\mathbb{P}(\langle \mathbf{1}_C, \eta^A_s\rangle \equiv 1)\big|_{\mathbf{1}_C = \widehat{\eta}^B_t}\big].$$

The first step shows that if there are sufficiently many types of random matrices J which can trigger a change in $\langle \mathbf{1}_C, \eta^A_r\rangle$ for $r \in [0, s]$ by *one* occurrence, which amounts to choosing appropriate J's such that $\langle \mathbf{1}_C, (\mathrm{Id} + J)\mathbf{1}_A\rangle = 1$, then $\mathbb{P}(\langle \mathbf{1}_C, \eta^A_s\rangle \equiv 1) \simeq \frac{1}{2}$. Roughly speaking, these Bernoulli variables are defined by the corresponding Poisson processes conditioned to ring at most once by time s. The second step then aims to find the good A's by using appropriate initial conditions[3] and the good C's by using the extinction versus unbounded growth condition. See [4, Section 3.2] for these two steps. The work in [4] also shows conditions for the extinction versus unbounded growth condition in terms of the survival of (η_t) and a certain recurrence property of $(\widehat{\eta}_t)$ when $\alpha \in (0, 1)$ [4, Sections 3.3 and 3.5].

4 References

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[3]The following initial conditions μ are enough for the required convergence: μ is translation invariant and is supported in the set of configurations $\mathbf{1}_A$ where η^A_t can be any configuration with positive probability when restricted to an arbitrary finite set for every $t > 0$.