Infinite Rémy bridges

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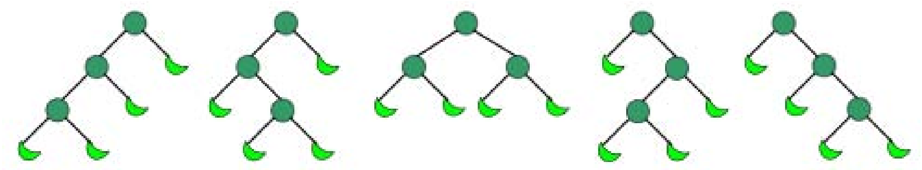

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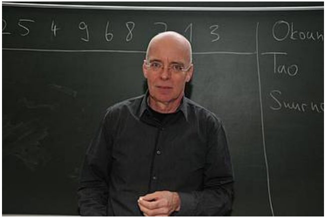

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Binary trees

- Write $\{0,1\}^\star := \bigsqcup_{k=0}^{\infty}\{0,1\}^k$ for the set of finite words drawn from the alphabet $\{0,1\}$ (with the empty word $\emptyset$ allowed).
- A binary tree is a finite subset $\mathbf{t} \subset \{0,1\}^\star$ with the properties:
  - $v_1 \ldots v_k \in \mathbf{t} \Longrightarrow v_1 \ldots v_{k-1} \in \mathbf{t}$,
  - $v_1 \ldots v_k 0 \in \mathbf{t} \Longleftrightarrow v_1 \ldots v_k 1 \in \mathbf{t}$.

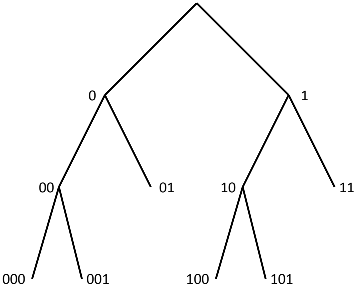


Figure: A binary tree is just a finite rooted tree in which every individual has zero or two children and we can distinguish left from right.

- Call the empty word $\emptyset$ the root of the tree.
- A binary tree has $2n+1$ vertices for some $n \in \mathbb{N}$: $n+1$ leaves and n interior vertices.
- The number of binary trees with $2n+1$ vertices is the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.

- Rémy's (1985) algorithm generates a sequence of random binary trees $T_1, T_2, \ldots$ such that T_1 is the unique binary tree $\aleph := \{\emptyset, 0, 1\}$ with 3 vertices and T_n is uniformly distributed on the set of binary trees with $2n+1$ vertices.

Example of one iteration of Rémy's algorithm

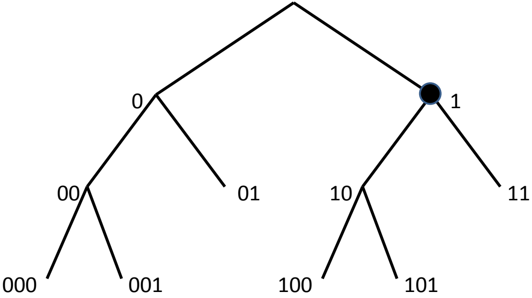


Figure: First step in an iteration of Rémy's algorithm: pick a vertex v uniformly at random.

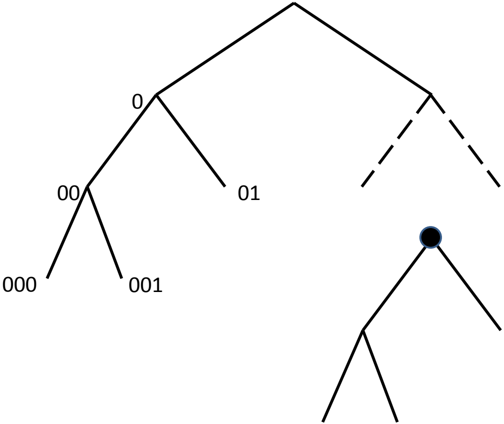


Figure: Second step in an iteration of Rémy's algorithm: cut off the subtree rooted at v and attach a copy of $\aleph$ to the end of the edge that previously led to v.

Example of one iteration of Rémy's algorithm – continued

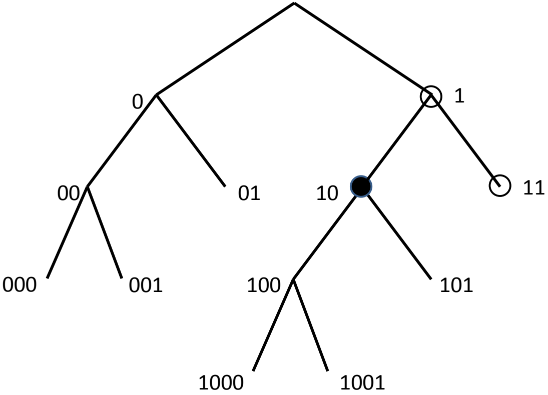


Figure: Third step in an iteration of Rémy's algorithm: re-attach the subtree rooted at v to one of the two leaves of the copy of $\aleph$, and re-label the vertices appropriately. The solid circle is the new location of v and the open circles are the clones of v.

- A segment in a metric space (X, d) is the image of an isometry $\alpha : [a, b] \to X$. The endpoints of the segment are $\alpha(a)$ and $\alpha(b)$.
- An $\mathbb{R}$-tree is a metric space (X, d) with the following properties.
    - For all $x, y \in X$ there is a segment in X with endpoints $\{x, y\}$.
    - If two segments of (X, d) intersect in a single point, which is an endpoint of both, then their union is a segment.
- Fact: If (X, d) is an $\mathbb{R}$-tree, then for all $x, y \in X$ there is a unique segment in X with endpoints $\{x, y\}$.

Connections between Rémy's chain and Aldous' Brownian CRT

- Marchal (2003) showed that Rémy's sequence of trees thought of as $\mathbb{R}$-trees with unit edge lengths converge almost surely in a suitable sense to a random $\mathbb{R}$-tree $\mathcal{T}$ called Aldous' Brownian continuum random tree (CRT) after rescaling the edge lengths by $n^{-\frac{1}{2}}$ at step n.
- Conversely, Le Gall (1999) showed that if one successively samples points in a conditionally independent manner from the Brownian CRT $\mathcal{T}$ using the associated mass measure on the leaves and thinks of the trees induced by the sampled leaves as (combinatorial) binary trees, then the resulting process is Rémy's chain.
- It follows from Hewitt-Savage that the limit Brownian CRT $\mathcal{T}$ generates the tail σ-field of the Rémy chain up to null sets.

- Conditioning the Rémy chain on the event $\{\lim_{n\to\infty} n^{-\frac{1}{2}} T_n = \mathbf{t}\}$ produces a Markov chain that has the same backward transition probabilities as $(T_n)_{n\in\mathbb{N}}$ itself and a trivial tail σ-field.
- We call Markov chain that starts at $\aleph$ and has the same backward transition probabilities as $(T_n)_{n\in\mathbb{N}}$ an infinite Rémy bridge.
- We say that an infinite Rémy bridge with a trivial tail σ-field is extremal.
- By general theory, any infinite Rémy bridge is a mixture of extremal ones.
- What are all the extremal infinite Rémy bridges?

Backward transition probabilities

- An infinite Rémy bridge evolves backwards in time as follows:
  - Pick a leaf uniformly at random.
  - Delete the chosen leaf and its sibling.
  - Close up the gap if there is one.

Example of a backward transition for a Rémy bridge

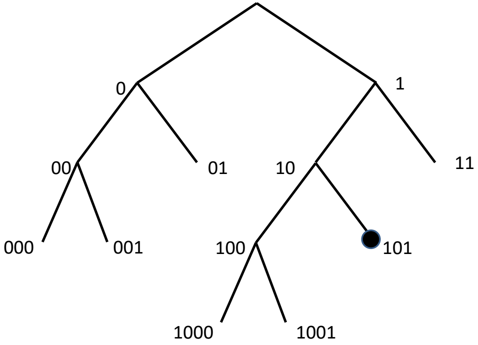


Figure: First step in a backward transition of an infinite Rémy bridge: pick a leaf w uniformly at random.

Example of a backward transition – continued

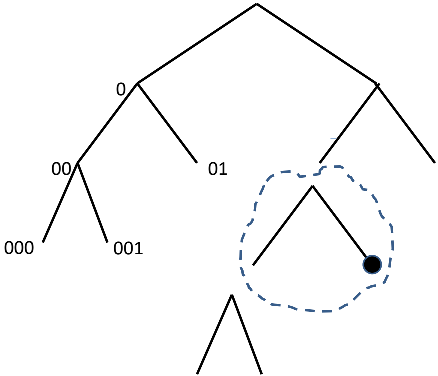


Figure: Second step in a backward transition of an infinite Rémy bridge: delete the chosen leaf w and its sibling.

Example of a backward transition – continued

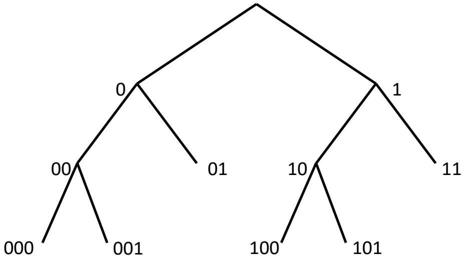


Figure: Third step in a backward transition of an infinite Rémy bridge: close up the gap.

- The radix sort tree for a collection of distinct inputs $z_1, z_2, \ldots, z_n \in \{0,1\}^\infty$ is the tree $R(z_1, z_2, \ldots, z_n)$ with leaves $\zeta_1, \zeta_2, \ldots, \zeta_n \in \{0,1\}^\star$, where ζ_i is the shortest prefix of z_i that is not a prefix of ζ_j for $j \neq i$.
- Note that $R(z_1, \ldots, z_n) = R(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ for any permutation σ of $[n]$.
- The possible radix sort trees are the trees such that if $v_1 \ldots v_{\ell-1} v_\ell$ is a leaf, then $v_1 \ldots v_{\ell-1} \bar{v}_\ell$ is a vertex, where $\bar{0} := 1$ and $\bar{1} := 0$.

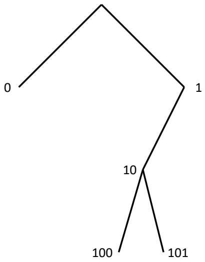


Figure: The radix sort tree for the inputs $z_1 = 100 * **$, $z_2 = 0 * **$, $z_3 = 101 * **$.

- Write Φ for the map that takes a tree, removes the vertices with outdegree 1, and "closes up the gaps" to produce a binary tree.
- The PATRICIA ("Practical Algorithm To Retrieve Information Coded In Alphanumeric") tree for a collection of inputs $z_1, z_2, \ldots, z_n$ is the binary tree $\bar{R}(z_1, z_2, \ldots, z_n) := \Phi \circ R(z_1, z_2, \ldots, z_n)$.

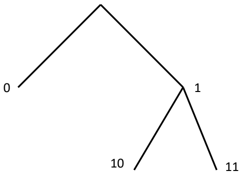


Figure: The PATRICIA tree for the inputs $z_1 = 100 * **$, $z_2 = 0 * **$, $z_3 = 101 * **$.

- Given i.i.d. $\{0,1\}^\infty$-valued random variables $Z_1, Z_2, \ldots$ with common distribution some diffuse probability measure ν, set

$$^\nu\bar{R}_n := \bar{\mathbf{R}}(Z_1, \ldots, Z_{n+1}).$$

- The Markov chain $(^\nu\bar{R}_n)_{n\in\mathbb{N}}$ is an extremal infinite Rémy bridge.

Yet another example: the zig-zag chain

- Let $((U_n, \eta_n))_{n \in \mathbb{N}}$ be an infinite sequence of independent identically distributed $[0,1] \times \{0,1\}$-valued random variables such that
  - U_n has the uniform distribution on $[0,1]$,
  - $\mathbb{P}\{\eta_n = 0\} = 1 - p$ and $\mathbb{P}\{\eta_n = 1\} = p$,
  - U_n and η_n are independent.
- Let σ_n be the random permutation of $[n]$ such that $U_{\sigma_n(1)} < U_{\sigma_n(2)} < \ldots < U_{\sigma_n(n)}$.
- For $k \in [n]$, put $\epsilon_{n,k} = \eta_{\sigma_n(k)}$.
- Set $S_n := \bigcup_{k \in [n]} \{\epsilon_{n,1} \ldots \epsilon_{n,k},\ \epsilon_{n,1} \ldots \epsilon_{n,(k-1)} \bar{\epsilon}_{n,k}\}$ for $n \in \mathbb{N}$
- The tree S_n consists of a single path of length n with obligatory leaves attached.
- The process evolves by inserting an additional independent random step into the path at a uniformly chosen position. The step is to the left with probability $1 - p$ and to the right with probability p.
- The Markov chain $(S_n)_{n \in \mathbb{N}}$ is an extremal infinite Rémy bridge.

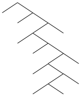


Figure: A possible realization at time $n = 9$ of the zig-zag chain.

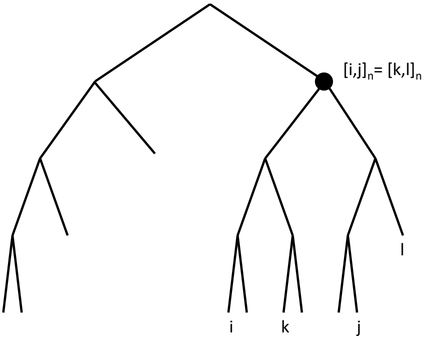
Consistently labeling the leaves in an infinite Rémy bridge

- Fix an infinite Rémy bridge $(T_n^\infty)_{n\in\mathbb{N}}$.
- By Kolmogorov's extension theorem, there is a Markov process $(\tilde{T}_n^\infty)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ the random element $\tilde{T}_n^\infty$ is a leaf-labeled binary tree with $n+1$ leaves labeled by $[n+1]$ and the following hold.
  - The binary tree obtained by removing the labels of $\tilde{T}_n^\infty$ is T_n^∞.
  - For every $n \in \mathbb{N}$, the conditional distribution of $\tilde{T}_n^\infty$ given T_n^∞ is uniform over the $(n+1)!$ possible labelings of T_n^∞.
  - In going backward from time $n+1$ to time n, $\tilde{T}_{n+1}^\infty$ is transformed into $\tilde{T}_n^\infty$ as follows:
    - The leaf labeled $n+2$ is deleted, along with its sibling.
    - If the sibling of the leaf labeled $n+2$ is also a leaf, then the common parent (which is now a leaf) is assigned the sibling's label.

What does the labeling do for us?

- We will use the labeling and a projective limit construction to build an infinite binary-tree-like structure for which $\mathbb{N}$ plays the role of the leaves.
- The "subtree spanned by the leaves labeled by [n+1]" in the projective limit will be essentially $\tilde{T}_n^\infty$, and hence the projective limit will encode the whole of $(\tilde{T}_n^\infty)_{n\in\mathbb{N}}$ (and hence the whole of $(T_n^\infty)_{n\in\mathbb{N}}$).

- If $i, j \in \mathbb{N}$ are the labels of two leaves T_n^∞ that are represented as the words $u_1 \dots u_k$ and $v_1 \dots v_\ell$ in $\{0,1\}^\star$, then set $[i,j]_n := u_1 \dots u_m = v_1 \dots v_m$, where $m := \max\{h : u_1 \dots u_h = v_1 \dots v_h\}$.
- That is, $[i,j]_n$ is the most recent common ancestor in T_n^∞ of the leaves labeled i and j.



- Define an equivalence relation $\equiv$ on the Cartesian product $\mathbb{N} \times \mathbb{N}$ by declaring that $(i', j') \equiv (i'', j'')$ if and only if $[i', j']_n = [i'', j'']_n$ for some (and hence all) n such that $i', j', i'', j'' \in [n+1]$.
- Write $\langle i, j \rangle$ for the equivalence class of the pair (i, j).
- Think of $\langle i, j \rangle$ as the being the most recent common ancestor of the leaves i and j and of such points being interior vertices of a tree-like object.

- Define a partial order $<_L$ on the set of equivalence classes by declaring for $(i', j'), (i'', j'') \in \mathbb{N} \times \mathbb{N}$ that $\langle i', j' \rangle <_L \langle i'', j'' \rangle$ if and only if for some (and hence all) n such that $i', j', i'', j'' \in [n+1]$ we have $[i', j']_n = u_1 \ldots u_k$ and $[i'', j'']_n = u_1 \ldots u_k 0 v_1 \ldots v_\ell$ for some $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in \{0, 1\}$.
- Interpret the ordering $\langle i', j' \rangle <_L \langle i'', j'' \rangle$ as the "vertex" $\langle i'', j'' \rangle$ being below and to the left of the "vertex" $\langle i', j' \rangle$.

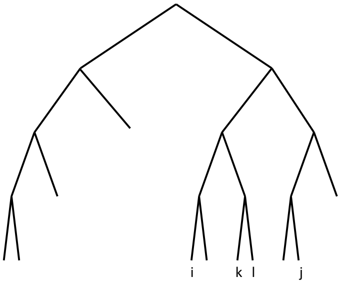


Figure: Configuration leading to $\langle i, j \rangle <_L \langle k, \ell \rangle$ in the projective limit.

Ordering equivalence classes – below and to the right

- Similarly, define another partial order $<_R$ by declaring that $\langle i', j' \rangle <_R \langle i'', j'' \rangle$ if and only if for some (and hence all) n such that $i', j', i'', j'' \in [n+1]$ we have $[i', j']_n = u_1 \dots u_k$ and $[i'', j'']_n = u_1 \dots u_k 1 v_1 \dots v_\ell$ for some $u_1, \dots, u_k, v_1, \dots, v_\ell \in \{0, 1\}$.
- Interpret the ordering $\langle i', j' \rangle <_R \langle i'', j'' \rangle$ as the "vertex" $\langle i'', j'' \rangle$ being below and to the right of the "vertex" $\langle i', j' \rangle$.

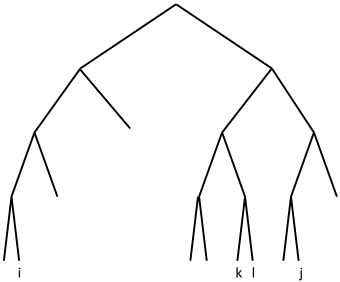


Figure: Configuration leading to $\langle i, j \rangle <_R \langle k, \ell \rangle$ in the projective limit.

- Define a third partial order $<$ on the set of equivalence classes of $\mathbb{N} \times \mathbb{N}$ by declaring that $\langle i', j' \rangle < \langle i'', j'' \rangle$ if either $\langle i', j' \rangle <_L \langle i'', j'' \rangle$ or $\langle i', j' \rangle <_R \langle i'', j'' \rangle$.
- Interpret the ordering $\langle i', j' \rangle < \langle i'', j'' \rangle$ as the "vertex" $\langle i'', j'' \rangle$ being below the "vertex" $\langle i', j' \rangle$.

The equivalence relation $\equiv$ and the partial orders $<_L$, $<_R$, and $<$ have the following properties.

(A) For $i, j \in \mathbb{N}$, $(i, j) \equiv (j, i)$.

(B) For $i, j, k \in \mathbb{N}$ $(i, j) \not\equiv (k, k)$ unless $i = j = k$.

(C) Fix $f, g, h, i, j, k \in \mathbb{N}$. If $\langle f, g\rangle <_L \langle h, i\rangle$ and $\langle h, i\rangle <_R \langle j, k\rangle$, then $\langle f, g\rangle <_L \langle j, k\rangle$. Similarly, if $\langle f, g\rangle <_R \langle h, i\rangle$ and $\langle h, i\rangle <_L \langle j, k\rangle$, then $\langle f, g\rangle <_R \langle j, k\rangle$.

(D) For $h, i, j, k \in \mathbb{N}$,

- $\langle h, i\rangle <_L \langle j, k\rangle$ if and only if $\langle h, i\rangle <_L j$ and $\langle h, i\rangle <_L k$;
- $\langle h, i\rangle <_R \langle j, k\rangle$ if and only if $\langle h, i\rangle <_R j$ and $\langle h, i\rangle <_R k$;
- $\langle h, i\rangle = \langle j, k\rangle$ if and only if either $\langle h, i\rangle \leq_L j$ and $\langle h, i\rangle \leq_R k$ or $\langle h, i\rangle \leq_L k$ and $\langle h, i\rangle \leq_R j$.

(E) For $h, i, j, k \in \mathbb{N}$, $\langle h, i\rangle < \langle j, k\rangle$ if and only if $\langle h, i\rangle <_L \langle j, k\rangle$ or $\langle h, i\rangle <_R \langle j, k\rangle$.

(F) For distinct $i, j, k \in \mathbb{N}$ there exist $\ell, m \in \{i, j, k\}$ such that $\langle \ell, m\rangle < p$ for all $p \in \{i, j, k\}$.

- A didendritic system is a set $\mathcal{N} \times \mathcal{N}$ for some non-empty (possibly infinite) set $\mathcal{N}$ that is equipped with an equivalence relation $\equiv$, equivalence classes $\langle \cdot, \cdot \rangle$, and partial order $<_L$, $<_R$, and $<$ on the equivalence classes such that (A) – (F) hold (with $\mathbb{N}$ replaced by $\mathcal{N}$).
- We have coined the word "didendritic" from the Greek roots "$\delta\iota\varsigma$" = "two, twice or double" and "$\delta\epsilon\nu\delta\rho\iota\tau\eta\varsigma$" = "of or pertaining to a tree, tree-like" as an adjective meaning "binary tree-like".
- A finite didendritic system is the same things as a leaf-labeled binary tree.
- A didendritic system on $\mathbb{N}$ is the same things as a sequence of leaf-labeled trees $(\tilde{\mathbf{t}}_n)_{n\in\mathbb{N}}$ with the property that the leaves of $\tilde{\mathbf{t}}_n$, $n \in \mathbb{N}$, are labeled by $[n+1]$ and the sequence is is consistent in the sense that $\tilde{\mathbf{t}}_n$ is produced from $\tilde{\mathbf{t}}_{n+1}$ by:
  - deleting leaf labeled $n+2$ along with its sibling;
  - if the sibling was also a leaf, assigning its label to the common parent (which is now a leaf).

- Given a didendritic system $\mathbf{D} = (\mathcal{N}, \equiv, \langle \cdot, \cdot \rangle, <_L, <_R, <)$ and a permutation σ of $\mathcal{N}$ the didendritic system $\mathbf{D}^\sigma = (\mathcal{N}, \equiv^\sigma, \langle \cdot, \cdot \rangle^\sigma, <_L^\sigma, <_R^\sigma, <^\sigma)$ is defined by
  - $(i', j') \equiv^\sigma (i'', j'')$ if and only if $(\sigma(i'), \sigma(j')) \equiv (\sigma(i''), \sigma(j''))$,
  - $\langle i, j \rangle^\sigma$ is the equivalence class of the pair (i, j) for the equivalence relation $\equiv^\sigma$,
  - $\langle h, i \rangle^\sigma <_L^\sigma \langle j, k \rangle^\sigma$ if and only if $\langle \sigma(h), \sigma(i) \rangle <_L \langle \sigma(j), \sigma(k) \rangle$,
  - $\langle h, i \rangle^\sigma <_R^\sigma \langle j, k \rangle^\sigma$ if and only if $\langle \sigma(h), \sigma(i) \rangle <_R \langle \sigma(j), \sigma(k) \rangle$,
  - $\langle h, i \rangle^\sigma <^\sigma \langle j, k \rangle^\sigma$ if and only if $\langle \sigma(h), \sigma(i) \rangle < \langle \sigma(j), \sigma(k) \rangle$.

Exchangeable random didendritic systems

- A random didendritic system $\mathbf{D} = (\mathbb{N}, \equiv, \langle \cdot, \cdot \rangle, <_L, <_R, <)$ is exchangeable if for each permutation σ of $\mathbb{N}$ such that $\sigma(i) = i$ for all but finitely many $i \in \mathbb{N}$ the random didendritic system $\mathbf{D}^\sigma$ has the same distribution as $\mathbf{D}$.

- The random didendritic system on $\mathbb{N}$ corresponding to the labeled version of an infinite Rémy bridge is exchangeable.
- Conversely, the sequence of random labeled binary trees produced from an exchangeable random didendritic system by successively restricting to $[n+1]$, $n \in \mathbb{N}$, is the labeled version of an infinite Rémy bridge.

- An exchangeable random didendritic system $\mathbf{D}$ is ergodic if

$$\mathbb{P}(\{\mathbf{D} \in A\} \triangle \{\mathbf{D}^\sigma \in A\}) = 0$$

for some measurable set A for all permutations σ of $\mathbb{N}$ with $\sigma(i) = i$ for all but finitely many $i \in \mathbb{N}$ implies that

$$\mathbb{P}\{\mathbf{D} \in A\} \in \{0, 1\}.$$

- Any exchangeable random didendritic system is a mixture of ergodic exchangeable random didendritic systems.
- An infinite Rémy bridge is extremal if and only if the corresponding exchangeable random didendritic system is ergodic.

Building a didendritic system on $\mathbb{N}$ from a $\mathbb{R}$-tree: Step 1

- Consider a complete separable $\mathbb{R}$-tree $\mathbf{S}$ and a distinguished point $\rho \in \mathbf{S}$.
- Define a partial order $\prec$ on $\mathbf{S}$ by declaring that $x \prec y$ if $[\rho, x] \subsetneq [\rho, y]$.
- Given $x, y \in \mathbf{S}$ define $x \curlywedge y \in \mathbf{S}$ by $[\rho, x \curlywedge y] = [\rho, x] \cap [\rho, y]$.
- Observe that $x \curlywedge y \preceq x$, $x \curlywedge y \preceq y$, and for any $z \in \mathbf{S}$ with $z \preceq x$ and $z \preceq y$ we have $z \preceq x \curlywedge y$.

Building a didendritic system on $\mathbb{N}$ from a $\mathbb{R}$-tree: Step 2

- Let $\{x_n : n \in \mathbb{N}\}$ be a subset of $\mathbf{S}$.
- Suppose that for distinct $i, j, k \in \mathbb{N}$, one of

$$x_i \curlywedge x_j = x_i \curlywedge x_k \prec x_j \curlywedge x_k,$$

$$x_j \curlywedge x_k = x_j \curlywedge x_i \prec x_k \curlywedge x_i,$$

or

$$x_k \curlywedge x_i = x_k \curlywedge x_j \prec x_i \curlywedge x_j$$

holds.

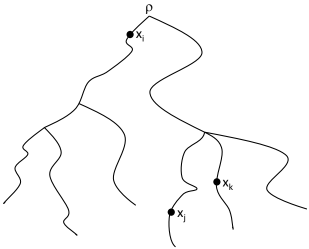


Figure: Although this $\mathbb{R}$-tree is not "binary", the required condition holds for the given choice of x_i, x_j, x_k.

- Define an equivalence relation $\equiv$ on $\mathbb{N} \times \mathbb{N}$ by declaring that
  - $(i, i) \equiv (j, k)$ if and only if $x_i = x_j = x_k$ (equivalently, $i = j = k$),
  - for $h \neq i$ and $j \neq k$, $(h, i) \equiv (j, k)$ if and only if $x_h \curlywedge x_i = x_j \curlywedge x_k$.
- Write $\langle i, j \rangle$ for the $\equiv$-equivalence class of (i, j).
- For simplicity, write i for $\langle i, i \rangle$.

Building a didendritic system on $\mathbb{N}$ from a $\mathbb{R}$-tree: Step 4

- Define a partial order $<$ on $\{\langle i,j\rangle : i,j \in \mathbb{N}\}$ by declaring that
  - for $\langle i,j\rangle \neq k$, $\langle i,j\rangle < k$ if and only if $i \neq j$ and $x_i \curlywedge x_j \preceq x_k$ (in particular, $\langle i,j\rangle < i$ and $\langle i,j\rangle < j$ for $i \neq j$),
  - for $h \neq i$ and $j \neq k$, $\langle h,i\rangle < \langle j,k\rangle$ if and only if $x_h \curlywedge x_i \prec x_j \curlywedge x_k$.

- Suppose further that for distinct $i, j \in \mathbb{N}$ there are elements $w(i,j)$ of the set $\{\curvearrowright, \curvearrowleft\}$ with the following properties:
  - For $i \neq j$, $w(i,j) = \curvearrowright$ if and only if $w(j,i) = \curvearrowleft$.
  - For distinct $i, j, k \in \mathbb{N}$, if $x_i \curlywedge x_j = x_i \curlywedge x_k \prec x_j \curlywedge x_k$, then $w(i,j) = w(i,k)$.
- FACT: There is a unique pair of partial orders $<_L$ and $<_R$ on $\{\langle i, j\rangle : i, j \in \mathbb{N}\}$ such that

$$\langle i, j\rangle <_L i \text{ and } \langle i, j\rangle <_R j \iff w(i,j) = \curvearrowright .$$

- END RESULT: The objects $\mathbb{N}$, $\equiv$, $\langle \cdot, \cdot \rangle$, $<_L$, $<_R$, $<$ form a didendritic system

Building an ergodic exchangeable random didendritic system: Step 1

- Start with a complete separable $\mathbb{R}$-tree $\mathbf{S}$ and a distinguished point $\rho \in \mathbf{S}$, and suitably randomize the above construction as follows.
- Fix a diffuse probability measure μ on $\mathbf{S}$ and take $\{x_n : n \in \mathbb{N}\}$ to be a realization of $\{\xi_n : n \in \mathbb{N}\}$, where $(\xi_n)_{n \in \mathbb{N}}$ are i.i.d. with common distribution μ.
- Require that μ is such that almost surely for distinct $i, j, k \in \mathbb{N}$, one of

  $$\xi_i \curlywedge \xi_j = \xi_i \curlywedge \xi_k \prec \xi_j \curlywedge \xi_k,$$

  $$\xi_j \curlywedge \xi_k = \xi_j \curlywedge \xi_i \prec \xi_k \curlywedge \xi_i,$$

  or

  $$\xi_k \curlywedge \xi_i = \xi_k \curlywedge \xi_j \prec \xi_i \curlywedge \xi_j$$

  holds.
- Define the equivalence relation $\equiv$, equivalence classes $\langle \cdot, \cdot \rangle$, and partial order $<$ as above.

Building an ergodic exchangeable random didendritic system: Step 2

- Construct a sequence of i.i.d. $[0,1]$-valued r.v. $(U_n)_{n \in \mathbb{N}}$ with common uniform distribution.
- Define $\{\curvearrowright, \curvearrowleft\}$-valued r.v. $w(i,j)$, $i,j \in \mathbb{N}$, $i \neq j$, by setting

$$w(i,j) := W(\xi_i, U_i, \xi_j, U_j)$$

  for a function $W : \mathbf{S} \times [0,1] \times \mathbf{S} \times [0,1] \to \{\curvearrowright, \curvearrowleft\}$ such that almost surely:
    - for $i \neq j$, $W(\xi_i, U_i, \xi_j, U_j) = \curvearrowright$ if and only if $W(\xi_j, U_j, \xi_i, U_i) = \curvearrowleft$;
    - for distinct i, j, k, if $\xi_i \curlywedge \xi_j = \xi_i \curlywedge \xi_k \prec \xi_j \curlywedge \xi_k$, then $W(\xi_i, U_i, \xi_j, U_j) = W(\xi_i, U_i, \xi_k, U_k)$.
- Define partial orders $<_L$ and $<_R$ as above.

Conclusion

- The above construction builds an ergodic exchangeable random didendritic system (and hence an extremal infinite Rémy bridge).
- Conversely, any ergodic exchangeable random didendritic system (and hence any extremal infinite Rémy bridge) arises from this construction — this converse takes a lot of work and involves the Aldous–Hoover–Kallenberg theory of exchangeable arrays. Moreover, there are parallels with graph limits and graphons.