From annihilating Brownian motions to the symbiotic branching model

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joint work with M. Ortgiese and F. Völlering (University of Bath)

Genealogies of Interacting Particle Systems
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PROBABILISTIC STRUCTURES
IN EVOLUTION

DFG SPP 1590

Annihilating Brownian Motions (aBMs)

- Consider a system of finitely many particles performing independent Brownian motions until two of them meet.
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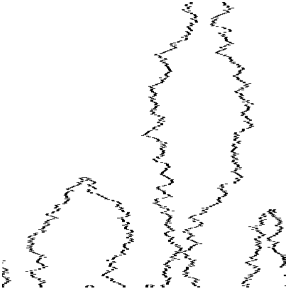

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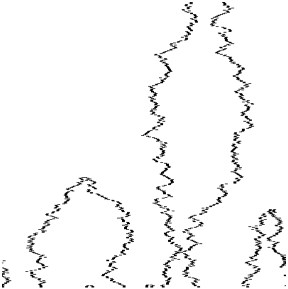

- Write $(\mathbf{X}_t^{\mathbf{x}})_{t\geq 0}$ for such a system of aBMs started from $\mathbf{x} \subset \mathbb{R}$. Construction straightforward as long as $\mathbf{x}$ is finite.

From finitely many to infinitely many

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- Let $(\mathbf{Y}_t^{\mathbf{x}_n})_{t\geq 0}$ resp. $(\mathbf{Y}_t^{\mathbf{x}})_{t\geq 0}$ denote a system of *coalescing* Brownian motions (cBMs) started from $\mathbf{x}_n$ resp. $\mathbf{x}$ such that

$$\mathbf{Y}_t^{\mathbf{x}_n} \subseteq \mathbf{Y}_t^{\mathbf{x}_{n+1}} \subseteq \cdots \subseteq \mathbf{Y}_t^{\mathbf{x}} \qquad \text{for all } t > 0,\ n \in \mathbb{N}.$$

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- For $y \in \mathbf{Y}_t^{\mathbf{x}}$, let

  $$C(t, y) := \#\text{BMs which have coalesced in the path leading to } (t, y)$$

  and define

  $$\mathbf{X}_t^{\mathbf{x}} := \{y \in \mathbf{Y}_t^{\mathbf{x}} \mid C(t, y) \text{ is odd}\}.$$

Entrance laws

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- Characterize all sequences $\mathbf{x}_n \in \mathcal{D}$ such that $(\mathbf{X}_t^{\mathbf{x}_n})_{t>0}$ converges to some limiting annihilating system $(\mathbf{X}_t)_{t>0}$ with $\mathbf{X}_t \in \mathcal{D}$ for all $t > 0 \rightsquigarrow$ *entrance laws*.

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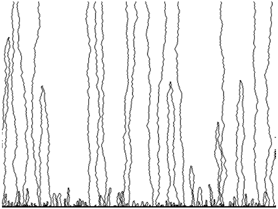
- Is there more than one limit which starts densely everywhere?
- For cBMs, the limit is unique $\rightsquigarrow$ coalescing point set of the Brownian web started from $\mathbb{R}$, 'maximal entrance law' for coalescing Brownian motions ([Arratia 1979]).

Simulations with discrete starting configurations

Credits to Florian Völlering...



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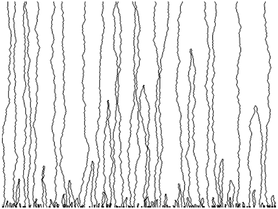

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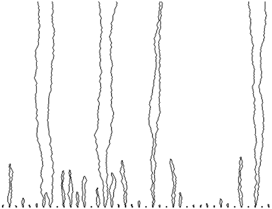

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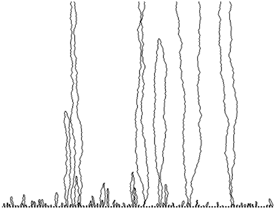

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$$\frac{dr(\mathbf{X}^{\mathbf{x}}, t)}{dz}(z) := \mathbb{1}_{\{(t,z) \text{ is red}\}}, \quad \frac{db(\mathbf{X}^{\mathbf{x}}, t)}{dz}(z) := \mathbb{1}_{\{(t,z) \text{ is blue}\}}.$$

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  Topology: Vague convergence, i.e. $u_n \to u$ in $\mathcal{M}_1(\mathbb{R})$ iff $\langle u_n, \phi \rangle \to \langle u, \phi \rangle$ for all $\phi \in \mathcal{C}_c(\mathbb{R})$.

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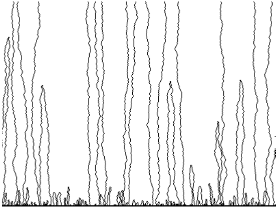
- Path-space topology of $\mathcal{C}([0,\infty); \mathcal{M}_1(\mathbb{R}))$.

Simulations (again)



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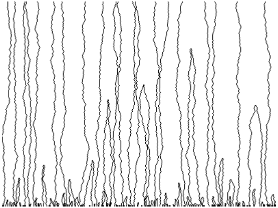

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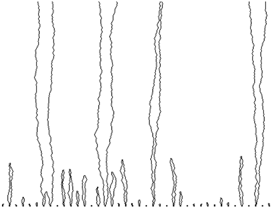

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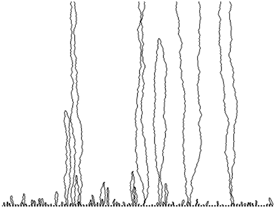

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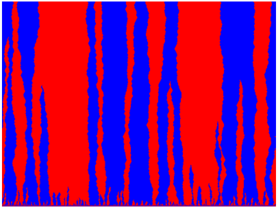

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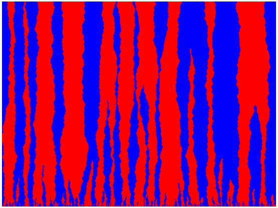

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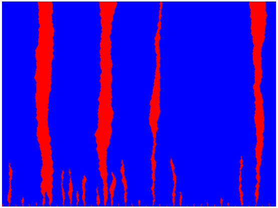

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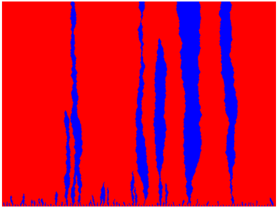

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Voter model

Classical (nearest-neighbor) voter model $(u_t)_{t\geq 0}$ on $\mathbb{Z}$: Markov process in $\{0,1\}^{\mathbb{Z}}$ with transitions

$$u(x) \text{ flips at rate } \frac{1}{2}\left(\mathbb{1}_{u(x-1)\neq u(x)} + \mathbb{1}_{u(x+1)\neq u(x)}\right)$$

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where $(\mathbf{Y}_t)_{t\geq 0}$ denotes a system of *instantly* coalescing random walks.

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Continuous-space analogue?

Continuous-space stepping stone / voter model

Theorem 1 ([Evans 1997], [Donnelly, Evans et. al. 2000])

There exists a unique Feller semigroup $(P_t)_{t\geq 0}$ on $\mathcal{M}_1(\mathbb{R})$ such that the corresponding Feller process $(u_t)_{t\geq 0}$ is characterized by the following moment duality: For all $u_0 \in \mathcal{M}_1(\mathbb{R})$ and $\mathbf{y} \subset \mathbb{R}$ finite we have

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The process $(u_t)_{t\geq 0}$ has continuous sample paths, and for all $t > 0$ fixed, we have almost surely

$$u_t(x) \in \{0,1\} \qquad \text{for almost all } x \in \mathbb{R}.$$

Stepping stone model: discrete case

A system of interacting Wright-Fisher diffusions on $\mathbb{Z}$:

$$\begin{cases} du_t^{(\gamma)}(x) = \frac{1}{2}\Delta u_t^{(\gamma)}(x)\,dt + \sqrt{\gamma u_t^{(\gamma)}(x)(1-u_t^{(\gamma)}(x))}\,dW_t(x), \\ u_0(x) \in [0,1], \quad x \in \mathbb{Z}. \end{cases}$$

Here Δ is the discrete Laplacian, $\{(W_t(x))_{t\geq 0} : x \in \mathbb{Z}\}$ is a system of independent Brownian motions, and $\gamma \in (0,\infty)$ is a parameter.

Notation: $\mathrm{dSSM}(\gamma)_{u_0}$.

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Fact:

$$\mathrm{dSSM}(\gamma) \xrightarrow[\gamma\uparrow\infty]{f.d.d.} \text{voter model} =: \mathrm{dSSM}(\infty).$$

Stepping stone model: continuous case

Stochastic heat equation with Wright-Fisher noise:

$$\begin{cases} \frac{\partial}{\partial t} u_t^{(\gamma)}(x) = \frac{1}{2}\Delta u_t^{(\gamma)}(x) + \sqrt{\gamma u_t^{(\gamma)}(x)(1-u_t^{(\gamma)}(x))}\,\dot{W}_t(x), \\ u_0(x) \in [0,1], \quad x \in \mathbb{R}. \end{cases}$$

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$$\mathbb{E}_{u_0}\left[\prod_{x\in\mathbf{y}} u_t^{(\gamma)}(x)\right] = \mathbb{E}_{\mathbf{y}}\left[\prod_{x\in\mathbf{Y}_t^{(\gamma)}} u_0(x)\right], \qquad t \geq 0 \qquad (3)$$

where $(\mathbf{Y}_t^{(\gamma)})_{t\geq 0}$ is a system of *delayed* coalescing Brownian motions.

Convergence of the continuous-space stepping stone model

Theorem 2 (H., Ortgiese, Völlering 2016)

For any $u_0(\cdot) \in \mathcal{M}_1(\mathbb{R})$*, the process* $(u_t^{(\gamma)})_{t\geq 0}$ *converges as* $\gamma \to \infty$ *in* $\mathcal{C}([0,\infty); \mathcal{M}_1(\mathbb{R}))$ *to the process* $(u_t)_{t\geq 0}$ *from Theorem 1.*

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Remark: As argued in [Athreya & Sun 2011], it is also true that the (discrete-space) voter model converges to cSSM(∞) under diffusive space-/time rescaling.

The relation between cSSM(∞) and aBMs

For $u \in \mathcal{M}_1(\mathbb{R})$, define the set of interface points

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Theorem 3 (H., Ortgiese, Völlering 2016)

Let $(u_t)_{t\geq 0}$ be the 'continuous-space voter model' cSSM$(\infty)_{u_0}$.

a) *If $\mathcal{I}(u_0) \in \mathcal{D}$ and we color the support of u_0 in red, then*

$$\mathcal{L}\left((u_t)_{t\geq 0} \,\middle|\, \mathbb{P}_{u_0}\right) = \mathcal{L}\left(r(\mathbf{X}, t)_{t\geq 0} \,\middle|\, \mathbb{P}_{\mathcal{I}(u_0)}\right),$$

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b) *['Coming down from infinity']*
Let $u_0 \in \mathcal{M}_1(\mathbb{R})$. Then for all $t_0 > 0$, almost surely $\mathcal{I}(u_{t_0}) \in \mathcal{D}$, and the evolution of $(u_t)_{t\geq t_0}$ is decribed (in law) by a system of aBMs started from $\mathcal{I}(u_{t_0})$ as in a).

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See also [Tribe 1995] for the case that $\mathcal{I}(u_0)$ is finite.

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- Now Theorem 3 gives, for all $\mathbf{x} \in \mathcal{D}$,

$$\mathcal{L}\left((\mathbf{X}_t)_{t\geq 0} \,\middle|\, \mathbb{P}_{\mathbf{x}}\right) = \mathcal{L}\left((\mathcal{I}[u_t])_{t\geq 0} \,\middle|\, \mathbb{P}_{\mathcal{I}^{-1}(\mathbf{x})}\right) \quad \text{on } \mathcal{C}([0,\infty); D).$$

Classification of entrance laws

Theorem 4 (H., Ortgiese, Völlering 2017)

a) *Let $(\mu^{(n)})_n$ be a sequence of probability measures on $\mathcal{D}$. Then $\mathcal{L}\big((\mathbf{X}_t)_{t>0}\,\big|\,\mathbb{P}_{\mu^{(n)}}\big)$ converges weakly in $\mathcal{C}((0,\infty);\mathcal{D})$ iff the sequence $(\mu^{(n)}\circ\mathcal{I})_n$ of probability measures on $\mathcal{V}^d$ converges weakly to some probability measure ν_0 on $\mathcal{V}$, in which case*

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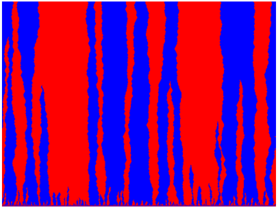
b) *There is a bijective correspondence between probability entrance laws $(\mu_t)_{t>0}$ for aBMs on $\mathcal{D}$ and probability measures ν_0 on $\mathcal{V}$, given by the formula*

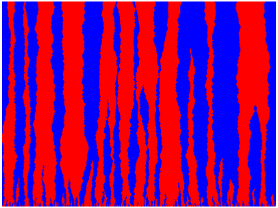
$$\mu_t = \mathcal{L}\left(\mathcal{I}[u_t] \,|\, \mathbb{P}_{\nu_0}\right), \qquad t > 0. \tag{5}$$

Moreover, any such entrance law is a suitable limit as in a).

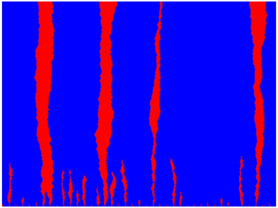
Back to the examples



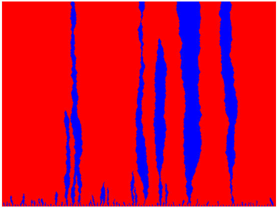
(a) $\mathbf{x}_n^{(1)} = \frac{1}{n}\mathbb{Z} \to [\frac{1}{2}]$



(b) $\mathbf{x}_n^{(2)} \sim PPP(n) \to [\frac{1}{2}]$



(c) $\mathbf{x}_n^{(3)} = \frac{1}{n}\mathbb{Z} + \{0, \frac{1}{n^2}\} \to [0]$



(d) $\mathbf{x}_n^{(4)} = \frac{1}{n}\mathbb{Z} + \{0, \frac{1}{4n}\} \to [\frac{1}{4}]$

Generalization: The symbiotic branching model

Symbiotic Branching Model ([Etheridge & Fleischmann 2004]):

$$\begin{cases} \frac{\partial}{\partial t} u_t^{(\gamma)}(x) = \frac{1}{2}\Delta u_t^{(\gamma)}(x) + \sqrt{\gamma u_t^{(\gamma)}(x) v_t^{(\gamma)}(x)}\, \dot{W}_t^{(1)}(x), \\ \frac{\partial}{\partial t} v_t^{(\gamma)}(x) = \frac{1}{2}\Delta v_t^{(\gamma)}(x) + \sqrt{\gamma u_t^{(\gamma)}(x) v_t^{(\gamma)}(x)}\, \dot{W}_t^{(2)}(x), \\ u_0(\cdot), v_0(\cdot) \geq 0, \qquad x \in \mathbb{R} \end{cases}$$

where $\gamma > 0$ is the branching rate and the white noises are *correlated* with parameter $\rho \in [-1, 1]$.
Notation: $\mathrm{cSBM}(\rho, \gamma)_{u_0, v_0}$ (resp. $\mathrm{dSBM}(\rho, \gamma)_{u_0, v_0}$ for the discrete-space analogue).

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The continuous-space stepping stone model corresponds to the special case $\rho = -1$ and $u_0(\cdot) + v_0(\cdot) \equiv 1$:

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For $\rho = 0$: Mutually catalytic branching, [Dawson & Perkins 1998].

Symbiotic branching model for $\rho = -1$

Let $\mathcal{M}_b(\mathbb{R}) := \{u(x)\,dx \mid u : \mathbb{R} \to \mathbb{R}^+ \text{ measurable and bounded}\}$.

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Let $\rho = -1$ and consider $(u_0, v_0) \in \mathcal{M}_b(\mathbb{R})^2$.

a) *As $\gamma \to \infty$, the process $(u_t^{(\gamma)}, v_t^{(\gamma)})_{t \geq 0}$ converges in law in $\mathcal{C}([0,\infty); \mathcal{M}_b(\mathbb{R})^2)$ to a Feller process $(u_t, v_t)_{t \geq 0}$ with separated types, i.e. for all $t > 0$ we have almost surely*

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b) *Suppose $w_0 := u_0 + v_0 \neq 0$. Then in analogy with Theorem 3, the dynamics of the limit cSBM$(-1,\infty)_{u_0,v_0}$ from a) is described in law by a system of aBMs with drift*

$$I_t = B_t + \int_0^t \frac{w_s'(I_s)}{w_s(I_s)}\,ds,$$

where $\frac{\partial}{\partial t} w_t = \frac{1}{2}\Delta w_t$, $w_0 = u_0 + v_0$.

Moment duality

Theorem 6 (H., Ortgiese, Völlering 2016)

Let $\rho = -1$ and consider $(u_0, v_0) \in \mathcal{M}_b(\mathbb{R})^2$. The limit $(u_t, v_t)_{t\geq 0}$ from Theorem 5 is characterized by a moment duality: For all $n \in \mathbb{N}$, a.e. $\mathbf{y} \in \mathbb{R}^n$ and all measures μ on $\{1,2\}^n$ we have

$$\mathbb{E}_{u_0,v_0}\left[H(u_t, v_t; \mathbf{y}, \mu)\right] = \mathbb{E}_{\mathbf{y},\mu}\left[H(u_0, v_0; B_t, M_t)\right], \quad t \geq 0, \qquad (6)$$

where $(B_t)_{t\geq 0} = (B_t^{(1)}, \dots, B_t^{(n)})_{t\geq 0}$ is an n-dimensional Brownian motion and $(M_t)_{t\geq 0}$ is a process taking values in the measures on $\{1,2\}^n$ and depending only on the collisions between $(B_s^{(i)})_{0\leq s\leq t}$.

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$$H(u, v; \mathbf{y}, \mu) = \sum_{m\in\{1,2\}^n} \mu(m) \prod_{\substack{i\in\{1,\ldots,n\},\\ m_i=1}} u(y_i) \prod_{\substack{j\in\{1,\ldots,n\},\\ m_j=2}} v(y_j)$$

for $(u, v) \in \mathcal{M}_b(\mathbb{R})^2$, $\mathbf{y} \in \mathbb{R}^n$, μ measure on $\{1,2\}^n$.

Further results and outlook

- For $\rho \in (-1, 0)$, we can still prove convergence to a (measure-valued) infinite rate limit cSBM(ρ, ∞)

$$(u_t^{(\gamma)}, v_t^{(\gamma)})_{t\geq 0} \xrightarrow{\gamma\uparrow\infty} (u_t, v_t)_{t\geq 0}$$

with *separated types*: For all $t > 0$ fixed, we have almost surely

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- Characterization via an abstract *martingale problem*.
- The moment duality (6) continues to hold for all $\rho \in (-1,0)$ and $n \in \mathbb{N}$ such that $\rho + \cos(\pi/n) < 0$ ([H., Ortgiese, Völlering 2016]).

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- For *discrete space*, convergence to an infinite rate-limit

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- **Open problems:** 'Explicit' characterization of cSBM(ρ,∞) for $\rho \in (-1,0)$? (E.g. in terms of interfaces, SPDE. . .) Positive correlations $\rho \geq 0$?

Thank you!

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