Brownian Web and Net, part I

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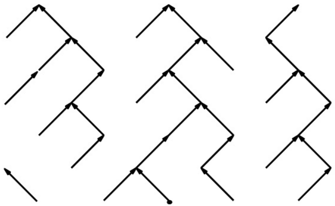
Coalescing simple random walk on $\mathbb{Z}$



One walker starting from every site in the lattice

$\mathbb{Z}^2_{\text{even}} := \{(x,n) : x+n \text{ is even}\}$.

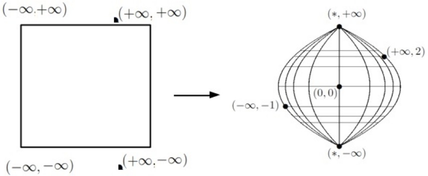
For each $(x,n) \in \mathbb{Z}^2_{\text{even}}$, an independent arrow is drawn from (x,n) to either $(x-1, n+1)$ or $(x+1, n+1)$, with probability $1/2$.

To find the scaling limit if space and time are scaled by $1/\sqrt{n}$ and $1/n$. $\rightsquigarrow$ Brownian web

Space of compact sets of paths

Let R_c^2 be the completion of $\mathbb{R}^2$ under the metric

$$\rho((x_1,t_1),(x_2,t_2)) = |\tanh(t_1)-\tanh(t_2)| \vee \left| \frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|} \right|.$$



A path $\pi \in R_c^2$ is map with starting point σ_π, $\pi : [\sigma_\pi, \infty] \to [-\infty, \infty] \cup \{*\}$ such that $\pi(\infty) = *$, $\pi(-\infty) = *$ and the map $t \to (\pi(t), t)$ is continuous in (R_c^2, ρ)

Space of compact sets of paths

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$$\rho((x_1,t_1),(x_2,t_2)) = |\tanh(t_1) - \tanh(t_2)| \vee \left| \frac{\tanh(x_1)}{1+|t_1|} - \frac{\tanh(x_2)}{1+|t_2|} \right| .$$

Let Π the set of path in $\mathbb{R}_c^2$ with the metric

$$d(\pi_1,\pi_2) = |\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \vee \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left| \frac{\tanh(\pi_1(t \vee \sigma_{\pi_1}))}{1+|t|} - \frac{\tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1+|t|} \right| .$$

Space of compact sets of paths

Denote by $\mathcal{H}$ the space of compact subsets of (Π, d) with the Hausdorff metric

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{\pi_1 \in K_1} \inf_{\pi_2 \in K_2} d(\pi_1, \pi_2) \vee \sup_{\pi_2 \in K_2} \inf_{\pi_1 \in K_1} d(\pi_1, \pi_2),$$

and $\mathcal{B}_{\mathcal{H}}$ is the Borel σ-algebra associated.

For $K \in \mathcal{H}$ and $A \subset R_c^2$, $K(A)$ will denote the set of paths in K with starting point in A.

Construction of the Brownian web

Theorem

There exists an $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued random variable $\mathcal{W}$, called the Brownian web, whose distribution is uniquely determined by:

(a) *For each deterministic $z \in \mathbb{R}^2$, a.s. there is a unique path $\pi_z \in \mathcal{W}(z)$.*

(b) *For any finite deterministic set of points $z_1, \cdots, z_k \in \mathbb{R}^2$, the collection $(\pi_{z_1}, \cdots, \pi_{z_k})$ is distributed as coalescing Brownian motions.*

(c) *For any deterministic countable dense subset $D \subset \mathbb{R}^2$, a.s. $\mathcal{W}$ is the closure of $\{\pi_z, z \in D\}$ in (Π, d).*

Proof

It can be find in [FINR03, FINR04]. The main steps are

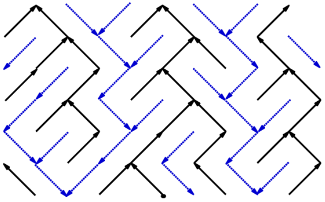
1. Let $\mathcal{D} = \{(x, t) : x, t \in \mathbb{Q}\}$ and construct the collection of coalescing Brownian motions $\mathcal{W}(\mathcal{D}) = \{\pi_z\}_{z \in \mathcal{D}}$ where π_z is a Brownian motion starting at z.
2. Show that $\mathcal{W}(\mathcal{D})$ is pre-compact in (Π, d) and then $\mathcal{W} = \overline{\mathcal{W}(\mathcal{D})}$ is a random compact set.
3. Show that properties (a) and (b) hold for $\mathcal{W}$.

Dual coalescing simple random walk on $\mathbb{Z}$

Downward arrows connecting points in the lattice

$\mathbb{Z}^2_{\text{odd}} := \{(x, n) : x+n \text{ is odd}\}$

such that the upward and backward arrows do not cross each other.



The scaling limit of the joint collection of forward and backward coalescing random walks $\rightsquigarrow$ double Brownian web

Double Brownian web

Given a $z = (x, t) \in R_c^2$ let denote by $-z = (-x, -t)$. Given a path $\pi \in \Pi$ we denote by $\widehat{\pi} := -\pi$ with starting point $\widehat{\sigma}_{\widehat{\pi}} = -\sigma_{\pi}$. We denote by $\widehat{\Pi}$ the associated backward paths with metric $\widehat{d}$ inherited from (Π, d) under the map $-$.

Double Brownian web

Theorem (STW00, FINR06)

There exists an $\mathcal{H} \times \widehat{\mathcal{H}}$*-valued r.v.* $(\mathcal{W}, \widehat{\mathcal{W}})$*, whose distribution is determined by*

(a) $\mathcal{W}$ *and* $-\widehat{\mathcal{W}}$ *are distributed as Brownian webs.*

(b) *a.s. no path* $\pi_z \in \mathcal{W}$ *crosses any path* $\hat{\pi}_{\hat{z}} \in \widehat{\mathcal{W}}$ *in the sense that* $z = (x, t)$ *and* $\hat{z} = (\hat{x}, \hat{t})$ *with* $t < \hat{t}$ *and* $(\pi_z(s_1) - \hat{\pi}_{\hat{z}}(s_1))(\pi_z(s_2) - \hat{\pi}_{\hat{z}}(s_2)) < 0$ *for some* $t < s_1 < s_2 < \hat{t}$.

Furthermore, for each $z \in \mathbb{R}^2$*,* $\widehat{\mathcal{W}}(z)$ *a.s. consists of a single path* $\widehat{\pi}_z$ *which is the unique path in* $\widehat{\Pi}$ *that doesn't cross any path in* $\mathcal{W}$*.*

Coalescing point set

Given the Brownian web $\mathcal{W}$ and a closed set $A \subset \mathbb{R}$ we define the coalescing point set by

$$\xi_t^A := \{y \in \mathbb{R} : y = \pi(t) \text{ for some } \pi \in \mathcal{W}(A \times \{0\})\}.$$

Proposition

For all $t > 0$ and $a < b$ we have

$$\mathbb{E}\left[|\xi_t^{\mathbb{R}} \cap [a,b]|\right] = \frac{b-a}{\sqrt{\pi t}}.$$

In other words, $\xi_t^{\mathbb{R}}$ becames a.s. locally finite for $t > 0$.
$\rightsquigarrow$ *We have a coming down from infinity phenomenon.*

Special points.

Given a $z = (x, t) \in \mathbb{R}^2$ we say that a path π enters z if $\sigma_\pi < t$ and $\pi(t) = x$, and π leaves z if $\sigma_\pi \le t$ and $\pi(t) = x$.

Two paths π and π' leaving z are equivalent, denoted by $\pi \sim^z_{out} \pi'$, if $\pi = \pi'$ on $[t, \infty)$. Two paths π and π' entering z are equivalent, denoted by $\pi \sim^z_{in} \pi'$, if $\pi = \pi'$ on $[t - \epsilon, \infty)$ for some $\epsilon > 0$.

Let denote by $m_{in}(z)$ and $m_{out}(z)$ the number of equivalence classes of paths in $\mathcal{W}$ entering and leaving z, le define $\hat{m}_{in}(z)$ and $\hat{m}_{out}(z)$ similarly by $\widehat{W}$.

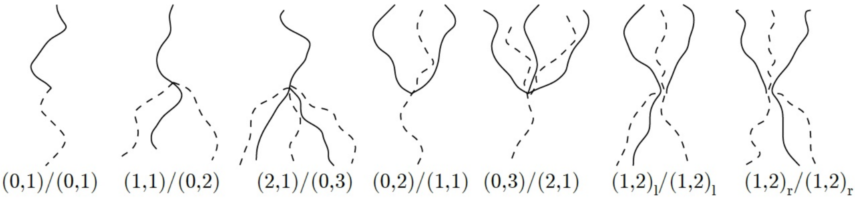
Special points of the Brownian web

Theorem (TW98,FINR06)

Let $(\mathcal{W}, \widehat{\mathcal{W}})$ be a double Brownian web. Then, a.s.

$$m_{out}(z) = \hat{m}_{in}(z) + 1 \qquad \text{and} \qquad \hat{m}_{out}(z) = m_{in}(z) + 1,$$

and z is one of the following types according to $(m_{in}(z), m_{out}(z))/(\hat{m}_{in}(z), \hat{m}_{out}(z))$:



If $S_{i,j}$ is the set of points in $\mathbb{R}^2$ that are of type (i, j).

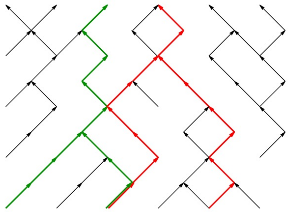
Then a.s. $S_{0,1}$ has full Lebesgue measure in $\mathbb{R}^2$.

$S_{1,1}$ and $S_{0,2}$ have Hausdorff dimension $3/2$ each.

$S_{1,2}$ has Hausdorff dimension 1.

and $S_{2,1}$ and $S_{0,3}$ are both countable and dense in $\mathbb{R}^2$.

Branching-coalescing random walks



For each $(x, n) \in \mathbb{Z}^2_{\text{even}}$, an arrow is drawn from (x, n) to either $(x-1, n+1)$ or $(x+1, n+1)$, with probability $(1-\epsilon)/2$, and with probability ϵ arrows are drawn to both $(x-1, n+1)$ or $(x+1, n+1)$.

To find the scaling limit if space and time are scaled by ϵ and ϵ^2, respectably and the branching probability by $b\epsilon$.
$\rightsquigarrow$ Brownian net with branching parameter b.

leftmost-rightmost r.walks $\rightsquigarrow$ left-right Brownian web $(\mathcal{W}^l, \mathcal{W}^r)$.

Left-right coalescing Brownian motions

The joint law of $(\mathcal{W}^l, \mathcal{W}^r)$ is characterized by: if $l_{z_1} \in \mathcal{W}^l(z_1), \cdots,$ $l_{z_k} \in \mathcal{W}^l(z_k)$ and $r_{z'_1} \in \mathcal{W}^r(z'_1), \cdots, r_{z'_n} \in \mathcal{W}^r(z'_n)$

- The paths $(l_{z_1}, \cdots, l_{z_k}; r_{z'_1}, \cdots, r_{z'_n})$ evolve independently when they are apart.
- the leftmost paths $(l_{z_1}, \cdots, l_{z_k})$ coalesce when they meet, the same is true for the rightmost paths $(r_{z'_1}, \cdots, r_{z'_n})$.
- Every pair $(l_{z_i}, r_{z'_j})$ solves the SDE

$$\begin{aligned} \mathrm{d}L_t &= \mathbf{1}_{\{L_t \neq R_t\}} \mathrm{d}B^1_t + \mathbf{1}_{\{L_t = R_t\}} \mathrm{d}B^s_t - \mathrm{d}t \\ \mathrm{d}R_t &= \mathbf{1}_{\{L_t \neq R_t\}} \mathrm{d}B^2_t + \mathbf{1}_{\{L_t = R_t\}} \mathrm{d}B^s_t + \mathrm{d}t. \end{aligned}$$

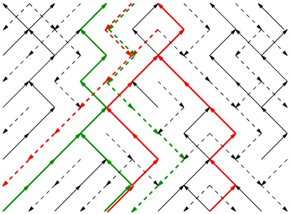
Moreover, $L_t \le R_t$ for all $t \ge T = \inf\{u : \sigma_R : L_u \le R_u\}$.

Left-right Brownian web and its dual

Theorem (SS08)

There exists an $\mathcal{H}^2$-valued r.v. $(\mathcal{W}^l, \mathcal{W}^r)$, whose distribution is determined by

- (i) *The left Brownian web $\mathcal{W}^l$ (resp. $\mathcal{W}^r$) is a Brownian web with drift -1 (resp. 1).*
- (ii) *For $z_1, \cdots z_k, z'_1 \cdots z'_n \in \mathbb{R}^2$ the paths $(l_{z_1}, \cdots l_{z_k}; r_{z'_1}, \cdots, r_{z'_n})$ is a family of left-right coalescing Brownian motions.*



Moreover, a.s. there exists a dual left-right Brownian web $(\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r) \in \widehat{\mathcal{H}}^2$.

Hopping construction of Brownian net

Given π_1 and π_2 with $\pi_1(t) = \pi_2(t)$, we say that t is the crossing time between π_1 and π_2 if there exists $t^- < t^+$ with $(\pi_1(t^-) - \pi_2(t^-))(\pi_1(t^+) - \pi_2(t^+)) < 0$ and

$$t = \inf\{s \in (t^-, t^+) : (\pi_1(t^-) - \pi_2(t^-))(\pi_1(s) - \pi_2(s)) < 0\}.$$

In this case we define the path π obtained by hopping π_1 to π_2 at time t by $\pi = \pi_1$ in $[\sigma_{\pi_1}, t]$ and $\pi = \pi_2$ in $[t, \infty)$.

Given a set of paths K, we denote by $\mathcal{H}_{cross}(K)$ the set of paths obtained by hopping a finite number of times among paths in K at crossing times.

Hopping characterization of the Brownian web

Brownian net $\rightsquigarrow \mathcal{N} := \overline{\mathcal{H}_{cross}(\mathcal{W}^l \cup \mathcal{W}^r)}$.

Theorem (Brownian net, SS08)

There exists an $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$-valued r.v. $\mathcal{N}$, whose distribution is determined by:

(i) *For each $z \in \mathbb{R}^2$, $\mathcal{N}(z)$ a.s. contains a unique left-most path l_z and right-most path r_z.*

(ii) *For $z_1, \cdots z_k, z'_1 \cdots z'_n \in \mathbb{R}^2$ the paths $(l_{z_1}, \cdots, l_{z_k}; r_{z'_1}, \cdots, r_{z'_n})$ is a family of left-right coalescing Brownian motions.*

(iii) *For any countable dense sets $\mathcal{D}^l, \mathcal{D}^r \subset \mathbb{R}^2$*

$$N = \overline{\mathcal{H}_{cross}(\{l_z : z \in \mathcal{D}^l\} \cup \{r_z : z \in \mathcal{D}^r\})} \text{ a.s.}$$

Remarks

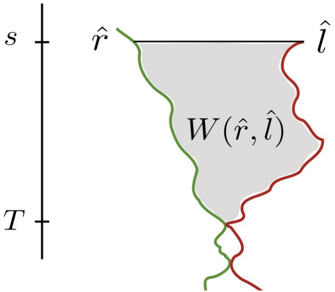
Let $(\mathcal{W}^l, \mathcal{W}^r)$ be a left-right Brownian web and $\mathcal{N}$ its associated Brownian net.

- a.s. for any $\pi_1, \pi_2 \in \mathcal{N}$ with $\pi_1(t) = \pi_2(t)$ and $\sigma_{\pi_1}, \sigma_{\pi_1} < t$, the hopping path defined as $\pi = \pi_1$ on $[\sigma_{\pi_1}, t]$ and $\pi = \pi_2$ on $[t, \infty)$ is in $\mathcal{N}$.
- a.s. no path in $\mathcal{W}^l$ can cross from left to right any path in $\mathcal{W}^r$, $\widehat{\mathcal{W}}^r$ or $\mathcal{N}$. (π_1 cross π_2 from left to right if there exist $s < t$ such that $\pi_1(s) < \pi_2(s)$ and $\pi_1(t) > \pi_2(t)$)
- Similarly, paths in $\mathcal{W}^r$ cannot cross from right to left paths in $\mathcal{W}^l$, $\widehat{\mathcal{W}}^l$ or $\mathcal{N}$.

Wedges

Let $(\mathcal{W}^l, \mathcal{W}^r, \widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r)$ be a double left-right Brownian web. For any $\hat{r} \in \widehat{\mathcal{W}}^r$ and $\hat{l} \in \widehat{\mathcal{W}}^l$ that are ordered $\hat{r}(s) < \hat{l}(s)$ at time $s = \hat{\sigma}_{\hat{r}} \wedge \hat{\sigma}_{\hat{l}}$. Let

$$T = \sup\{t < s : \hat{r}(t) = \hat{l}(t)\}$$



We call the open set

$$W = W(\hat{r}, \hat{l}) = \{(x, u) \in \mathbb{R}^2 : T < u < s, \hat{r}(u) < x < \hat{l}(u)\}$$

a wedge of $(\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r)$ with boundary $\hat{r}$ and $\hat{l}$ and bottom point $z = (\hat{r}(T), T)$. A path π enters W from outside if exist $\sigma_\pi \leq s < t$ such that $(\pi(s), s) \notin \overline{W}$ and $(\pi(t), t) \in W$.

Wedge construction of the Brownian net

Theorem (SS08)

Let $(\mathcal{W}^l, \mathcal{W}^r, \widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r)$ *be a left-right Brownian web with its dual. Then, a.s.*

$$\mathcal{N} = \{\pi : \pi \text{ doesn't enter any wedge of } (\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r) \text{ from outside}\}$$

is the Brownian net associated with $(\mathcal{W}^l, \mathcal{W}^r)$.

Wedge characterisation of the Brownian web

The wedge characterisation can be applied to the Brownian web $\mathcal{W}$ with its dual $\widehat{W}$.

In other words, W, the Brownian web, is a.s. equal to the set of continuous paths π that doesn't enter from outside any wedge that it is formed with two paths in $\widehat{W}$.

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