Brownian Web and Net, part II

Nic Freeman & Sandra Palau

Institute of Mathematical Sciences, NUS

2$^{\text{nd}}$ August 2017

References in [magenta] from:
E. Schertzer, R. Sun and J.M. Swart. *The Brownian web, the Brownian net, and their universality.* Cambridge University Press, 2017. (arXiv:1506.00724)

Convergence to the Brownian web

Consider a sequence X_n of random compact sets of continuous paths. Three steps:

1. Tightness

   Ignore everything outside of large finite box. Within box, rel. compactness $\Leftrightarrow$ equicontinuity.
   $\to$ Low level criteria based on uniform control of small-time behaviour of paths (w.h.p). [e.g. P6.1]

Convergence to the Brownian web

Consider a sequence X_n of random compact sets of continuous paths. Three steps:

1. Tightness

   Ignore everything outside of large finite box. Within box, rel. compactness $\Leftrightarrow$ equicontinuity.
   $\rightarrow$ Low level criteria based on uniform control of small-time behaviour of paths (w.h.p). [e.g. P6.1]

2. Lower bound

   Need enough paths for coalescing Brownian motions 'from every point':

   (I) There exists $\pi_{n,z} \in X_n$ for each $z \in \mathbb{R}^2$ such that, for any deterministic $z_1, \ldots, z_k \in \mathbb{R}^2$, $(\pi_{n,z_i})_{i=1}^k$ converges in distribution to coalescing Brownian motions starting at $(z_i)_{i=1}^k$.

Convergence to the Brownian web

Consider a sequence X_n of random compact sets of continuous paths. Three steps:

1. Tightness

   Ignore everything outside of large finite box. Within box, rel. compactness $\Leftrightarrow$ equicontinuity.
   $\rightarrow$ Low level criteria based on uniform control of small-time behaviour of paths (w.h.p). [e.g. P6.1]

2. Lower bound

   Need enough paths for coalescing Brownian motions 'from every point':

   (I) There exists $\pi_{n,z} \in X_n$ for each $z \in \mathbb{R}^2$ such that, for any deterministic $z_1, \ldots, z_k \in \mathbb{R}^2$, $(\pi_{n,z_i})_{i=1}^k$ converges in distribution to coalescing Brownian motions starting at $(z_i)_{i=1}^k$.

   Happily: for non-crossing X_n, condition (I) $\Rightarrow$ tightness! [P6.4]

Convergence to the Brownian web

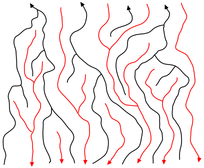
3. Upper bound

Need to avoid having more paths than the BW. Two strategies:

a. For non-crossing paths X_n, find a ‘suitable’ dual system $\widehat{X}_n$. [T6.6, EFS15, RSS16b]

More precisely:

(U’) For each n there exists $\widehat{X}_n \in \widehat{\mathcal{H}}$ whose path a.s. do not cross those of $\hat{X}_n$, and whose starting points are dense as $n \to \infty$. Also:
$>$ Paths of $\widehat{X}_n$ do not enter wedges of X_n from outside.
$>$ Condition (I) holds for $\widehat{X}_n$ (automatically); this convergence must be joint with convergence of meeting times



Convergence to the Brownian web

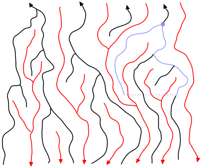
3. Upper bound

Need to avoid having more paths than the BW. Two strategies:

a. For non-crossing paths X_n, find a 'suitable' dual system $\widehat{X}_n$. [T6.6, EFS15, RSS16b]

More precisely:

(U') For each n there exists $\widehat{X}_n \in \widehat{\mathcal{H}}$ whose path a.s. do not cross those of $\hat{X}_n$, and whose starting points are dense as $n \to \infty$. Also:
$>$ Paths of $\widehat{X}_n$ do not enter wedges of X_n from outside.
$>$ Condition (I) holds for $\widehat{X}_n$ (automatically); this convergence must be joint with convergence of meeting times



Convergence to the Brownian web

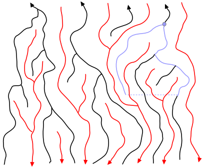
3. Upper bound

Need to avoid having more paths than the BW. Two strategies:

a. For non-crossing paths X_n, find a 'suitable' dual system $\widehat{X}_n$. [T6.6, EFS15, RSS16b]

More precisely:

(U') For each n there exists $\widehat{X}_n \in \widehat{\mathcal{H}}$ whose path a.s. do not cross those of $\hat{X}_n$, and whose starting points are dense as $n \to \infty$. Also:
> Paths of $\widehat{X}_n$ do not enter wedges of X_n from outside.
> Condition (I) holds for $\widehat{X}_n$ (automatically); this convergence must be joint with convergence of meeting times



b. Control $\eta(t, h, a, b)$, the number of distinct positions of paths, at time $t + h$, that passed through (a, b) at time t.

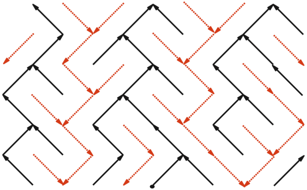
Several variants, [T6.2, T6.3, T6.5, FINR04, NSR05, etc]. Finer control required for crossing case.

Convergence of coalescing RWs to the BW

Theorem. *[T6.6, R6.7] Let (X_n) be a sequence of $\mathcal{H}$ valued random variables. Suppose that each X_n consists of non-crossing paths, and that conditions (I) and (U') are satisfied. Then X_n converges in distribution to the Brownian web.*

Convergence of coalescing RWs to the BW

Theorem. [T6.6, R6.7] *Let (X_n) be a sequence of $\mathcal{H}$ valued random variables. Suppose that each X_n consists of non-crossing paths, and that conditions (I) and (U') are satisfied. Then X_n converges in distribution to the Brownian web.*

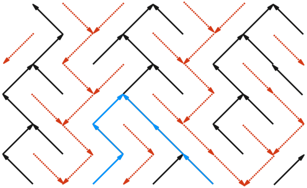


For coalescing SSRWs on $\mathbb{Z}^2_{even}$, with diffusive rescaling X_n, convergence to the BW is then straightforward:

- Dual $\hat{X}_n$ has same distribution as X_n.
- Finite number of coalescing SSRWs converges to coalescing BMs.
- Meeting times also converge, jointly.
- Paths in X_n don't enter wedges of $\hat{X}_n$ (obvious from picture!).

Convergence of coalescing RWs to the BW

Theorem. *[T6.6, R6.7] Let (X_n) be a sequence of $\mathcal{H}$ valued random variables. Suppose that each X_n consists of non-crossing paths, and that conditions (I) and (U') are satisfied. Then X_n converges in distribution to the Brownian web.*



For coalescing SSRWs on $\mathbb{Z}^2_{even}$, with diffusive rescaling X_n, convergence to the BW is then straightforward:

- Dual $\hat{X}_n$ has same distribution as X_n.
- Finite number of coalescing SSRWs converges to coalescing BMs.
- Meeting times also converge, jointly.
- Paths in X_n don't enter wedges of $\hat{X}_n$ (obvious from picture!).

Convergence of coalescing RWs to the BW

Now (switch to continuous time and) consider more general coalescing RWs X_n on $\mathbb{Z}$, with jump distribution J. Assume $\mathbb{E}[J] = 0, \mathrm{Var}(J) < \infty$.

Convergence of coalescing RWs to the BW

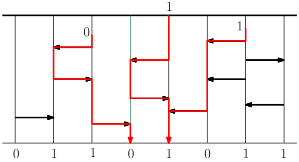
Now (switch to continuous time and) consider more general coalescing RWs X_n on $\mathbb{Z}$, with jump distribution J. Assume $\mathbb{E}[J] = 0, \mathrm{Var}(J) < \infty$.

Theorem. *[NSR05, BMSV06] X_n converges (in law) to the BW*

1. *if $\mathbb{E}[|J|^{3+\epsilon}] < \infty$,*
2. *but not if $\mathbb{E}[|J|^{3-\epsilon}] = \infty$.*

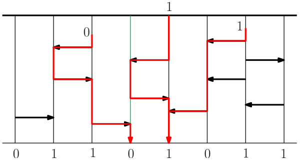
Voter-like population models (web)

Instantaneously coalescing SSRWs are dual to nearest neighbour voter model, rescales to BW. [NRS05]



Voter-like population models (web)

Instantaneously coalescing SSRWs are dual to nearest neighbour voter model, rescales to BW. [NRS05]

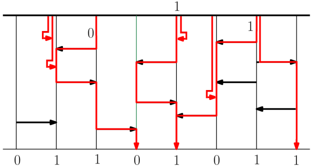


Continuum of individuals at each site: (dual of) 'spatial Fleming-Viot process'.

→ Effect on dual: now pairs of RWs within same site coalesce at rate 1.

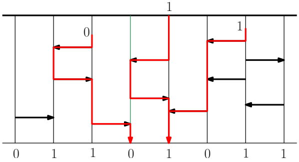
Also rescales to BW.

Reason: once nearby, coalescence quickly.



Voter-like population models (web)

Instantaneously coalescing SSRWs are dual to nearest neighbour voter model, rescales to BW. [NRS05]

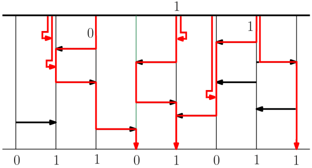


Continuum of individuals at each site: (dual of) 'spatial Fleming-Viot process'.

→ Effect on dual: now pairs of RWs within same site coalesce at rate 1.

Also rescales to BW.

Reason: once nearby, coalescence quickly.

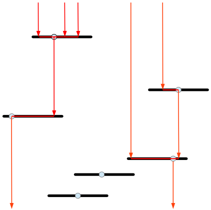


Move into continuous space: (dual of) 'spatial Λ-Fleming-Viot process'.

→ Effect on dual: pairs of RWs in $\mathbb{R}$ affected by same event (drawn in black) coalesce instantaneously.

Also rescales to BW. [EFS15]

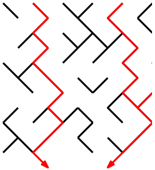
Reason: when paths not nearby, independent; once nearby, coalescence quickly.



Supercritical orientated percolation

Supercritical bond percolation on $\mathbb{Z}^2_{\text{even}}$, edges directed downwards.

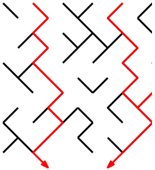
Points in infinite cluster have positive density when $p > p_c$.
For X_n use only right-most paths (to ∞) starting from such points.



Supercritical orientated percolation

Supercritical bond percolation on $\mathbb{Z}^2_{\text{even}}$, edges directed downwards.

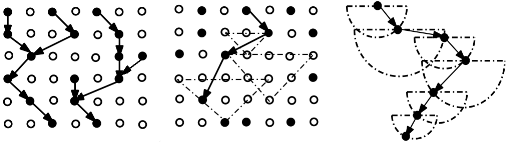
Points in infinite cluster have positive density when $p > p_c$.
For X_n use only right-most paths (to ∞) starting from such points.



Converges to BW. [SS13]

[K89]: single r-most path satisfies CLT; can be extended to convergence to BM.
Complication: The r-most paths are (long-range) correlated.
[SS13]: r-most paths are approximately independent until nearby, then coalesce quickly.

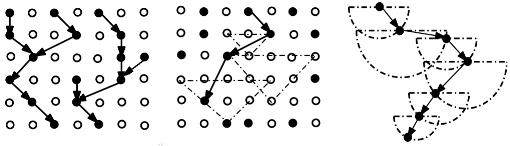
Drainage networks and spanning forests



From left to right:

1. [RSS16b] Howard's drainage network, converges to BW.
   Each vertex of $\mathbb{Z}^2$ is a water-source, with probability $p \in (0, 1)$.
   Single directed edge from each water-source to nearest (strictly) downwards water-source.
   Key idea: approximate self-duality, martingale approach to (I) for dual.

Drainage networks and spanning forests

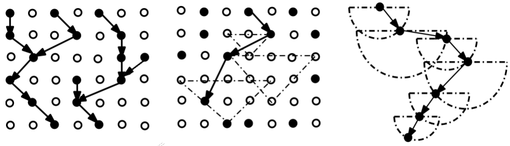


From left to right:

1. [RSS16b] Howard's drainage network, converges to BW.
   Each vertex of $\mathbb{Z}^2$ is a water-source, with probability $p \in (0, 1)$.
   Single directed edge from each water-source to nearest (strictly) downwards water-source.
   Key idea: approximate self-duality, martingale approach to (I) for dual.

2. [RSS16a] Discrete spanning forest, converges to BW.
   Variant of 1, graph distance on (square lattice) $\mathbb{Z}^2$ instead of Euclidean.
   Key ideas: approximate self-duality, control $\eta(t, h, a, b)$ for dual

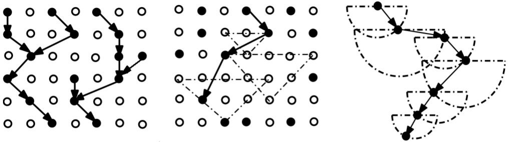
Drainage networks and spanning forests



From left to right:

1. [RSS16b] Howard's drainage network, converges to BW.
   Each vertex of $\mathbb{Z}^2$ is a water-source, with probability $p \in (0, 1)$.
   Single directed edge from each water-source to nearest (strictly) downwards water-source.
   Key idea: approximate self-duality, martingale approach to (I) for dual.
2. [RSS16a] Discrete spanning forest, converges to BW.
   Variant of 1, graph distance on (square lattice) $\mathbb{Z}^2$ instead of Euclidean.
   Key ideas: approximate self-duality, control $\eta(t, h, a, b)$ for dual
3. Directed spanning forest, converges to BW [RSS, to appear].
   Continuous space/time version, introduced in [BB07] in the study of the radial spanning tree.
   Key ideas: approximate self-duality and a variant of (U') based on
   – convergence of meeting times of forward paths,
   – non-overlap of dual with forwards model.

Drainage networks and spanning forests



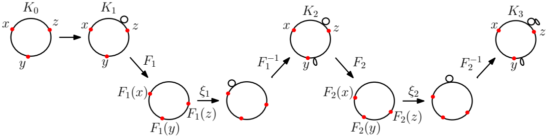
From left to right:

1. [RSS16b] Howard's drainage network, converges to BW.
   Each vertex of $\mathbb{Z}^2$ is a water-source, with probability $p \in (0, 1)$.
   Single directed edge from each water-source to nearest (strictly) downwards water-source.
   Key idea: approximate self-duality, martingale approach to (I) for dual.
2. [RSS16a] Discrete spanning forest, converges to BW.
   Variant of 1, graph distance on (square lattice) $\mathbb{Z}^2$ instead of Euclidean.
   Key ideas: approximate self-duality, control $\eta(t, h, a, b)$ for dual
3. Directed spanning forest, converges to BW [RSS, to appear].
   Continuous space/time version, introduced in [BB07] in the study of the radial spanning tree.
   Key ideas: approximate self-duality and a variant of (U') based on
   – convergence of meeting times of forward paths,
   – non-overlap of dual with forwards model.

Key idea in all cases: approximate independence until nearby, then coalesce quickly

Planar Aggregation

Hastings-Levitov model:

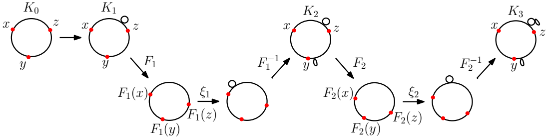


Defines a 'weak flow' $F_n(\cdot)$ on $\mathbb{S}^1$.

A weak flow is essentially a stochastic flow in which an order structure is used to allow (binary) branching. The BW can be formulated as a weak flow on $\mathbb{S}^1$.

Planar Aggregation

Hastings-Levitov model:



Defines a 'weak flow' $F_n(\cdot)$ on $\mathbb{S}^1$.

A weak flow is essentially a stochastic flow in which an order structure is used to allow (binary) branching.
The BW can be formulated as a weak flow on $\mathbb{S}^1$.

Let δ = particle radius. As $\delta \to 0$, with time sped up by δ^{-3}, the weak flow $F_n(\cdot)$ converges to BW on $\mathbb{S}^1$ in 'weak flow topology'. [NT15]

Alternative topologies

Recall that convergence of (non-simple) coalescing RWs $\rightarrow$ BW needed a $3 + \epsilon$ moment condition, for tightness.

Workaround: Change topology.

Main issue: BW has (1,2) points ($\dim_{\mathcal{H}} = 1$ and a.s. dense in $\mathbb{R}^2$) $\rightarrow$ want to allow for binary branching.

Alternative topologies

Recall that convergence of (non-simple) coalescing RWs $\rightarrow$ BW needed a $3 + \epsilon$ moment condition, for tightness.

Workaround: Change topology.

Main issue: BW has (1,2) points ($\dim_{\mathcal{H}} = 1$ and a.s. dense in $\mathbb{R}^2$) $\rightarrow$ want to allow for binary branching.

1. Weak flows [NT15]
   Usual stochastic flow property $X_{s,t} = X_{s,u} \circ X_{u,t}$ does not allow branching.
   Key idea: Use a 'weak flow' of order preserving (i.e. non-strictly increasing) functions.
   Left/right branch represented using left/right-continuous versions.

Alternative topologies

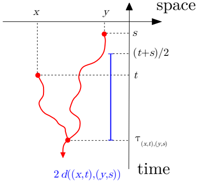
Recall that convergence of (non-simple) coalescing RWs $\to$ BW needed a $3+\epsilon$ moment condition, for tightness.

Workaround: Change topology.

Main issue: BW has (1,2) points ($\dim_{\mathcal{H}} = 1$ and a.s. dense in $\mathbb{R}^2$) $\to$ want to allow for binary branching.

1. Weak flows [NT15]
   Usual stochastic flow property $X_{s,t} = X_{s,u} \circ X_{u,t}$ does not allow branching.
   Key idea: Use a 'weak flow' of order preserving (i.e. non-strictly increasing) functions.
   Left/right branch represented using left/right-continuous versions.

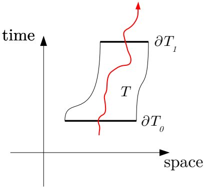
2. Marked Metric Measures [DGP11, GSW15]
   Metric $d\big((x,t),(y,s)\big) = t + s - 2\tau_{(x,t),(y,s)}$ on $\mathbb{R}^2$.
   Characterize a set of paths using the distributions of the distance matrices between finitely many sampled points of $\mathbb{R}^2$.
   Needs enrichment of $\mathbb{R}^2$ to handle $(1,2)$ points.



Alternative topologies

3. Tube topology [BGS15]

A tube $(T, \partial T_0, \partial T_1)$: homeomorphic to $[0,1]^2$, with flat top ∂T_1 and flat bottom ∂T_0.
Crossing a tube: enter by crossing ∂T_0, stay inside T until exit by crossing ∂T_1.

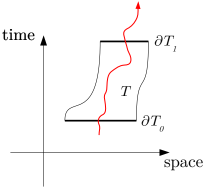


$\mathcal{T}$ = set of all tubes, with Hausdorff metric (coordinate-wise on T, ∂T_1, ∂T_2)
$\mathrm{Cr}(X) = \{$tubes crossed by at least one path in $X\}$.

Alternative topologies

3. Tube topology [BGS15]

A tube $(T, \partial T_0, \partial T_1)$: homeomorphic to $[0,1]^2$, with flat top ∂T_1 and flat bottom ∂T_0.
Crossing a tube: enter by crossing ∂T_0, stay inside T until exit by crossing ∂T_1.



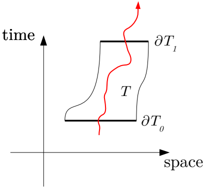
$\mathcal{T}$ = set of all tubes, with Hausdorff metric (coordinate-wise on T, ∂T_1, ∂T_2)
$\mathrm{Cr}(X) = \{$tubes crossed by at least one path in $X\}$.

Lemma: If X is compact (in uniform topology), then $\mathrm{Cr}(X)$ is a closed subset of $\mathcal{T}$.
So, define $\mathscr{T}$ = set of closed subsets of T, equipped with Fell topology.

Alternative topologies

3. Tube topology [BGS15]

A tube $(T, \partial T_0, \partial T_1)$: homeomorphic to $[0,1]^2$, with flat top ∂T_1 and flat bottom ∂T_0.
Crossing a tube: enter by crossing ∂T_0, stay inside T until exit by crossing ∂T_1.



$\mathcal{T}$ = set of all tubes, with Hausdorff metric (coordinate-wise on T, ∂T_1, ∂T_2)
$\mathrm{Cr}(X) = \{$tubes crossed by at least one path in $X\}$.

Lemma: If X is compact (in uniform topology), then $\mathrm{Cr}(X)$ is a closed subset of $\mathcal{T}$.
So, define $\mathscr{T}$ = set of closed subsets of T, equipped with Fell topology.

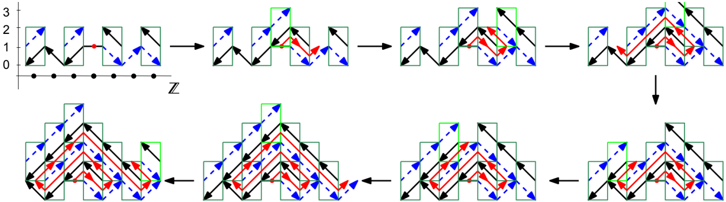
In fact $\mathrm{Cr} : \mathcal{H} \mapsto \mathscr{H}$ is a continuous map, where $\mathscr{H}$ is a compact subset of $\mathscr{T}$.
$\rightarrow$ Tightness is free!

True self-repelling motion

[TW98] began the modern study of BW.

([FINR04] introduced paths topology, [SS08] introduced BN.)

Special case of true self-avoiding walk:
On $\mathbb{Z}$: rectangular blocks arranged into columns; vertices as center points of cols, edges as borders between cols.
walk is vertex valued

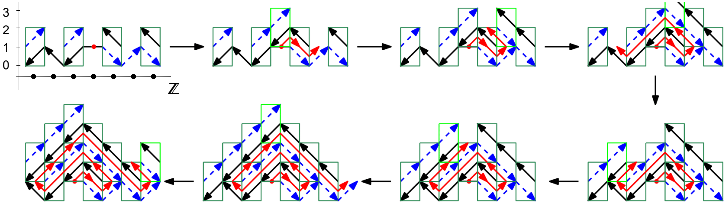


True self-repelling motion

[TW98] began the modern study of BW.

([FINR04] introduced paths topology, [SS08] introduced BN.)

Special case of true self-avoiding walk:
On $\mathbb{Z}$: rectangular blocks arranged into columns; vertices as center points of cols, edges as borders between cols.
walk is vertex valued



Self-avoiding walk is represented in red.
Edge-local time L is recorded by the black/blue arrows.
Dynamics:

- Move either one edge left or one edge right, on each time-step.
- If L(left edge) $<$ L(right edge), move left – and vice versa.
  If L(left edge) $=$ L(right edge), toss coin. In this case: on a rightwards step, leave a  behind.
  on a leftwards step, leave a  behind.

TSW = projection of walk onto $\mathbb{Z}$.
TSM = continuum limit of TSW.
DBW appears as the environment (black/blue arrows).

Convergence to the Brownian net

Consider a sequence X_n of random compact sets of continuous paths.
Similar style of conditions as for convergence to the BW, but more structure needed.
Currently, only known for non-crossing paths.

Three steps:

Convergence to the Brownian net

Consider a sequence X_n of random compact sets of continuous paths.
Similar style of conditions as for convergence to the BW, but more structure needed.
Currently, only known for non-crossing paths.

Three steps:

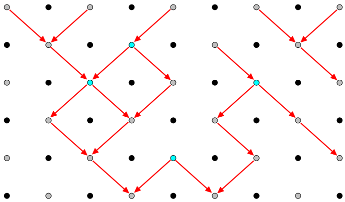
1. Identify subsets of non-crossing left-most (resp. right-most) paths:
   - (C) There exist non-crossing subsets $W_n^l, W_n^r \subseteq X_n$, such that
     - $>$ No path $\pi \in X_n$ crosses any $\pi^l \in W_n^l$ from right to left.
     - $>$ No path $\pi \in X_n$ crosses any $\pi^r \in W_n^r$ from left to right.
   - (H) X_n contains any path obtained by hopping between paths of W_n^l, W_n^r at crossing times.

Convergence to the Brownian net

Consider a sequence X_n of random compact sets of continuous paths.
Similar style of conditions as for convergence to the BW, but more structure needed.
Currently, only known for non-crossing paths.

Three steps:

1. Identify subsets of non-crossing left-most (resp. right-most) paths:
   - (C) There exist non-crossing subsets $W_n^l, W_n^r \subseteq X_n$, such that
     - $>$ No path $\pi \in X_n$ crosses any $\pi^l \in W_n^l$ from right to left.
     - $>$ No path $\pi \in X_n$ crosses any $\pi^r \in W_n^r$ from left to right.
   - (H) X_n contains any path obtained by hopping between paths of W_n^l, W_n^r at crossing times.

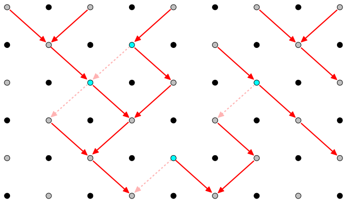


Convergence to the Brownian net

Consider a sequence X_n of random compact sets of continuous paths.
Similar style of conditions as for convergence to the BW, but more structure needed.
Currently, only known for non-crossing paths.

Three steps:

1. Identify subsets of non-crossing left-most (resp. right-most) paths:
   - (C) There exist non-crossing subsets $W_n^l, W_n^r \subseteq X_n$, such that
     $>$ No path $\pi \in X_n$ crosses any $\pi^l \in W_n^l$ from right to left.
     $>$ No path $\pi \in X_n$ crosses any $\pi^r \in W_n^r$ from left to right.
   - (H) X_n contains any path obtained by hopping between paths of W_n^l, W_n^r at crossing times.

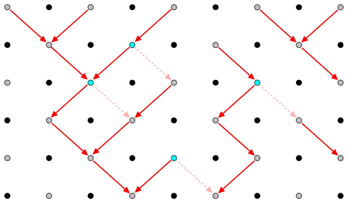


Convergence to the Brownian net

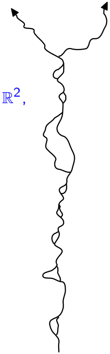
Consider a sequence X_n of random compact sets of continuous paths.
Similar style of conditions as for convergence to the BW, but more structure needed.
Currently, only known for non-crossing paths.

Three steps:

1. Identify subsets of non-crossing left-most (resp. right-most) paths:
   - (C) There exist non-crossing subsets $W^l_n, W^r_n \subseteq X_n$, such that
     $>$ No path $\pi \in X_n$ crosses any $\pi^l \in W^l_n$ from right to left.
     $>$ No path $\pi \in X_n$ crosses any $\pi^r \in W^r_n$ from left to right.
   - (H) X_n contains any path obtained by hopping between paths of W^l_n, W^r_n at crossing times.



Convergence to the Brownian net



2. Lower bound:

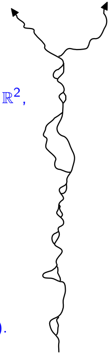
Left/right-most paths must converge to left-right Brownian motions.

(I_{net}) There exist $l_{n,z} \in W_n^l$ and $r_{n,z}$ for all $z \in \mathbb{R}^2$, such that for any deterministic $z_1, \ldots, z_k \in \mathbb{R}^2$,

$$(l_{n,z_1}, \ldots, l_{n,z_k}, r_{n,z_1}, \ldots, z_{n,z_k})$$

converge in distribution to left-right coalescing Brownian motions started from $z_1, \ldots, z_k$.

Convergence to the Brownian net



2. Lower bound:

Left/right-most paths must converge to left-right Brownian motions.

(I_{net}) There exist $l_{n,z} \in W_n^l$ and $r_{n,z}$ for all $z \in \mathbb{R}^2$, such that for any deterministic $z_1, \ldots, z_k \in \mathbb{R}^2$,

$$(l_{n,z_1}, \ldots, l_{n,z_k}, r_{n,z_1}, \ldots, z_{n,z_k})$$

converge in distribution to left-right coalescing Brownian motions started from $z_1, \ldots, z_k$.

Remarks:

- $(\mathrm{I}_{net}) \Rightarrow$ (I) for W_n^l and W_n^r (+drift) $\Rightarrow$ tightness for W_n^l and W_n^r $\Rightarrow$ tightness for $\mathcal{H}_{cross}(W_n^l, W_n^r)$.
- Reason for $n^{-1/2}$ scaling for the branching rate (in e.g. SSRW case):

  Two (newly branched) SSRW paths are born at separation $n^{-1/2}$.
  The probability that such paths achieve macroscopic rescaled distance is order $n^{-1/2}$.
  So want order $n^{1/2}$ branches in 1 unit of rescaling time (= time n) $\Rightarrow$ want to branch at rate $n^{-1/2}$.

Convergence to the Brownian net

3. Upper bound:

Find 'suitable' dual systems $\widehat{W}_n^l, \widehat{W}_n^r$. More precisely:

(U'$_{net}$) There exists $\widehat{W}_n^l, \widehat{W}_n^r \in \widehat{\mathcal{H}}$ such that

> Starting points of paths in $\widehat{W}_n^l$ (resp. $\widehat{W}_n^r$) are dense as $n \to \infty$.

> Paths in $\widehat{W}_n^l$ and paths in W_n^l do not cross. Paths in $\widehat{W}_n^r$ and paths in W_n^r do not cross.

> Paths in X_n do not enter wedges of $(\widehat{W}_n^l, \widehat{W}_n^r)$ from the outside.

> Condition (I$_{net}$) holds automatically for $(\widehat{W}_n^l, \widehat{W}_n^r)$; this convergence must occur jointly with convergence of meeting times.

Convergence to the Brownian net

3. Upper bound:

Find 'suitable' dual systems $\widehat{W}^l_n, \widehat{W}^r_n$. More precisely:

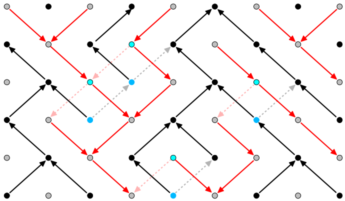
(U'$_{net}$) There exists $\widehat{W}^l_n, \widehat{W}^r_n \in \widehat{\mathcal{H}}$ such that

> Starting points of paths in $\widehat{W}^l_n$ (resp. $\widehat{W}^r_n$) are dense as $n \to \infty$.

> Paths in $\widehat{W}^l_n$ and paths in W^l_n do not cross. Paths in $\widehat{W}^r_n$ and paths in W^r_n do not cross.

> Paths in X_n do not enter wedges of $(\widehat{W}^l_n, \widehat{W}^r_n)$ from the outside.

> Condition (I$_{net}$) holds automatically for $(\widehat{W}^l_n, \widehat{W}^r_n)$; this convergence must occur jointly with convergence of meeting times.



Convergence to the Brownian net

3. Upper bound:

Find 'suitable' dual systems $\widehat{W}^l_n$, $\widehat{W}^r_n$. More precisely:

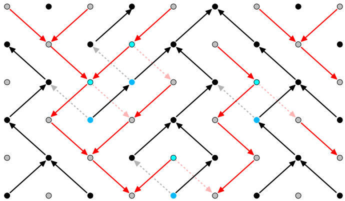
(U'$_{net}$) There exists $\widehat{W}^l_n, \widehat{W}^r_n \in \widehat{\mathcal{H}}$ such that

> Starting points of paths in $\widehat{W}^l_n$ (resp. $\widehat{W}^r_n$) are dense as $n \to \infty$.

> Paths in $\widehat{W}^l_n$ and paths in W^l_n do not cross. Paths in $\widehat{W}^r_n$ and paths in W^r_n do not cross.

> Paths in X_n do not enter wedges of $(\widehat{W}^l_n, \widehat{W}^r_n)$ from the outside.

> Condition (I$_{net}$) holds automatically for $(\widehat{W}^l_n, \widehat{W}^r_n)$; this convergence must occur jointly with convergence of meeting times.



Convergence to the Brownian net

3. Upper bound:

Find 'suitable' dual systems $\widehat{W}_n^l$, $\widehat{W}_n^r$. More precisely:

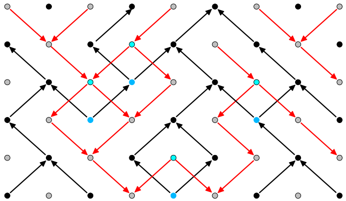
(U'$_{net}$) There exists $\widehat{W}_n^l, \widehat{W}_n^r \in \widehat{\mathcal{H}}$ such that

> Starting points of paths in $\widehat{W}_n^l$ (resp. $\widehat{W}_n^r$) are dense as $n \to \infty$.

> Paths in $\widehat{W}_n^l$ and paths in W_n^l do not cross. Paths in $\widehat{W}_n^r$ and paths in W_n^r do not cross.

> Paths in X_n do not enter wedges of $(\widehat{W}_n^l, \widehat{W}_n^r)$ from the outside.

> Condition (I$_{net}$) holds automatically for $(\widehat{W}_n^l, \widehat{W}_n^r)$; this convergence must occur jointly with convergence of meeting times.



Convergence to the Brownian net

3. Upper bound:

Find 'suitable' dual systems $\widehat{W}^l_n, \widehat{W}^r_n$. More precisely:

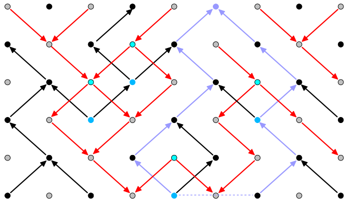
(U'$_{net}$) There exists $\widehat{W}^l_n, \widehat{W}^r_n \in \widehat{\mathcal{H}}$ such that

> Starting points of paths in $\widehat{W}^l_n$ (resp. $\widehat{W}^r_n$) are dense as $n \to \infty$.

> Paths in $\widehat{W}^l_n$ and paths in W^l_n do not cross. Paths in $\widehat{W}^r_n$ and paths in W^r_n do not cross.

> Paths in X_n do not enter wedges of $(\widehat{W}^l_n, \widehat{W}^r_n)$ from the outside.

> Condition (I$_{net}$) holds automatically for $(\widehat{W}^l_n, \widehat{W}^r_n)$; this convergence must occur jointly with convergence of meeting times.



Convergence of coalescing SSRWs to the Brownian net

Theorem. *[T6.6, R6.7] Let (X_n) be a sequence of $\mathcal{H}$ valued random variables. Suppose that each X_n consists of non-crossing paths, and that conditions (C), (H), (I_{net}) and (U_{net}) are satisfied. Then X_n converges in distribution to the Brownian net.*

Coalescing SSRW on $\mathbb{Z}^2_{even}$, with branching at rate $n^{-1/2}$ converges to the Brownian net.

Voter-like population models (net)

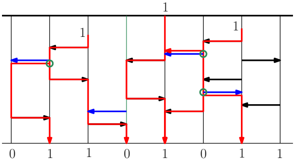
Add branching events to (dual of) Voter model:

(Forwards in time: biased Voter model)

→ Effect on dual: *branching*-coalescing SSRWs

Rescales to BN. [SSS]

Crossing case: in progress [Sun, Swart, Yu]



Voter-like population models (net)

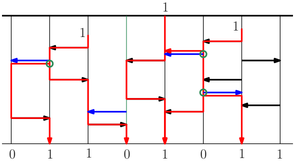
Add branching events to (dual of) Voter model:

(Forwards in time: biased Voter model)

→ Effect on dual: *branching*-coalescing SSRWs

Rescales to BN. [SSS]

Crossing case: in progress [Sun, Swart, Yu]



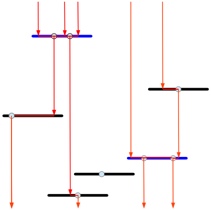
Add branching events to (dual of) SΛFV process:

(Forwards in time: SΛFV process with selection)

→ Effect on dual: *branching*-coalescing RWs in $\mathbb{R}$

Rescales to BN. [EFS15]

[EFS15] Key idea: approximate self-duality, mimic SSRW



Voter-like population models (net)

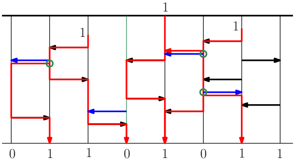
Add branching events to (dual of) Voter model:

(Forwards in time: biased Voter model)

→ Effect on dual: *branching*-coalescing SSRWs

Rescales to BN. [SSS]

Crossing case: in progress [Sun, Swart, Yu]



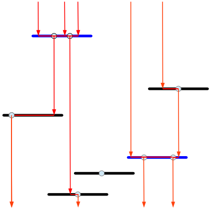
Add branching events to (dual of) SΛFV process:

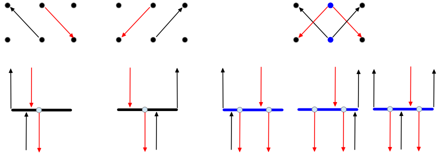
(Forwards in time: SΛFV process with selection)

→ Effect on dual: *branching*-coalescing RWs in $\mathbb{R}$

Rescales to BN. [EFS15]

[EFS15] Key idea: approximate self-duality, mimic SSRW

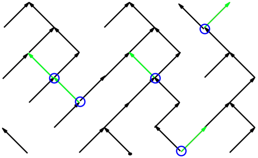




Coupling the Brownian web and net

Drift-less connection between BW and BN:

Discrete picture: [NRS10, SSS14]



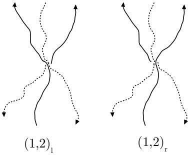
BN $\mapsto$ BW:
At each branch point,
Delete either left or right arrow, chosen by a fair coin toss.

BW $\mapsto$ BN:
Sample branch points (w.p. $n^{-1/2}$).
Include left and right arrows at each branch point.

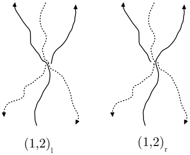
Coupling the Brownian web and net

Drift-less connection between BW and BN:

Continuous picture: [NRS10, SSS14]



Coupling the Brownian web and net



Drift-less connection between BW and BN:

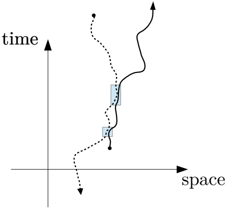
Continuous picture: [NRS10, SSS14]

For the BW, there exists a natural ‘local time’ measure ℓ on $S_{1,2}$: [P4.1]

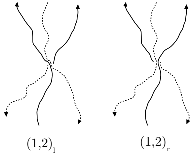
$$\ell\left(\left\{(x,t)\in\mathbb{R}^2\,;\,\sigma_\pi<t<\sigma_{\hat\pi},\,\pi(t)=x=\hat\pi(t)\right\}\right)=\lim_{\epsilon\to0}\frac{|\{t\in\mathbb{R}\,;\,\sigma_\pi<t<\sigma_{\hat\pi},\,|\pi(t)-\hat\pi(t)|<\epsilon\}|}{\epsilon}$$

for any $\pi\in\mathcal{W}$, $\hat\pi\in\widehat{\mathcal{W}}$.

ℓ is a.s. non-atomic, σ-finite,
concentrated on the set $S_{1,2}$ of $(1,2)$ points of $\mathcal{W}$.



Coupling the Brownian web and net



Drift-less connection between BW and BN:

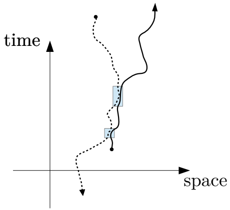
Continuous picture: [NRS10, SSS14]

For the BW, there exists a natural 'local time' measure ℓ on $S_{1,2}$: [P4.1]

$$\ell\left(\left\{(x,t)\in\mathbb{R}^2\,;\,\sigma_\pi<t<\sigma_{\hat\pi},\,\pi(t)=x=\hat\pi(t)\right\}\right)=\lim_{\epsilon\to 0}\frac{|\{t\in\mathbb{R}\,;\,\sigma_\pi<t<\sigma_{\hat\pi},\,|\pi(t)-\hat\pi(t)|<\epsilon\}|}{\epsilon}$$

for any $\pi\in\mathcal{W}$, $\hat\pi\in\widehat{\mathcal{W}}$.

ℓ is a.s. non-atomic, σ-finite,
concentrated on the set $S_{1,2}$ of $(1,2)$ points of $\mathcal{W}$.



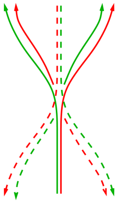
BW $\mapsto$ BN: [T4.2]

Take Poisson point process P with intensity ℓ,
allow incoming paths to branch at points of P ('marked (1,2) points').

Coupling the Brownian web and net

Next, want BN $\mapsto$ BW.
Need to identify an equivalent of marked (1,2) points, within the net.
$\rightarrow$ special points of the net.

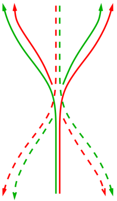
Separation points:
right-most paths
left-most paths



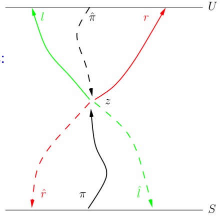
Coupling the Brownian web and net

Next, want BN $\mapsto$ BW.
Need to identify an equivalent of marked (1,2) points, within the net.
$\rightarrow$ special points of the net.

Separation points:
right-most paths
left-most paths



(S, U)-relevant separation points:



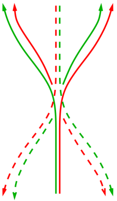
Lemma: For deterministic $S < U$ the set of (S, U)-relevant separation points is a.s. locally finite. [P4.7, SSS09]
In other words, for all but finitely many separation points, l and r meet again within time $\epsilon > 0$.
Sketch proof: If $R_{S,U} = \{(S, U)\text{-relevant separation points}\}$, can calculate $\mathbb{E}[|R_{S,U} \cap [a, b] \times (S, U)|]$;
using density of $z = (x, t)$ such that l, r (born at z) have a dual path in between them during (t, U).

A separation point is 'relevant' if it is (S, U)-relevant for some $S < U$.

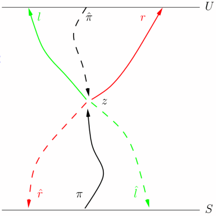
Coupling the Brownian web and net

Next, want BN $\mapsto$ BW.
Need to identify an equivalent of marked (1,2) points, within the net.
$\rightarrow$ special points of the net.

Separation points:
right-most paths
left-most paths



(S, U)-relevant separation points:



Lemma: For deterministic $S < U$ the set of (S, U)-relevant separation points is a.s. locally finite. [P4.7, SSS09]
In other words, for all but finitely many separation points, l and r meet again within time $\epsilon > 0$.
Sketch proof: If $R_{S,U} = \{(S, U)\text{-relevant separation points}\}$, can calculate $\mathbb{E}[|R_{S,U} \cap [a, b] \times (S, U)|]$;
using density of $z = (x, t)$ such that l, r (born at z) have a dual path in between them during (t, U).

A separation point is 'relevant' if it is (S, U)-relevant for some $S < U$.

BN $\mapsto$ BW: [T4.2]

For each relevant separation point:
Sample a random sign, include in BW only paths which turn in that direction.

The dynamical Brownian web

The dynamical Brownian web $\lambda \mapsto \mathcal{W}_\lambda$:
constructed by flipping the sign of (1,2) points with 'intensity' $\lambda > 0$.

Take P_λ to be a Poisson point process on $S_{1,2}$, where P_λ has intensity $\lambda\ell$.
Coupled so that $\lambda \mapsto P_\lambda$ is increasing.
Define $\mathcal{W}_\lambda$ by flipping the signs of the $(1, 2)$ points of $\mathcal{W}$ that are in P_λ.

(First constructed as limit of discreet dynamical webs in [HW09b].)

The dynamical Brownian web

The dynamical Brownian web $\lambda \mapsto \mathcal{W}_\lambda$:
constructed by flipping the sign of (1,2) points with 'intensity' $\lambda > 0$.

Take P_λ to be a Poisson point process on $S_{1,2}$, where P_λ has intensity $\lambda\ell$.
Coupled so that $\lambda \mapsto P_\lambda$ is increasing.
Define $\mathcal{W}_\lambda$ by flipping the signs of the $(1,2)$ points of $\mathcal{W}$ that are in P_λ.

(First constructed as limit of discreet dynamical webs in [HW09b].)

Exceptional times:

[FNRS09] show that a.s. there is a dense set of λ such that the path from 0 fails the law of the iterated logarithm.
Reason: When a path hits a switched $(1,2)$ point it goes on an excursion away from its original path.
Size of this excursion is heavy tailed, with infinite mean.

The dynamical Brownian web

The dynamical Brownian web $\lambda \mapsto \mathcal{W}_\lambda$:
constructed by flipping the sign of (1,2) points with 'intensity' $\lambda > 0$.

Take P_λ to be a Poisson point process on $S_{1,2}$, where P_λ has intensity $\lambda\ell$.
Coupled so that $\lambda \mapsto P_\lambda$ is increasing.
Define $\mathcal{W}_\lambda$ by flipping the signs of the $(1, 2)$ points of $\mathcal{W}$ that are in P_λ.

(First constructed as limit of discreet dynamical webs in [HW09b].)

Exceptional times:

[FNRS09] show that a.s. there is a dense set of λ such that the path from 0 fails the law of the iterated logarithm.
Reason: When a path hits a switched $(1, 2)$ point it goes on an excursion away from its original path.
Size of this excursion is heavy tailed, with infinite mean.

Connection to black noise:

Informally: A process is a 'black noise' if re-sampling an arbitrarily small, but evenly spread, fraction of its underlying randomness results in a new, independent sample.
The BW is a black noise [EF16, T04a/b]. This stems from the dense $(1, 2)$ points.

Open questions

1. Convergence criteria to the net, for crossing paths?
2. Construction of Lévy webs?
3. Characterization of net as a branching-coalescing point set?
4. Universality class of BN with killing?