

Phase Transition for the Contact Process in a Random Environment on $\mathbb{Z}^d \times \mathbb{Z}^+$

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History of contact process

- First introduced by T. E. Harris (1974).
- A model to describe the spread of diseases.
- Two classical books:
T. M. Liggett (1985), T. M. Liggett (1999).

References:

- [1] T. E. Harris: Contact interactions on a lattice, *Ann. Probab.* **2** 969-988 (1974).
- [2] T. M. Liggett: *Interacting Particle Systems*. New York: Springer-Verlag, 1985.
- [3] T. M. Liggett: *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, Berlin, Heidelberg, 1999.

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Basic definitions of contact process

- $G = (V, E)$: a connected undirected graph.
- The process $(\xi_t : t \geq 0)$: a continuous-time Markov process.
- State space: $\{A : A \subseteq V\}$.
- At each t , each vertex is either **healthy** or **infected**.
 ξ_t is the collection of infected vertices at time t .
- Transition rates:

$$\begin{cases} \xi_t \rightarrow \xi_t \setminus \{x\} \text{ for } x \in \xi_t \text{ at rate } 1, \\ \xi_t \rightarrow \xi_t \cup \{x\} \text{ for } x \notin \xi_t \text{ at rate } \lambda \cdot |\{y \in \xi_t : x \sim y\}|. \end{cases}$$

- $(\xi_t^A : t \geq 0)$: the process with initial state A .
- Absorbing state: \emptyset .

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Basic definitions of contact process

- We say that the process **survives** if the infection will persist with positive probability; otherwise we say that it **dies out**.



$$\text{survival} \left\{ \begin{array}{l} \text{strong survival} \\ \text{weak survival} \end{array} \right.$$

- Strong survival: every site will be infected infinitely many times with positive probability.
- Weak survival: the infection will persist with positive probability, but every site will be infected only **finite** times with probability 1. (Infection moves away to **infinity**.)

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$$\begin{cases} \lambda_1 := \inf\{\lambda : \text{the process survives}\}, \\ \lambda_2 := \inf\{\lambda : \text{the process survives strongly}\}. \end{cases}$$

- On an infinite connected graph of bounded degrees,

$$0 < \lambda_1 \leq \lambda_2 < +\infty.$$

- “Three” phases:

$$\begin{cases} \xi_t \text{ **dies out** if } \lambda < \lambda_1, \\ \xi_t \text{ **survives weakly** if } \lambda_1 < \lambda < \lambda_2, \\ \xi_t \text{ **survives strongly** if } \lambda > \lambda_2. \end{cases}$$

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Question 1: When $\lambda_1 = \lambda_2$? When $\lambda_1 < \lambda_2$?

- On integer lattices \mathbb{Z}^d ($d \geq 1$), $\lambda_1 = \lambda_2 \triangleq \lambda_c$.
- On homogeneous trees \mathbb{T}_d ($d \geq 3$), $\lambda_1 < \lambda_2$.

References:

- [1] C. E. Bezuidenhout and G. R. Grimmett: The critical contact process dies out, *Ann. Probab.* **18** 1462-1482 (1990).
- [2] A. M. Stacey: The existence of an intermediate phase for the contact process on trees, *Ann. Probab.* **24** 1711-1726 (1996).

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Two critical values

- **Conjecture:** For the contact process on an infinite locally finite connected quasi-transitive graph G , $\lambda_1 < \lambda_2$ **if and only if** G is non-amenable, that is, the Cheeger's constant of G is **positive**.
- Cheeger's constant:

$$\iota(G) := \inf \left\{ \frac{|\partial H|}{|H|} : H \subset G, H \text{ is connected}, 1 \leq |H| < \infty \right\}.$$

Reference:

- [1] R. Pemantle and A. M. Stacey: The branching random walk and contact process on Galton-Watson and nonhomogeneous trees, *Ann. Probab.* **29** 1563-1590 (2001).

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Question 2: What is the asymptotic behavior of the process **at** the critical value?

(Die out/Survive weakly/Survive strongly?)

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Contact process at the critical value

- For \mathbb{Z}^d ($d \geq 1$), the contact process **dies out** at the common critical value.
- For \mathbb{T}_d ($d \geq 3$), the contact process **dies out** at λ_1 .
- For \mathbb{T}_d ($d \geq 3$), the contact process **survives weakly** at λ_2 .

References:

- [1] C. E. Bezuidenhout and G. R. Grimmett: The critical contact process dies out, *Ann. Probab.* **18** 1462-1482 (1990).
- [2] G. Morrow, R. B. Schinazi and Y. Zhang: The critical contact process on a homogeneous tree, *J. Appl. Probab.* **31** 250-255 (1994).
- [3] Y. Zhang: The complete convergence theorem of the contact process on trees, *Ann. Probab.* **24** 1408-1443 (1996).

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Complete convergence theorem (CCT)

- Complete convergence theorem (CCT):

$$\mathbf{P}(\xi_t^A \in \cdot) \rightarrow \alpha_A \cdot \bar{\nu} + [1 - \alpha_A] \cdot \delta_\emptyset$$

weakly as $t \rightarrow \infty$, where $\bar{\nu}$ is the **upper invariant measure** and $\alpha_A := \mathbf{P}(\xi_t^A \neq \emptyset, \forall t \geq 0)$ is the survival probability.

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Question 3: Does CCT hold?

Complete convergence theorem (CCT)

- For \mathbb{Z}^d ($d \geq 1$), CCT holds for all λ .
- For \mathbb{T}_d ($d \geq 3$), CCT holds if $\lambda > \lambda_2$, and CCT does not hold if $\lambda_1 < \lambda \leq \lambda_2$.

References:

- [1] C. E. Bezuidenhout and G. R. Grimmett: The critical contact process dies out, *Ann. Probab.* **18** 1462-1482 (1990).
- [2] R. Durrett and R. B. Schinazi: Intermediate phase for the contact process on a tree, *Ann. Probab.* **23** 668-673 (1995)
- [3] Y. Zhang: The complete convergence theorem of the contact process on trees, *Ann. Probab.* **24** 1408-1443 (1996).
- [4] M. Salzano and R. H. Schonmann: A new proof that for the contact process on homogeneous trees local survival implies complete convergence, *Ann. Probab.* **26** 1251-1258 (1998).

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Complete convergence theorem (CCT)

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- **Conjecture:** For the supercritical contact process on **any** homogeneous graph G , CCT holds if and only if $\lambda > \lambda_2$.

Reference:

- [1] Y. Zhang: The complete convergence theorem of the contact process on trees, *Ann. Probab.* **24** 1408-1443 (1996).

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Contact processes in random environments (general setting)

- Transition rates:

$$\begin{cases} \xi_t \rightarrow \xi_t \setminus \{x\} \text{ for } x \in \xi_t \text{ at rate } \delta_x, \\ \xi_t \rightarrow \xi_t \cup \{x\} \text{ for } x \notin \xi_t \text{ at rate } \sum_{y \in \xi_t, y \sim x} \lambda_{(y,x)}. \end{cases}$$

That means, the recovery rates and infection rates are random variables.

Reference:

- [1] T. M. Liggett: The survival of one-dimensional contact processes in random environments, *Ann. Probab.* **20** 696-723 (1992).

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- Half space: $\mathbb{Z}^d \times \mathbb{Z}^+$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$.
- $\delta_x \equiv 1$.
- $\lambda_{(x,y)} = \lambda_{(y,x)}$ for x and y being neighbors. Denote them by λ_e if $e = (x, y)$.
- $\{\lambda_e : e \in E\}$ i.i.d. $\sim \mu$.

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We will consider the three problems described in the introduction part concerning this model.

1. Are the two critical values equal?
2. What is the asymptotic behavior of the process **at** the critical value?
3. Does CCT hold?

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- **Theorem 1:** Suppose μ puts mass 1 on $[0, +\infty)$. Then CCT holds almost surely.
- This result is a generalization of our earlier work which deals with the supercritical bond percolation cluster case, where μ follows the Bernoulli distribution.

References:

- [1] X. X. Chen and Q. Yao (2012): The complete convergence theorem holds for contact processes in a random environment on $\mathbb{Z}^d \times \mathbb{Z}^+$, *Stoch. Proc. Appl.* **122** 3066-3099.
- [2] X. X. Chen and Q. Yao (2009): The complete convergence theorem holds for contact processes on open clusters of $\mathbb{Z}^d \times \mathbb{Z}^+$, *J. Statist. Phys.* **135** 651-680.

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(Work in progress)

- In order to deal with the first two problems, we need to parameterize the model as follows.

$\{\lambda_e : e \in E\}$ i.i.d. and have the same distribution as ηX , where X is a fixed nonnegative random variable, and $\eta > 0$ is the parameter.

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- By monotonicity, we can define two critical values as follows:

$$\begin{cases} \eta_1 := \inf\{\eta : \text{the process survives}\}, \\ \eta_2 := \inf\{\eta : \text{the process survives strongly}\}. \end{cases}$$

- Obviously $\eta_1 \leq \eta_2$.
- $\eta_1 > 0$ or $= 0$? $\eta_2 < +\infty$ or $= +\infty$? Depends on the distribution of X .

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- **Theorem 2:** $\eta_1 = \eta_2$ and they depend only on the distribution of X almost surely. We denote by η_c the common value.
- **Theorem 3:** The process at the critical parameter η_c **dies out** almost surely.
- **Corollary:** For any $\eta > 0$, CCT holds almost surely.

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- For simplicity, we only consider the half plane case when $d = 1$.
- The idea is enlightened by C. E. Bezuidenhout and G. R. Grimmett (1990) (the “BG” renormalization argument).
- The idea is used in the percolation in the half space by D. J. Barsky, G. R. Grimmett and C. M. Newman (1991).

References:

- [1] C. E. Bezuidenhout and G. R. Grimmett: The critical contact process dies out, *Ann. Probab.* **18** 1462-1482 (1990).
- [2] D. J. Barsky, G. R. Grimmett and C. M. Newman: Percolation in half spaces: equalities of critical probabilities and continuity of the percolation probability, *Probab. Th. Rel. Fields* **90** 111-148 (1991).

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- Idea: Use **annealed law** and **coupling** to construct infection path, then go back to quenched law.
- Difficulties:
 - (1) randomness;
 - (2) inhomogeneity;
 - (3) not Markovian under the annealed law.

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Idea of proof—Theorem 1

- **Griffeath's lemma:** CCT holds iff

$$(1) \quad \mathbf{P}(x \in \xi_t^A \text{ i.o.}) = \mathbf{P}(\xi_t^A \neq \emptyset, \forall t \geq 0)$$

for all x and A .

$$(2) \quad \lim_{n \rightarrow \infty} \liminf_{t \rightarrow \infty} \mathbf{P}(\xi_t^{B(n)} \cap B(n) \neq \emptyset) = 1,$$

where $B(n)$ is the ball of radius n and a fixed center.

Reference:

- [1] D. Griffeath: Limit theorems for nonergodic set-valued Markov processes, *Ann. Probab.* **6** 379-387 (1978).

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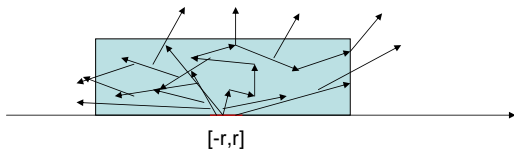
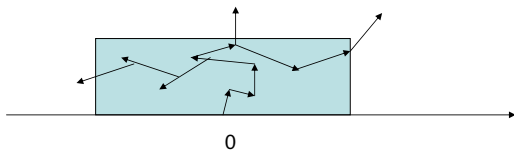
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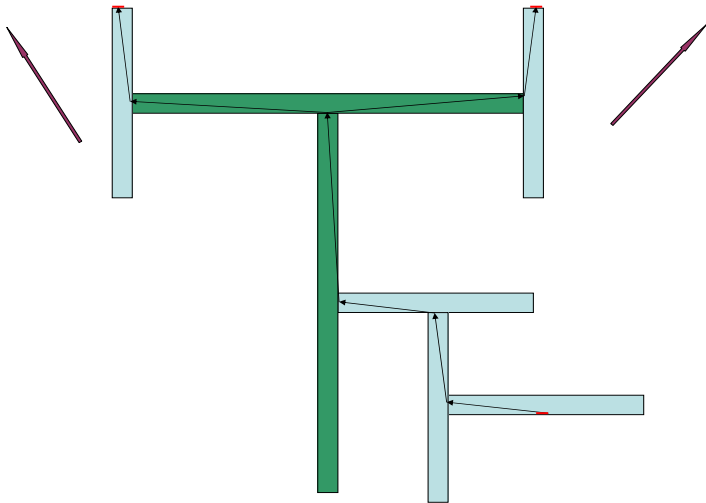
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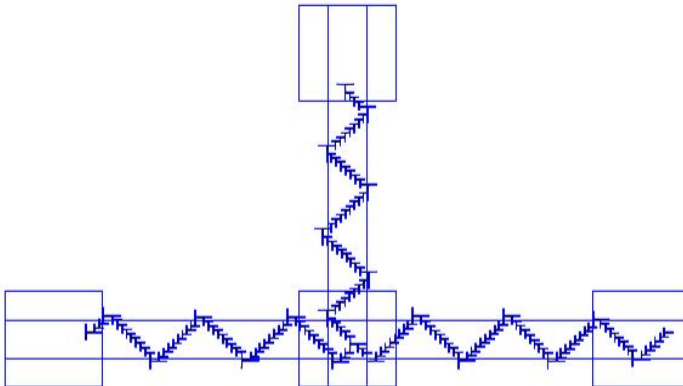
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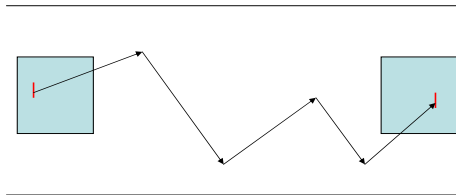
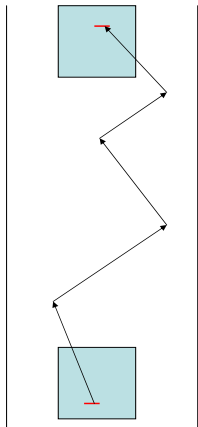
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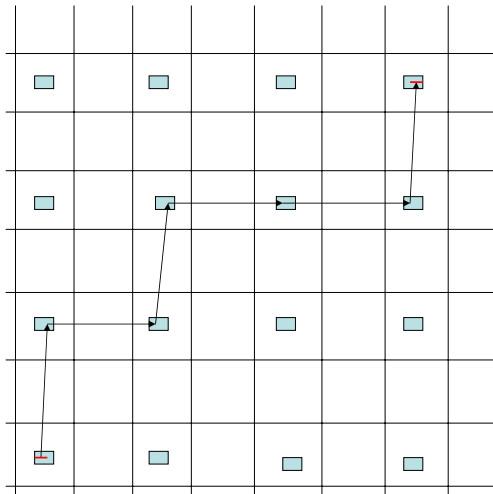
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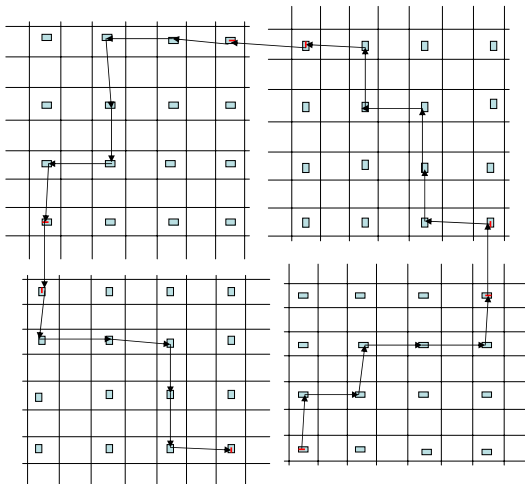
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Idea of proof—Theorems 2 and 3

- Survival is equivalent to some “block condition” which depends only on finite space–time region. Therefore, the value defined in the block condition is continuous in η .
- If the process survives at the critical value, then the block condition also holds. Furthermore, it holds for some larger parameter, and the process also survives, this contradicts with the definition of the critical value.

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- Method and results are still true for higher dimensions $d \geq 2$.
- We believe that the results are true for the whole space case.

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- **Why our method only works for the half space case?**

In this model, we first fix the environment according to some law, and then run the contact process under the environment. It is NOT Markovian under the annealed law, but the events determined by disjoint subgraphs are independent in some sense. That's why we cannot construct the space-time block conditions and get the result in the whole space case. Instead, we use the independence (in some sense) in space to construct the space block conditions and get the result in the half space case.

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Possible ways to prove the whole space case:

- Prove directly. Not easy. In the percolation cluster case, it is of the same difficulty as the problem that whether there is percolation at the critical point in the whole space case, which is clear in the half space case.
- Prove the critical value in the whole space case is the same as it in the half space case, which is clear for the percolation case as well as the contact process case, but not known for the contact process on the percolation cluster case.
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Related works: O. Garet and R. Marchand (2012, 2014).

- They considered the **whole** space \mathbb{Z}^d case.
- The problems they were dealing with: **shape theorem** and **large deviations**.
- Their assumptions: the law of $\{\lambda_e : e \in E\}$ is **stationary** and **ergodic**. λ_e takes value in $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min} > \lambda_c(\mathbb{Z}^d)$, and $\lambda_{\max} < +\infty$.

References:

- [1] O. Garet and R. Marchand: Asymptotic shape for the contact process in random environment, *Ann. Appl. Probab.* **22** 1362-1410 (2012).
- [2] O. Garet and R. Marchand: Large deviations for the contact process in random environment, *Ann. Probab.* **42** 1438-1479 (2014).

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Thank you!

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