Extremal Particles in Branching Brownian Motion - Part 1

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Singapore - Genealogies of Interacting Particle Systems - 2017

Overview:

Part 1: Extremal Particles in Branching Brownian Motion.

Part 2: Branching Brownian Motion under Selection.

Part 1

- Branching Brownian motion (BBM) - Definition and basic properties.
- The F-KPP equation and its connection to BBM.
- The distribution of the maximum of BBM.
- Extremal particles in BBM
  - ∗ Genealogy of extremal particles of BBM (cf. Arguin, Bovier and Kistler, [ABK11]);
  - ∗ Poissonian statistics in the extremal process of BBM (cf. [ABK12]).

Remark:

Following the bibliography, further slides are given relating to

- The extremal process of BBM (cf. [ABK13]);
- BBM seen from its tip (cf. Aïdékon, Berestycki, Brunet and Shi, [ABBS13]);
- Additional details, with pointers given by $\looparrowright$.

Branching Brownian Motion

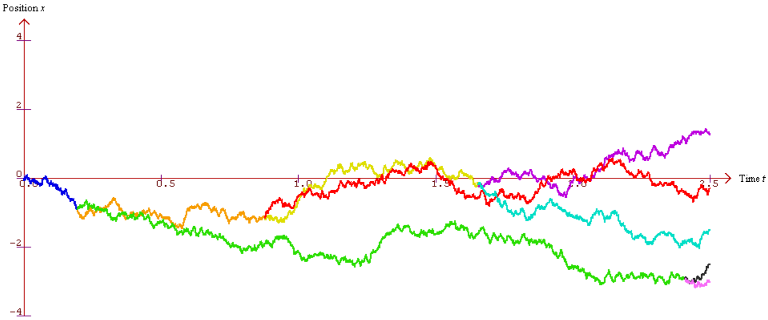
Definition (Branching Brownian Motion (BBM))

- $t = 0$: *single particle* $x_1(0)$ *starts at origin;*
- *moves as a Brownian motion (BM) in* $\mathbb{R}^1$ *until*
- *after (at) time* $\tau \sim Exp(1)$ *it splits into*
- *two identical particles that start (both) at* $x_1(\tau)$ *and*
- *move as two independent BMs each.*
- *Repeat.*

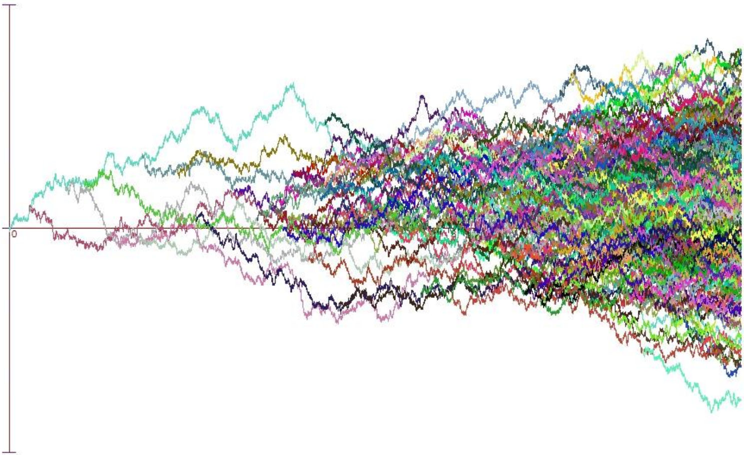
The resulting process is a collection of a (random) number $n(t)$ of particles

$$\left(x_1(t), x_2(t), \ldots, x_{n(t)}(t)\right)_{t\geq 0} = \left\{x_k(t) : k \leq n(t)\right\}_{t\geq 0}.$$

Note 1. $n(\tau) = 2$.
Note 2. Here we use w.l.o.g. binary branching.



(see homepage of Matt Roberts, Univ. of Bath)



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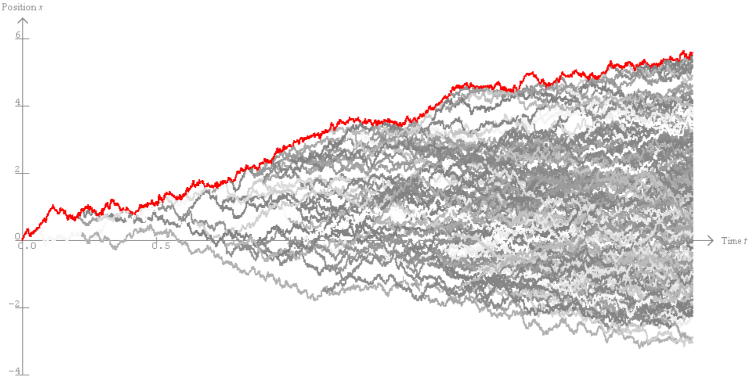

(see J. Berestycki (Lecture Notes, "Topics on BBM",

`http://www.stats.ox.ac.uk/~berestyc/Articles/EBP18_v2.pdf`), Image by Matt Roberts)

Remark *(see Bovier, [B15] for more respectively for references to literature)*

1. *There are connections to spin glass theory; in particular, Generalised Random Energy models (GREM).*
2. *Many results can be extended to branching random walk.*
3. *Connection to extremes of the free Gaussian random field in $d = 2$.*
4. *Can be extended to variable speed BBM.*

Remark (Genealogies of BBM)

Let

$$\begin{aligned} d(x_k(t), x_\ell(t)) &\equiv \inf\{0 \le s \le t : x_k(s) \neq x_\ell(s)\} = \text{ time (from 0) to MRCA} \\ &\equiv \text{ unique time where the } \textbf{m}\text{ost } \textbf{r}\text{ecent } \textbf{c}\text{ommon } \textbf{a}\text{ncestor split} \\ &= \text{ time of death of longest surviving ancestor of both particles.} \end{aligned}$$

A BBM can then also be constructed as follows. Construct first a continuous time Galton-Watson tree with binary branching. Let $\mathcal{I}(t)$ be the set of its leaves at time t. Then, conditional on the GW-process, BBM is a Gaussian process $x_k(t), k \in \mathcal{I}(t)$ with

$$\mathbb{E}[x_k(t)] = 0 \text{ and } Cov(x_k(t), x_\ell(t)) = d(x_k(t), x_\ell(t)).$$

First Properties

1. $\boxed{\mathbb{E}[n(t)] = e^t.}$
2. $e^{-t}n(t)$ is a martingale that converges, a.s. and in L^1, to an exponential r.v. of parameter 1.

The F-KPP equation

Consider the partial differential equation

$$u_t = \frac{1}{2}u_{xx} + u^2 - u, \quad u = u(t,x) \in [0,1], t \geq 0, x \in \mathbb{R}, \quad u(0,x) = f(x). \quad (1)$$

Set $v = 1 - u$. Then this is a special case of the Kolmogorov-Petrovskii-Piskunov-(KPP)-equation (also known as the Kolmogorov- or Fisher-equation).

Let $\left\{x_k(t) : k \leq n(t)\right\}_{t\geq 0}$ be a BBM starting at 0 and $f : \mathbb{R} \to [0,1]$. Then

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right]$$

is the solution to the F-KPP equation (1) with $u(0,x) = f(x)$.

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is the solution to the F-KPP equation (1) with $u(0,x) = f(x)$.

Idea of proof: Branching property: Let $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ denote the Heat kernel. Then (use that $\tau \sim \text{Exp}(1)$, i.e. $f_\tau(s) = 1_{\{s \geq 0\}} e^{-s}$ and $\mathbb{P}(\tau > t) = e^{-t}$)

$$u(t,x) = e^{-t} \int p_t(z) f(x-z) dz + \int_0^t e^{-s} \int p_s(z) u^2(t-s, x-z) dz ds.$$

Now differentiate w.r.t. t, use integration by parts and $\frac{\partial}{\partial t} p_t(x) = \frac{\partial^2}{\partial_x \partial_x} \frac{p_t(x)}{2}$. □

$$u_t = \frac{1}{2} u_{xx} + u^2 - u, \quad u(0,x) = f(x) \quad \text{has as solution} \quad u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right].$$

Example 1. Let $M(t) \equiv \max_{k \le n(t)} x_k(t)$. With $f(x) = 1_{[0,\infty)}(x)$ we obtain

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[\prod_{k=1}^{n(t)} 1_{[0,\infty)}(x - x_k(t))\right] \\ &= \mathbb{E}\left[\prod_{k=1}^{n(t)} 1_{\{x_k(t) \le x\}}\right] = \mathbb{P}\big(\max_{k \le n(t)} x_k(t) \le x\big) = \mathbb{P}(M(t) \le x) = F_{M(t)}(x). \end{aligned}$$

Example 2. Let $f(x) = e^{-\phi(x)}, \phi \in \mathcal{C}_c^+$. Set $\mathcal{P}_t \equiv \sum_{k=1}^{n(t)} \delta_{x_k(t)}$. Then

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} e^{-\phi(x - x_k(t))}\right] = \mathbb{E}\left[e^{-\int \phi(x-z)\mathcal{P}_t(dz)}\right].$$

The F-KPP equation revisited

Consider $v(t,x) \equiv 1 - u(t,x)$ instead of $u(t,x)$. Then $v(t,x)$ solves

$$v_t = \frac{1}{2} v_{xx} - v^2 + v = \frac{1}{2} v_{xx} + v(1-v), \qquad v(0,x) = 1 - u(0,x). \tag{2}$$

The Feynman-Kac formula yields the following representation for the linear equation

$$v_t = \frac{1}{2} v_{xx} + k(t,x) v, \qquad v_0(x) = v(0,x).$$

Namely,

$$\begin{aligned} v(t,x) &= \mathbb{E}\left[\exp\left(\int_0^t k(t-s, B^x(s)) ds\right) v(0, B^x(t))\right] \\ &= \mathbb{E}_x\left[\exp\left(\int_0^t k(t-s, B(s)) ds\right) v(0, B(t))\right]. \end{aligned}$$

Here, $(B^x(t))_{t \geq 0}$ is a BM, starting in $x \in \mathbb{R}$.
Bramson [B83] sets $k(t,x) \equiv 1 - v(t,x)$ with v solving (2). Then

$$v(t,x) = \mathbb{E}_x\left[\exp\left(\int_0^t \big(1 - v(t-s, B(s))\big) ds\right) v(0, B(t))\right]$$

("implicit description of v").

$$
\begin{aligned}
v(t,x) &= \mathbb{E}_x\left[\exp\left(\int_0^t k(t-s,B(s))ds\right) v(0,B(t))\right] \\
&= \mathbb{E}_x\left[\exp\left(\int_0^t \big(1-v(t-s,B(s))\big)ds\right) v(0,B(t))\right].
\end{aligned}
$$

Observations: (recall: $1-v(t,x)=u(t,x)$, $u(0,x)=1_{[0,\infty)}(x)$)

- "$v(t,x)$ is the weighted average of the different sample paths of BM."
- $0<k(t-s,B(s))<1$,
- can show: $\lim_{y\to-\infty} k(r,y)\le\epsilon$ for $r\ge r(\epsilon)$, $\lim_{y\to\infty} k(r,y)=1$;
- i.e., for y large, weighting of path nearly maximal, for y small, insignificant.
- Transition near $k(r,y)=1/2 \iff u(r,y)=1/2$
  (preview: $m(t)=\sup\{x: u(t,x)\le 1/2\}+O(1)$).

Bramson [B83] distinguishes paths $x(s), 0\le s\le t$ according to

$$
\exp\left(\int_0^t k(t-s,x(s))\right)ds \sim e^t \quad\text{or}\quad \exp\left(\int_0^t k(t-s,x(s))\right)ds \ll e^t.
$$

How to calculate $v(t,x)$?

$$\begin{aligned} v(t,x) &= \mathbb{E}\left[\exp\left(\int_0^t k(t-s,B^x(s))ds\right) v(0,B^x(t))\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\int_0^t k(t-s,B^x(s))ds\right) v(0,B^x(t)) \mid B^x(t)\right]\right] \\ &= \int_{-\infty}^{\infty} v(0,y)\frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}}\mathbb{E}\left[\exp\left(\int_0^t k(t-s,\mathfrak{z}^t_{x,y}(s))ds\right)\right] dy, \end{aligned}$$

where $\mathfrak{z}^t_{x,y}$ denotes a Brownian bridge starting at x (at time 0) and ending at y (at time t).

- A Brownian bridge has the distribution of a BM starting at x, conditional on being in y at time t.
- $\mathfrak{z}^t_{0,0}(s) \equiv B^0(s) - \frac{s}{t}B^0(t)$, $0 \le s \le t$ is
    - a Gaussian process (all finite-dim. distrib.s are normally distributed),
    - a.s. continuous on $[0,t]$, a strong Markov process and indep. of $B^0(t)$.
- $\mathfrak{z}^t_{x,y}(s) \stackrel{\mathcal{D}}{=} \mathfrak{z}^t_{0,0}(s) + \frac{s}{t}y + \frac{t-s}{t}x$, $0 \le s \le t$.
- $\mathrm{Var}(\mathfrak{z}^t_{0,0}(s)) = \frac{s(t-s)}{t}$ with a maximum in the middle, $\mathrm{Var}(\mathfrak{z}^t_{0,0}(t/2)) = t/4$.

(Brownian Bridge, cf. `https://www.researchgate.net/figure/228766780_fig2_Figure-Sample-path-examples-of-a-Brownian-bridge-for-different-initial-and-final-states`)

The distribution of the maximum of BBM

Let us return to Example 1. Then $u(t,x) = \mathbb{P}(M(t) \leq x) = \mathbb{P}\big(\max_{k \leq n(t)} x_k(t) \leq x\big)$ solves (1) with $u(0,x) = 1_{[0,\infty)}(x)$.

As a result: $0 < u(t,x) < 1$ for all $x \in \mathbb{R}, t > 0$.

Take as centering term,

$$m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t), \qquad \text{(in i.i.d. case } \sqrt{2}t - \frac{1}{2\sqrt{2}} \log(t) \looparrowright^{1}\text{)}$$

then $\boxed{m(t) = \sup\{x : u(t,x) \leq 1/2\} + O(1)}$ (cf. Bramson [B83], Roberts [R13]) and

$$\boxed{\mathbb{P}(M(t) - m(t) \leq x) = u(t, m(t)+x) \to w(x)} \quad \text{unif. in } x \text{ as } t \to \infty. \qquad (3)$$

Here, $w(x)$ is the unique (up to translation) solution of the equation

$$\frac{1}{2} w_{xx} + \sqrt{2} w_x + w^2 - w = 0$$

satisfying $0 < w(x) < 1$ for all $x \in \mathbb{R}$ and $w(x) \to 0$ as $x \to -\infty$, $w(x) \to 1$ as $x \to \infty$.

The derivative martingale

Lalley and Sellke [LS87] show: Let

$$Z(t) \equiv \sum_{k=1}^{n(t)} \left(\sqrt{2}t - x_k(t)\right) e^{-\sqrt{2}\left(\sqrt{2}t - x_k(t)\right)}$$

be the so-called derivative martingale, then

$$Z = \lim_{t\to\infty} Z(t) \tag{4}$$

exists and is strictly positive a.s. Moreover, for some $C > 0$,

$$\mathbb{P}(M(t)-m(t)\leq x) \overset{t\to\infty}{\to} \boxed{w(x) = \mathbb{E}\left[\exp\left(-CZe^{-\sqrt{2}x}\right)\right]} = \mathbb{E}\left[\exp\left(-e^{-\sqrt{2}\left(x-\frac{\log(CZ)}{\sqrt{2}}\right)}\right)\right]. \tag{5}$$

Note 1. The so-called Gumbel distribution has cumulative distribution function

$$F_G(x) = \mathbb{P}(G \leq x) = \exp\left(-e^{-(x-\mu)/\beta}\right) = \exp\left(-e^{\mu/\beta}e^{-x/\beta}\right)$$

with parameters $\mu \in \mathbb{R}, \beta > 0$. Hence, $w(x)$ represents a random shift of "the" Gumbel distribution with $\mu = 0$ and $\beta = 1/\sqrt{2}$.

Note 2. $\boxed{1 - w(x) \sim Cxe^{-\sqrt{2}x} \text{ for } x \to \infty.}$

(Gumbel distribution, cf. https://en.wikipedia.org/wiki/Gumbel_distribution#/media/File:Gumbel-Density.svg

https://en.wikipedia.org/wiki/Gumbel_distribution#/media/File:Gumbel-Cumulative.svg)

Additionally, [LS87] showed

Theorem

Suppose two independent BBMs $\left(X_1^A(t), \ldots, X_{n^A(t)}^A(t)\right)$ *and* $\left(X_1^B(t), \ldots, X_{n^B(t)}^B(t)\right)$ *are started at* 0 *respectively* $x < 0$*. Then, with probability* 1*, there exist finite random times* $t_n, n \in \mathbb{N}, t_n \to \infty$ *such that*

$$M^A(t_n) < M^B(t_n)$$

for all $n \in \mathbb{N}$*.*

Idea of proof. Use (3), i.e. $\lim_{t\to\infty} \mathbb{P}(M(t) - m(t) \leq x) = w(x)$. □

Corollary

Every particle born in a BBM has a descendant particle in the "lead" at some future time.

Extremal particles in BBM (cf. [ABK11] and [ABK12])

- Arguin, L.-P. and Bovier, A. and Kistler, N.
  Genealogy of extremal particles of branching Brownian motion.
  Comm. Pure Appl. Math. **64** *(2011), 1647–1676.*
- Arguin, L.-P. and Bovier, A. and Kistler, N.
  Poissonian statistics in the extremal process of branching Brownian motion.
  Ann. Appl. Probab. **22** *(2012), 1693–1711.*

We are interested in the limit ($t \to \infty$) of the extremal process

$$\mathcal{E}_t \equiv \sum_{k \le n(t)} \delta_{x_k(t)-m(t)} \equiv \sum_{k \le n(t)} \delta_{\overline{x_k}(t)}.$$

Recall: The *centering term* $m(t)$ satisfies

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log(t) = \sup\{x : u(t,x) \le 1/2\} + O(1).$$

By the previous Remark on Genealogies of BBM, for a *given* realization of the branching, the genealogical distances

$$d(x_k(t), x_\ell(t)) = \inf\{0 \le s \le t : x_k(s) \ne x_\ell(s)\} = \text{ time (from 0) to MRCA }.$$

Genealogy of extremal particles of BBM

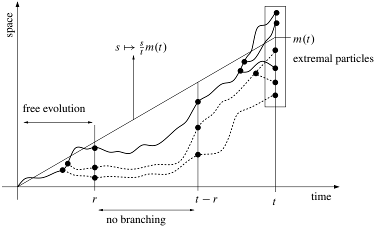
Theorem (Genealogy of Extremal Particles, Theorem 2.1, [ABK11])

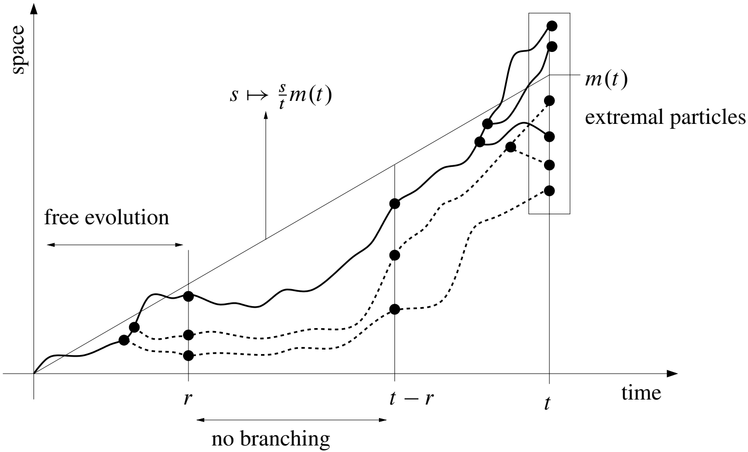
For any compact set $D \subset \mathbb{R}$,

$$\lim_{r\to\infty} \sup_{t>3r} \mathbb{P}\big(\exists 1 \leq k, \ell \leq n(t) : \overline{x_k}(t), \overline{x_\ell}(t) \in D \text{ and } d(x_k(t), x_\ell(t)) \in (r, t-r)\big) = 0.$$

Conclusion: The MRCA of extremal particles at time t splits/branches off with high probability at a time

1. in the interval $(0, r)$ ("very early branching") or
2. in the interval $(t-r, t)$ ("very late branching").





(Figure 2.4 of [ABK11])

Poissonian statistics in the extremal process of BBM

W.l.o.g., order the (centered) particles in decreasing order, i.e.

$$\overline{x_1}(t) \geq \overline{x_2}(t) \geq \cdots \geq \overline{x_{n(t)}}(t). \tag{6}$$

Let

$$\overline{D}(t) = \{\overline{D}_{k\ell}(t)\}_{k,\ell \leq n(t)} \equiv \left\{ \frac{d(x_k(t), x_\ell(t))}{t} \right\}_{k,\ell \leq n(t)}.$$

Definition

Let $0 < q < 1$. The q-thinning $\mathcal{E}_t^{(q)}$ of the pair $(\mathcal{E}_t, \overline{D}(t))$ is defined as follows:

- *Consider the equivalence classes of particles alive at time t with MRCA at a time later than $q \cdot t$, i.e.* $k \sim_q \ell \iff \overline{D}_{k\ell}(t) \geq q.$
- *Select the maximal (according to (6)) particle within each class.*
- *Then $\mathcal{E}_t^{(q)}$ is the point process of these representatives.*

Note 1. This extends to $q = q(t) \in (0, 1)$.

Note 2. The thinning map $(\mathcal{E}_t, \overline{D}(t)) \mapsto \mathcal{E}^{(q)}(t)$ is continuous (on the space of pairs (X, Q), X ordered positions, Q symm. matrix with entries in $[0, 1]$ and transitive op. $Q_{ij} \geq q$).

Theorem (Theorem 2, [ABK12])

For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also,

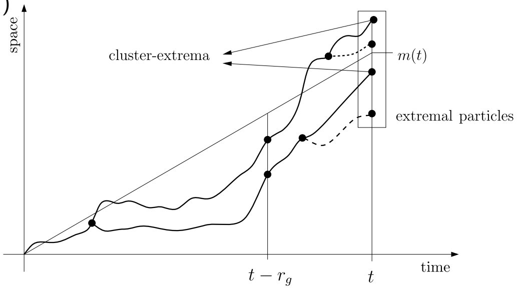
$$\lim_{r\to\infty} \lim_{t\to\infty} \mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0.$$

Moreover, conditionally on Z, (the limit of the derivative martingale, cf. (4)),

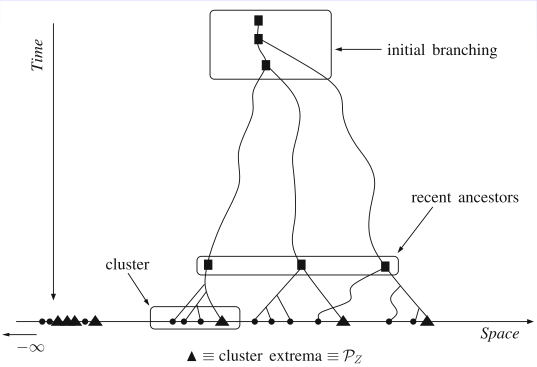
$$\mathcal{E}^0 = PPP\big(CZ\sqrt{2}e^{-\sqrt{2}x}dx\big) \qquad \text{(PPP stands for "Poisson Point Process")}$$

where $C > 0$ is the constant appearing in (5).

(Figure 2 in [ABK12])



(Figure 1 of [ABK13])



Conclusion:

The particles at the frontier of BBM for large times can be constructed as follows:

1. set down so-called cluster extrema according to $\mathcal{E}^0$, that is, according to a randomly shifted PPP with "exponential" ($x \in \mathbb{R}$) density;
2. attach to each cluster extrema a cluster.

Note 1. Particles in one cluster lie to the left (in space) of its corresponding cluster extrema (Poissonian particle).

Note 2. Heuristic for PP-structure: the ancestors of the extremal particles evolve independently for the time-interval $[r, t - r]$.

Remark (Invariance property)

The law of the limiting extremal process $\mathcal{E} = \sum_{i \in \mathbb{N}} \delta_{e_i}$ satisfies the following invariance property: For any $s \ge 0$,

$$\mathcal{E} \stackrel{\mathcal{D}}{=} \sum_{i,k} \delta_{e_i + x_k^{(i)}(s) - \sqrt{2}s},$$

where $\{x_k^{(i)}(s) : k \le n^{(i)}(s), s \ge 0\}_{i \in \mathbb{N}}$ are i.i.d. BBMs. Indeed, use that for $t \to \infty$,

$$m(t) = m(t-s) + \sqrt{2}s + o(1).$$

Then rewrite

$$\begin{aligned}\mathcal{E}_t \equiv \sum_{i \le n(t)} \delta_{x_i(t) - m(t)} &= \sum_{i \le n(t-s)} \sum_{k \le n^{(i)}(s)} \delta_{x_i(t-s) + x_k^{(i)}(s) - m(t)} \\ &= \sum_{i \le n(t-s)} \sum_{k \le n^{(i)}(s)} \delta_{x_i(t-s) - m(t-s) + x_k^{(i)}(s) - \sqrt{2}s + o(1)}\end{aligned}$$

and take $t \to \infty$.

Idea of proof of Theorem 2, [ABK12]

Theorem 2, [ABK12]

For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also, $\lim_{r\to\infty} \lim_{t\to\infty} \mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0$.

Moreover, conditionally on Z, $\mathcal{E}^0 = PPP\left(CZ\sqrt{2}e^{-\sqrt{2}x}dx\right)$. $(*)$

Genealogy of Extremal Particles, Theorem 2.1, [ABK11]

For any compact set $D \subset \mathbb{R}$,

$\lim_{r\to\infty} \sup_{t>3r} \mathbb{P}\left(\exists 1 \le k, \ell \le n(t) : \overline{x_k}(t), \overline{x_\ell}(t) \in D \text{ and } d(x_k(t), x_\ell(t)) \in (r, t-r)\right) = 0.$ $(**)$

- Same limit $\mathcal{E}^0$ for $\frac{r}{t} < q < 1 - \frac{r}{t}$ with r big enough follows from $(**)$.
- It remains to show $(*)$. This is done via convergence of Laplace functionals, that is, for $\phi \in \mathcal{C}_c^+$ we claim that

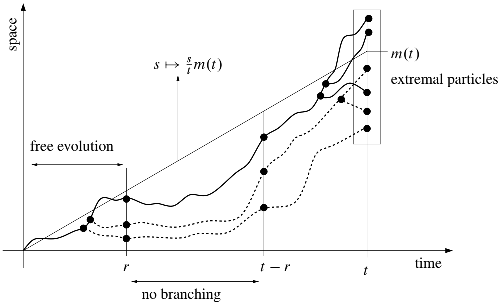
$$\lim_{r\to\infty} \lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi(x)\mathcal{E}_t^{(1-r/t)}(dx)}\right] = \mathbb{E}\left[e^{-CZ\int \left(1-e^{-\phi(x)}\right)\sqrt{2}e^{-\sqrt{2}x}dx}\right].$$

Note. If $X \sim PPP(\lambda)$, then $\mathbb{E}[e^{-\int \phi(x)X(dx)}] = e^{\int (e^{-\phi(x)}-1)\lambda(dx)}$ (cf. [B15], Appendix).

Conditional on the evolution of the BBM up to time r, we obtain

$$\mathcal{E}_t^{(1-r/t)} \overset{\mathcal{D}}{=} \left\{ x_j(r) + M^{(j)}(t-r) - m(t) \right\}_{j=1,\dots,n(r)},$$

where $\{x_k^{(j)}(t)\}_{k \le n^{(j)}(t)}$, $j \in \mathbb{N}$ are i.i.d. BBM with $M^{(j)}(t) \equiv \max_{k \le n^{(j)}(t)} x_k^{(j)}(t)$.



(Figure 2.4 of [ABK11])

Now,

$$m(t) = \sqrt{2}r + m(t-r) + o(1) \quad \text{for} \quad t \to \infty.$$

$\mathbb{P}(M(t) - m(t) \leq x) = u(t, m(t) + x) \to w(x)$ unif. in x as $t \to \infty$.
$m(r) = \sqrt{2}r - \frac{3}{2\sqrt{2}} \log(r)$.

As a result,

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi(x)\mathcal{E}_t^{(1-r/t)}(dx)}\right] = \lim_{t\to\infty} \mathbb{E}\left[\prod_{j=1}^{n(r)} \mathbb{E}\left[e^{-\phi\left(x_j(r)-\sqrt{2}r+M(t-r)-m(t-r)+o(1)\right)}\right]\right]$$
$$= \mathbb{E}\left[\prod_{j=1}^{n(r)} \mathbb{E}\left[e^{-\phi\left(x_j(r)-\sqrt{2}r+\bar{M}\right)}\right]\right],$$

where $\overline{M}$ has law w. $\looparrowright^2$

Note 1. $\boxed{\max_{j\leq n(r)}(x_j(r) - \sqrt{2}r) \to -\infty \text{ a.s. as } r \to \infty.}$

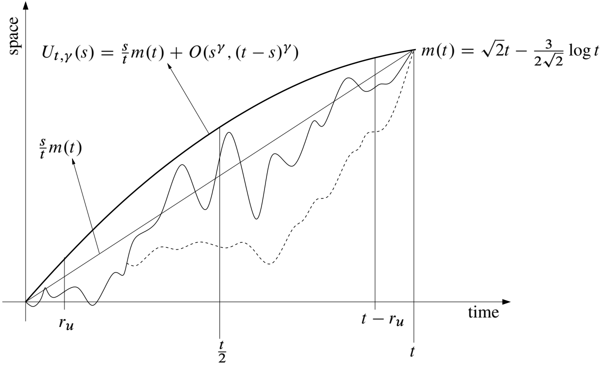
Note 2. Now use asymptotics for $\overline{M}$, i.e. $1 - w(x) \sim Cxe^{-\sqrt{2}x}$ for $x \to \infty$.

Idea of proof of Theorem 2.1, [ABK11]

Localization of Paths of Extremal Particles

Theorem (Genealogy of Extremal Particles, Theorem 2.1, [ABK11]) For any compact set $D \subset \mathbb{R}$,

$$\lim_{r\to\infty} \sup_{t>3r} \mathbb{P}\big(\exists 1 \le k, \ell \le n(t) : \ \overline{x_k}(t), \overline{x_\ell}(t) \in D \text{ and } d(x_k(t), x_\ell(t)) \in (r, t-r)\big) = 0.$$



(Figure 2.1 of [ABK11])

For $\gamma > 0$, set

$$f_{t,\gamma}(s) \equiv \begin{cases} s^\gamma, & 0 \le s \le \frac{t}{2}, \\ (t-s)^\gamma, & \frac{t}{2} \le s \le t. \end{cases}$$

Then the upper envelope at time t is defined as

$$U_{t,\gamma}(s) \equiv \frac{s}{t} m(t) + f_{t,\gamma}(s).$$

Theorem (Upper Envelope, Theorem 2.2 of [ABK11])

Let $0 < \gamma < 1/2$. Let also $y \in \mathbb{R}$ and $\epsilon > 0$ be given. There exists $r_u = r_u(\gamma, y, \epsilon)$ such that for $r \ge r_u$ and for any $t > 3r$,

$$\mathbb{P}(\exists k \le n(t) : x_k(s) > y + U_{t,\gamma}(s) \text{ for some } s \in [r, t-r]) < \epsilon.$$

Idea of proof. Discretize path and use $\mathbb{P}(M(t) > m(t) + x) \overset{t\to\infty}{\to} 1 - w(x) \sim Cxe^{-\sqrt{2}x}$ for $x \to \infty$. □

Remark (Why is this useful?)

For $\mathbb{E}[n(t)] = e^t$ *i.i.d. BMs,* $m(t) = \sqrt{2}t - \frac{1\cdot\log(t)}{2\sqrt{2}}$ *and*

$$\mathbb{E}[\#\{k \le n(t) : B_k(t) > m(t)\}] \sim \frac{e^t}{\sqrt{2\pi t}} e^{-\frac{m(t)^2}{2t}}$$

$$= \frac{e^t}{\sqrt{2\pi t}} e^{-\frac{2t^2 - \frac{2\sqrt{2}t\log(t)}{2\sqrt{2}} + \frac{\log(t)^2}{8}}{2t}} = \frac{1}{\sqrt{2\pi t}} e^{\frac{\log(t)}{2} - \frac{\log(t)^2}{16t}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\log(t)^2}{16t}} = O(1).$$

For BBM, $m(t) = \sqrt{2}t - \frac{3\cdot\log(t)}{2\sqrt{2}}$*. For* e^t *i.i.d. BMs we now get*

$$\mathbb{E}[\#\{k \le n(t) : B_k(t) > m(t)\}] = \frac{1}{\sqrt{2\pi t}} e^{3\frac{\log(t)}{2} - \frac{9\log(t)^2}{16t}} = O(t).$$

Now, for a BM B_t *starting in* 0*,*

$$\mathbb{P}(B_s \le U_{t,\gamma}(s), r \le s \le t - r | B_t = m(t))$$
$$= \mathbb{P}(\mathfrak{z}^t_{0,m(t)}(s) \le U_{t,\gamma}(s), r \le s \le t - r) \sim \frac{1}{t}.$$

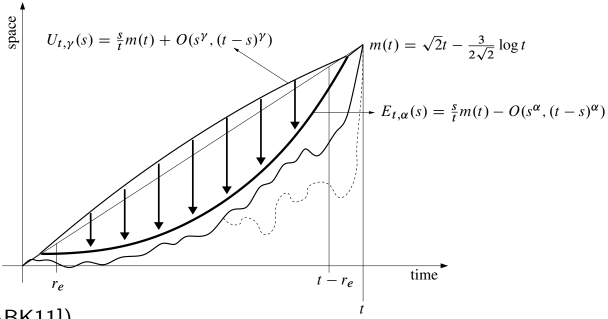
The upper envelope can be replaced by a lower "entropic envelope" E:
For $\alpha > 0$ let

$$E_{t,\alpha}(s) \equiv \frac{s}{t} m(t) - f_{t,\alpha}(s).$$

Theorem (Entropic Repulsion, Theorem 2.3 of [ABK11])

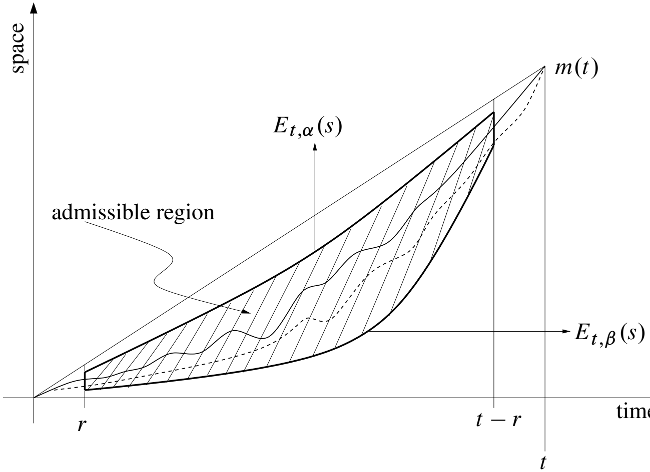
Let $D \subset \mathbb{R}$ be a compact set and $0 < \alpha < 1/2$. Set $\bar{D} \equiv \sup\{x \in D\}$. For any $\epsilon > 0$ there exists $r_e = r_e(\alpha, D, \epsilon)$ such that for $r \geq r_e$ and $t > 3r$,

$$\mathbb{P}(\exists k \leq n(t) : x_k(t) \in m(t) + D, \text{ but } \exists s \in [r, t-r] : x_k(s) \geq \bar{D} + E_{t,\alpha}(s)) < \epsilon.$$



(Figure 2.2 of [ABK11])

Together with a lower envelope we get with $0 < \alpha < \frac{1}{2} < \beta < 1$,



(Figure 2.3 of [ABK11])

Remark (How to use this $\looparrowright^3$)

The expected number of pairs of particles of BBM whose (respective) path $(x(s))_{0\le s\le t}$ satisfies some conditions for $s\in[r,t-r]$, say $\Sigma_t^{[r,t-r]}$, is

$$\mathbb{E}[\#\{(k,\ell): k\neq \ell, x_k(\cdot), x_\ell(\cdot)\in\Sigma_t^{[r,t-r]}\}]$$
$$=C(e^t)^2\int_0^t ds\, e^{-s}\int_{-\infty}^{\infty} dy\, p_s(y)\mathbb{P}(x(\cdot)\in\Sigma_t^{[r,t-r]}|x(s)=y)\mathbb{P}(x(\cdot)\in\Sigma_t^{[r\vee s,t]}|x(s)=y).$$

- *how many pairs at time t on average;*
- *condition on splitting at time s and*
- *at position y.*
- *If first particle satisfies the condition on $[0,t]$, then the second one automatically satisfies it on $[0,s]$.*

If we include a condition on genetic distance, we get

$$\mathbb{E}[\#\{(k,\ell): k\neq \ell, x_k(\cdot), x_\ell(\cdot)\in\Sigma_t^{[r,t-r]}, d(x_k(t),x_\ell(t))\in[r,t-r]\}]$$
$$=Ce^t\int_r^{t-r} ds\, e^{t-s}\int_{-\infty}^{\infty} dy\, p_s(y)\mathbb{P}(x\in\Sigma_t^{[r,t-r]}|x(s)=y)\mathbb{P}(x\in\Sigma_t^{[s,t-r]}|x(s)=y).$$

Bibliography

Arguin, L.-P. and Bovier, A. and Kistler, N.
Genealogy of extremal particles of branching Brownian motion.
Comm. Pure Appl. Math. **64** *(2011), 1647–1676.*

Arguin, L.-P. and Bovier, A. and Kistler, N.
Poissonian statistics in the extremal process of branching Brownian motion.
Ann. Appl. Probab. **22** *(2012), 1693–1711.*

Arguin, L.-P. and Bovier, A. and Kistler, N.
The extremal process of branching Brownian motion.
Probab. Theory Related Fields **157** *(2013), 535–574.*

Aïdékon, E. and Berestycki, J. and Brunet, É. and Shi, Z.
Branching Brownian motion seen from its tip.
Probab. Theory Related Fields **157** *(2013), 405–451.*

Bovier, A.
From spin glasses to branching Brownian motion—and back?
Lecture Notes in Math. **2144** *(2015), 1–64.*

Bramson, M.
Convergence of solutions of the Kolmogorov equation to travelling waves.
Mem. Amer. Math. Soc. **44**, *Number 285 (1983).*

Chauvin, B. and Rouault, A.
KPP Equation and Supercritical Branching Brownian Motion in the Subcritical Speed Area. Application to Spatial Trees.
Probab. Theory Related Fields **80** *(1988), 299–314.*

Gouéré, J.-B.
Branching Brownian motion seen from its leftmost particle (following Arguin-Bovier-Kistler and Aïdékon-Berestycki-Brunet-Shi).
Astérisque **361** *(2014), 271–298.*

Lalley, S. P. and Sellke, T.
A conditional limit theorem for the frontier of a branching Brownian motion.
Ann. Probab. **15** *(1987), 1052–1061.*

Neveu, J.
Multiplicative martingales for spatial branching processes.
Progr. Probab. Statist. **15** *(1988), 223–242.*

Roberts, M.I.
A simple path to asymptotics for the frontier of a branching Brownian motion.
Ann. Probab. **41** *(2013), 3518–3541.*

The extremal process of BBM (cf. [ABBS13] and [ABK13])

- ARGUIN, L.-P. and BOVIER, A. and KISTLER, N.
  The extremal process of branching Brownian motion.
  Probab. Theory Related Fields **157** *(2013), 535–574.*
- AÏDÉKON, E. and BERESTYCKI, J. and BRUNET, É. and SHI, Z.
  Branching Brownian motion seen from its tip.
  Probab. Theory Related Fields **157** *(2013), 405–451.*

Both articles give a description of the (weak w.r.t. $\phi \in \mathcal{C}_c^+$) limit ($t \to \infty$) of the extremal process

$$\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t)-m(t)} \equiv \sum_{k \leq n(t)} \delta_{\overline{x_k}(t)}.$$

See Gouéré [G14] for a (french) review that presents and compares both approaches.

[ABK13] The extremal process of BBM

Remark

- *Bovier [B15] discusses the following results in detail.*
- *There are also other representations.* $\looparrowright^4$
- *The proofs rely on the consideration of the respective Laplace functionals.*

Theorem (Theorem 3.1 (Existence of the limit), [ABK13])

The point process $\mathcal{E}_t$ converges in law to a point process $\mathcal{E}$.

Idea of proof: Example 2 for the F-KPP equation. $\square$

Theorem 2, [ABK12]

For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also, $\lim_{r\to\infty}\lim_{t\to\infty}\mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0$. Moreover, conditionally on Z, $\mathcal{E}^0 = PPP\big(CZ\sqrt{2}e^{-\sqrt{2}x}dx\big)$, where $C > 0$ is the constant appearing in (5).

$$\mathcal{E}_t \equiv \sum_{k\le n(t)} \delta_{x_k(t)-m(t)} \quad \text{with} \quad m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t).$$

Definition (Cluster-extrema)

Conditionally on the limiting derivative martingale Z, consider the PPP

$$\mathcal{P}_Z \equiv \sum_{i\in\mathbb{N}} \delta_{p_i} \stackrel{\mathcal{D}}{=} PPP\big(CZ\sqrt{2}e^{-\sqrt{2}x}dx\big) \tag{7}$$

with C as in (5).

Definition (Clusters)

Let $\bar{\mathcal{E}}_t \equiv \sum_{k\le n(t)} \delta_{x_k(t)-\sqrt{2}t}$. **Conditionally on** $\{\max_{k\le n(t)} x_k(t) - \sqrt{2}t \ge 0\}$, *the process $\bar{\mathcal{E}}_t$ converges to a point process $\bar{\mathcal{E}} = \sum_j \delta_{\xi_j}$. Now define the point process of the gaps by*

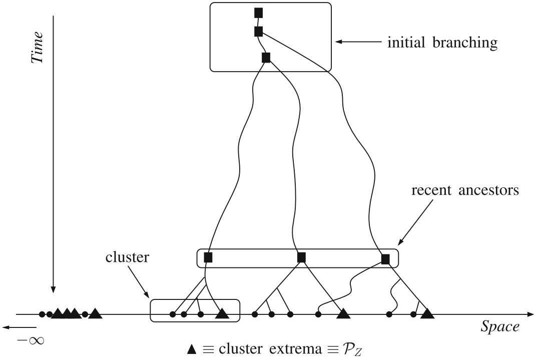
$$\mathcal{D} \equiv \sum_j \delta_{\Delta_j}, \quad \Delta_j \equiv \xi_j - \max_j \xi_j. \tag{8}$$

Note. $\mathcal{D}$ is a point process on $(-\infty, 0]$ with an atom at 0.

Theorem (Theorem 2.1 (Main Theorem), [ABK13])

Let $\mathcal{P}_Z$ be as in (7) and let $\{\mathcal{D}^{(i)} : i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (8). Then the point process $\mathcal{E}_t$ converges in law as $t \to \infty$ to a Poisson cluster point process $\mathcal{E}$ given by

$$\mathcal{E} \equiv \lim_{t\to\infty} \mathcal{E}_t \stackrel{\mathcal{D}}{=} \sum_{i,j} \delta_{p_i+\Delta_j^{(i)}}.$$



(Figure 1 of [ABK13])

[ABBS13] BBM seen from its tip

Remark

- *The proofs use path localization and path decomposition techniques.*
- *J. Berestycki (Lecture Notes, "Topics on BBM", http://www.stats.ox.ac.uk/~berestyc/Articles/EBP18_v2.pdf) gives a good introduction in the underlying concepts.*

Notation (Change in Scaling)

Note that a different scaling is used and instead of rightmost particles, leftmost are considered.

- *Particles now follow a BM* **with drift** 2 **and variance** 2*, that is, replace* B_t *by* $\sqrt{2}(B_t - \sqrt{2}t)$.
- *(The exponential clocks for splitting-events still ring at rate* 1 *and a particle splits in two.)*
- *Instead of* $M(t) = \max_{k \leq n(t)} x_k(t)$ *they consider* $\min_{k \leq n(t)} x_k(t)$.

$m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t)$.

The derivative mart. $Z(t) \equiv \sum_{k=1}^{n(t)} \left(\sqrt{2}t - x_k(t)\right) e^{-\sqrt{2}\left(\sqrt{2}t - x_k(t)\right)}$ satisfies $Z = \lim_{t\to\infty} Z(t)$.

Particles now follow a BM **with drift** 2 **and variance** 2, that is, replace B_t by $\sqrt{2}(B_t - \sqrt{2}t)$. Consider $\min_{k\le n(t)} x_k(t)$.

Remark (Consequences of Scaling)

- *$m(t)$ becomes $m'(t) = +\frac{3}{2}\log(t)$.*
- *$Z(t)$ becomes $\frac{1}{\sqrt{2}}\sum_{k=1}^{n(t)} x_k(t)e^{-x_k(t)}$. [ABBS13] use $Z'(t) = \sum_{k=1}^{n(t)} x_k(t)e^{-x_k(t)}$ instead and thus CZ becomes $(C/\sqrt{2})Z' = C'Z'$.*
- *$\mathbb{E}[Z'_t] = 0$ for all $t \ge 0$.*

From now onwards, we use the notation of [ABBS13].

Definition (The additive martingale)

The process $\mathcal{M}(t) \equiv \sum_{k=1}^{n(t)} e^{-x_k(t)}$ is a martingale with $\mathbb{E}[M_t] = 1$ for all $t \ge 0$, the so-called additive martingale.

$\mathcal{E}_t \equiv \sum_{k \le n(t)} \delta_{x_k(t) - m(t)}$.

Cluster-extrema $\mathcal{P}_Z \equiv \sum_{i \in \mathbb{N}} \delta_{p_i} \stackrel{\mathcal{D}}{=} PPP\left(CZ\sqrt{2}e^{-\sqrt{2}x}dx\right) = PPP\left(e^{-\sqrt{2}x + \log(CZ)} d(\sqrt{2}x)\right)$.

Theorem 2.1 (Main Theorem), [ABK13] Let $\mathcal{P}_Z$ be as in (7) and let $\{\mathcal{D}^{(i)} : i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (8). Then $\boxed{\mathcal{E} \equiv \lim_{t\to\infty} \mathcal{E}_t \stackrel{\mathcal{D}}{=} \sum_{i,j} \delta_{p_i + \Delta_j^{(i)}}.}$

Let

$$\bar{\mathcal{N}}(t) \equiv \sum_{k \le n(t)} \delta_{x_k(t) - m(t) + \log(CZ)}.$$

Theorem (Theorem 2.1, [ABBS13] ↬[5])

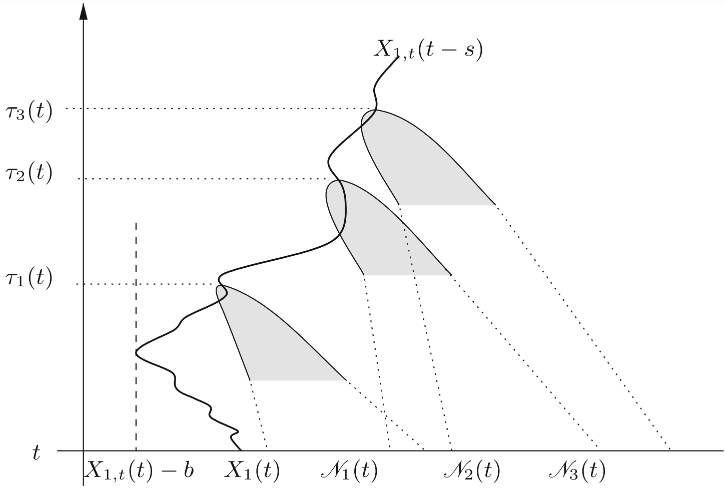
As $t \to \infty$ the pair $\{\bar{\mathcal{N}}(t), Z(t)\}$ converges jointly in distribution to $\{\mathcal{L}, Z\}$, $\mathcal{L}$ and Z are independent and $\mathcal{L}$ is obtained as follows.

(i) *Define $\mathcal{P}$ a Poisson point measure on $\mathbb{R}$, with intensity measure $e^x dx$.*

(ii) *For each atom x of $\mathcal{P}$, we attach a point measure $\mathcal{D}^{(x)}$ where $\mathcal{D}^{(x)}$ are independent copies of a certain decoration point measure $\mathcal{D}$.*

(iii) *$\mathcal{L}$ is then the point measure corresponding to*

$$\mathcal{L} \equiv \sum_{x \in \mathcal{P}} \sum_{y \in \mathcal{D}^{(x)}} \delta_{x+y}.$$



(Figure 1 of [ABBS13])

Notation

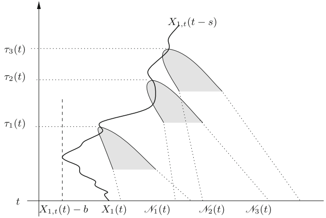
- *Order the particles in* **increasing** *order, i.e.* $x_1(t) \leq x_2(t) \leq \cdots \leq x_{n(t)}(t)$.
- *For* $s \leq t$, *let* $x_{1,t}(s)$ *denote the position at time* s *of the ancestor of* $x_1(t)$, *i.e.* $s \mapsto x_{1,t}(s)$ *is the path of the leftmost particle up until time* t.
- *Let*

$$Y_t(s) \equiv x_{1,t}(t-s) - x_1(t), \quad s \in [0,t]$$

the time reversed path back from the final position $x_1(t)$.

- *Denote by* $\cdots < \tau_2(t) < \tau_1(t) \leq t$ *the successive splitting times along the path of the leftmost particle (enumerated backwards).*
- *The time at which* $x_i(t)$ *and* $x_j(t)$ *share their MRCA is denoted by* $\tau_{i,j}(t)$.
- *Let* $\mathcal{N}_i(t) \equiv \sum_{1\leq j \leq n(t): \tau_{1,j}(t) = \tau_i(t)} \delta_{x_j(t) - x_1(t)}$.
- *Finally, let for* $0 < \eta < t$,

$$\mathcal{D}(t,\eta) \equiv \delta_0 + \sum_{i: \tau_i(t) > t-\eta} \mathcal{N}_i(t).$$



Theorem (Theorem 2.3, [ABBS13])

The following convergence holds jointly in distribution:

$$\lim_{\eta\to\infty}\lim_{t\to\infty}\big((Y_t(s), s\in[0,t]),\ \mathcal{D}(t,\eta),\ x_1(t)-m(t)\big) = \big((Y(s), s\geq 0),\ \mathcal{D},\ W\big),$$

where the r.v. W is independent of the pair $((Y(s), s\geq 0), \mathcal{D})$, and $\mathcal{D}$ is the point measure which appears in Theorem 2.1.

Note. $\mathbb{P}(W(x)\leq x) = 1 - w(-x/\sqrt{2}) \sim C'|x|e^x$ for $x\to-\infty$ (cf. (5) and below).

Construction of the decoration point measure $\mathcal{D}$

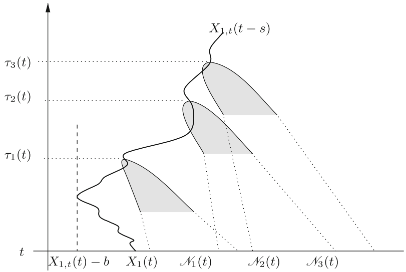
We will construct $\mathcal{D}$ conditional on Y.

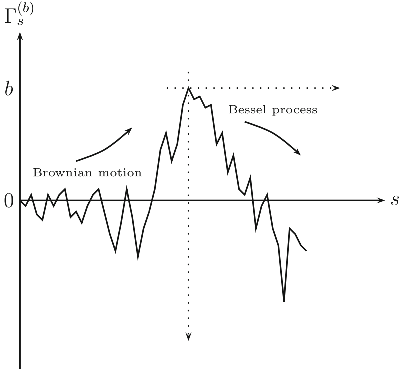
Notation

Let $b > 0$, $(B_t, t \geq 0)$ *a BM and* $(R_t, t \geq 0)$ *a three-dimensional Bessel process started from* $R_0 = 0$ *and independent of* B. *Let* $T_b \equiv \inf\{t \geq 0 : B_t = b\}$. *Set*

$$\Gamma_s^{(b)} \equiv \begin{cases} B_s, & s \in [0, T_b], \\ b - R_{s-T_b}, & s \geq T_b. \end{cases}.$$

(Figure 1 and 2 of [ABBS13])





(1) *Construction of Y.*

For $A \subset \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ measurable,

$$\mathbb{P}(Y \in A, -\inf_{s \geq 0} Y(s) \in db) = \frac{1}{c} \mathbb{E}\left[e^{-2\int_0^\infty \mathbb{P}\left(x_1(v) \leq \sqrt{2}\Gamma_v^{(b)}\right) dv} 1_{\{-\sqrt{2}\Gamma^{(b)} \in A\}} \right]$$

with normalizing constant c.

(2) *Construction of $\mathcal{D}$ conditional on Y.*

Conditionally on the path Y, let π be a PPP on $[0, \infty)$ with intensity $2 \cdot \mathbb{P}(Y(\tau) + x_1(\tau) > 0) d\tau$. For each point $\tau \in \pi$ start an independent BBM $(\mathcal{N}^*_{Y(\tau)}(u), u \geq 0)$ at position $Y(\tau)$ **conditioned** to have $\min \mathcal{N}^*_{Y(\tau)}(\tau) > 0$. Then

$$\mathcal{D} \equiv \delta_0 + \sum_{\tau \in \pi} \mathcal{N}^*_{Y(\tau)}(\tau).$$

(Recall that $Y(0) = 0$ and that the path Y moves backwards in time, whereas the BBMs move forward in time.)

Spinal decomposition

The process $\mathcal{M}(t) \equiv \sum_{k=1}^{n(t)} e^{-x_k(t)}$ is a martingale with $\mathbb{E}[M_t] = 1$ for all $t \geq 0$, the so-called additive martingale.

Let $\mathbb{Q}$ be the probability measure s.t.

$$\mathbb{Q}|_{\mathcal{F}_t} = \mathcal{M}(t) \cdot \mathbb{P}|_{\mathcal{F}_t},$$

where $\mathbb{P}$ refers to the distribution of BBM and $\mathcal{F}_t$ is the filtration of the BBM (under $\mathbb{P}$) up to time t.

Theorem (Theorem 5 of Chauvin and Rouault [CR88])

$\mathbb{Q}$ *is the law of the following branching diffusion.*

(1) *Let* $\Xi_s \in \{1, \ldots, n(s)\}$ *denote the label of a distinguished particle at time* $s \geq 0$ *with*

$$\mathbb{Q}(\Xi_t = i | \mathcal{F}_t) = \frac{e^{-x_i(t)}}{\mathcal{M}_t}.$$

The process $(\Xi_s, s \in [0, t])$ *is called the spine.*

(2) *The position of the spine* $(x_{\Xi_s}(s), s \in [0, t])$ *is a* **driftless** *BM of variance* 2.

(3) *The particle with label* Ξ_s *at time* s *branches at* **(accelerated) rate** 2 *and gives birth to BBMs (with distribution* $\mathbb{P}$*).*

Lemma (Many-to-one-principle)

Let $\Psi_i, i = 1, \ldots, n(t)$ be $\mathcal{F}_t$-measurable r.v.s. Then

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{i \le n(t)} \Psi_i\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\mathcal{M}(t)} \sum_{i \le n(t)} \Psi_i\right] = \mathbb{E}_{\mathbb{Q}}[e^{x_{\Xi_t}(t)} \Psi_{\Xi_t}].$$

Example

We obtain

$$\begin{aligned}\mathbb{P}\big(\exists i \le n(t) : (x_{i,t}(s), s \in [0,t]) \in A\big) &\le \mathbb{E}\Big[\sum_{i \le n(t)} 1_{(\{x_{i,t}(s), s \in [0,t]) \in A\}}\Big] \\ &= \mathbb{E}\Big[e^{x_{\Xi_t}(t)} 1_{\{(x_{\Xi_s}, s \in [0,t]) \in A\}}\Big] \\ &= \mathbb{E}\Big[e^{\sqrt{2}B_t} 1_{\{(\sqrt{2}B_s, s \in [0,t]) \in A\}}\Big].\end{aligned}$$

(cf. A. Bovier, Lecture Notes, "Extreme values of random processes",

`https://wt.iam.uni-bonn.de/fileadmin/WT/Inhalt/people/Anton_Bovier/lecture-notes/extreme.pdf`, Lemma 1.2.1)

Let $X_1(t), \dots, X_n(t)$ be $n \in \mathbb{N}$ i.i.d. normal r.v.s. Let

$$b_n \equiv \sqrt{2\log(n)} - \frac{\log(\log(n)) + \log(4\pi)}{2\sqrt{2\log(n)}} \quad \text{and} \quad a_n \equiv \sqrt{2\log(n)}.$$

Then, for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}(\max_{k\le n} X_k(t) \le b_n + x/a_n) = e^{-e^{-x}}.$$

For a BBM, $\mathbb{E}[n(t)] = e^t$. Set $n = e^t$ and consider $B_i, i \in \mathbb{N}$ independent standard BMs. Then, for $y \equiv x/\sqrt{2}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{k\le e^t} \frac{B_k(t)}{\sqrt{t}} \le \sqrt{2t} - \frac{\log(t)}{2\sqrt{2t}} + \frac{\log(4\pi) + 2x}{2\sqrt{2t}}\right)$$

$$= \lim_{t\to\infty} \mathbb{P}\left(\max_{k\le e^t} B_k(t) \le \sqrt{2}t - \frac{1 \cdot \log(t)}{2\sqrt{2}} + O(1) + y\right) = e^{-e^{-\sqrt{2}y}}.$$

Recall, that for BBM,

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t) \quad \text{and} \quad \mathbb{P}(M(t) - m(t) \le x) \overset{t\to\infty}{\to} \mathbb{E}\left[e^{-CZe^{-\sqrt{2}x}}\right].$$

$\looparrowright_2$ Idea of proof of Theorem 2, [ABK12] - Heuristic

$1 - w(x) \sim Cxe^{-\sqrt{2}x}$ for $x \to \infty$. Suppose heuristically $w'(x) \sim C\sqrt{2}xe^{-\sqrt{2}x}$.
$Z(t) = \sum_{k=1}^{n(t)} \left(\sqrt{2}t - x_k(t)\right) e^{-\sqrt{2}\left(\sqrt{2}t - x_k(t)\right)} \to Z$ for $t \to \infty$.

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi(x)\mathcal{E}_t^{(1-r/t)}(dx)}\right] = \mathbb{E}\left[\prod_{j=1}^{n(r)} \mathbb{E}\left[e^{-\phi\left(x_j(r)-\sqrt{2}r+\bar{M}\right)}\right]\right],$$

where $\overline{M}$ has law w and $\max_{j\le n(r)}(x_j(r) - \sqrt{2}r) \to -\infty$ a.s. as $r \to \infty$.

Rewrite the above to ($\log(ab) = \log(a) + \log(b)$, $\log(x) \sim -(1-x)$ for $0 < x < 1$)

$$\mathbb{E}\left[e^{\sum_{j=1}^{n(r)} \log\left(\mathbb{E}\left[e^{-\phi\left(x_j(r)-\sqrt{2}r+\bar{M}\right)}\right]\right)}\right] \sim \mathbb{E}\left[e^{-\sum_{j=1}^{n(r)} \mathbb{E}\left[1-e^{-\phi\left(x_j(r)-\sqrt{2}r+\bar{M}\right)}\right]}\right]$$

$$\sim \mathbb{E}\left[e^{-\sum_{j=1}^{n(r)} \int\left(1-e^{-\phi(x)}\right)\mathbb{P}(\bar{M}=-x_j(r)+\sqrt{2}r+dx)}\right]$$

$$\sim \mathbb{E}\left[e^{-\int\left(1-e^{-\phi(x)}\right)\sum_{j=1}^{n(r)} C\sqrt{2}\left(-x_j(r)+\sqrt{2}r+x\right)e^{-\sqrt{2}\left(-x_j(r)+\sqrt{2}r+x\right)}dx}\right]$$

$$\overset{r\to\infty}{\to} \mathbb{E}\left[e^{-\int C\sqrt{2}\left(1-e^{-\phi(x)}\right)Ze^{-\sqrt{2}x}dx}\right]. \qquad \square$$

$\looparrowright_3$ Idea of proof of Theorem 2.1, [ABK11] - Heuristic

- Let k, ℓ be such that $d(x_k(t), x_\ell(t)) = s \in [r, t-r]$. Consider the case

  $$s \leq t/2 \text{ and to simplify calculations } s = O(t)$$

  in what follows.

  Due to entropic repulsion: w.l.o.g.

  $$x_k(t), x_\ell(t) \in m(t) + D$$

  and

  $$x_k(s)(= x_\ell(s)) < \bar{D} + E_{t,\alpha}(s) = \bar{D} + \sqrt{2}s - \frac{s}{t}\frac{3}{2\sqrt{2}}\log(t) - s^\alpha$$

  $$\sim \sqrt{2}s - s^\alpha$$

  for some fixed $0 < \alpha < 1/2$ and for all $s \in [r, t-r]$.

- For particles k and ℓ to reach $m(t) + D$, the MRCA of k and l (at time s) must itself produce a BBM that after a time-interval of length $t - s$ has *height at least*

  $$\big(m(t) - \min(D)\big) - \big(\sqrt{2}s - s^\alpha\big) \sim \sqrt{2}(t-s) + s^\alpha.$$

- *Number of possible choices for ancestor:* At time s, there are on average e^s particles. The chance of one particle (= BM) to reach $(x_k(s) =)\sqrt{2}s - s^\alpha$ is of order
$$\frac{1}{\sqrt{2\pi s}} e^{-\frac{2s^2 - 2\sqrt{2}ss^\alpha + s^{2\alpha}}{2s}} = \frac{1}{\sqrt{2\pi s}} e^{-s} e^{\sqrt{2}s^\alpha} e^{-\frac{s^{2\alpha-1}}{2}},$$
where $2\alpha - 1 < 0$. In the product (more particles at this height as if we consider BBM) at time s we have on average at most of order $e^{\sqrt{2}s^\alpha}$ choices.
- *Chance for ancestor (at time s) to have* **a** *child at height $\sqrt{2}(t-s) + s^\alpha$ (at time t):* Starting with a single particle, the probability that BBM jumps this high in the time-interval $t - s$ is (use that $t - s \geq r \gg$ and $s = O(t)$ and $\mathbb{P}(M(t) - m(t) > x) = 1 - w(x) \sim Cxe^{-\sqrt{2}x}$ for $x \to \infty$)
$$\begin{aligned}&\mathbb{P}(M(t-s) \geq \sqrt{2}(t-s) + s^\alpha) \\ &= \mathbb{P}\left(M(t-s) - m(t-s) \geq \frac{3}{2\sqrt{2}} \log(t-s) + s^\alpha\right) \\ &\sim 1 - w\left(\frac{3}{2\sqrt{2}} \log(t-s) + s^\alpha\right) \sim C(s^\alpha + \delta)e^{-\sqrt{2}(s^\alpha+\delta)} \sim e^{-\sqrt{2}s^\alpha}.\end{aligned}$$
- *Chance to have* **two** *(that split immediately):* of order $\left(e^{-\sqrt{2}s^\alpha}\right)^2$.
- *Overall chance:* of order $e^{-\sqrt{2}s^\alpha}$, so negligible.

$\looparrowright_4$ Other Representations for the extremal process of BBM, (cf. [ABK13])

Theorem 2, [ABK12]

For any $0 < q < 1$, the processes $\mathcal{E}_t^{(q)}$ converge in law to the same limit, $\mathcal{E}^0$. Also, $\lim_{r\to\infty} \lim_{t\to\infty} \mathcal{E}_t^{(1-r/t)} = \mathcal{E}^0$. Moreover, conditionally on Z, $\mathcal{E}^0 = PPP\left(CZ\sqrt{2}e^{-\sqrt{2}x}dx\right)$.

Proposition (Proposition 3.2, [ABK13])

For $\phi \in \mathcal{C}_c^+(\mathbb{R})$ and any $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int \phi(y+x)\mathcal{E}_t(dy)}\right] = \mathbb{E}\left[e^{-C(\phi)Ze^{-\sqrt{2}x}}\right]$$

where, for $v(t,y)$ the solution of F-KPP with initial condition $v(0,y) = e^{-\phi(y)}$,

$$C(\phi) = \lim_{t\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty (1 - v(t, y + \sqrt{2}t)) y e^{\sqrt{2}y} dy$$

is a strictly positive constant depending on ϕ only.

$$\mathcal{P}_Z \equiv \sum_{i\in\mathbb{N}} \delta_{p_i} \stackrel{\mathcal{D}}{=} PPP(CZ\sqrt{2}e^{-\sqrt{2}x}dx)$$

Let $\bar{\mathcal{E}}_t \equiv \sum_{k\le n(t)} \delta_{x_k(t)-\sqrt{2}t}$. **Conditionally on** $\{\max_{k\le n(t)} x_k(t) - \sqrt{2}t \ge 0\}$, the process $\bar{\mathcal{E}}_t$ converges to a point process $\bar{\mathcal{E}} = \sum_j \delta_{\xi_j}$. Now define the point process of the gaps by

$\boxed{\mathcal{D} \equiv \sum_j \delta_{\Delta_j}, \quad \Delta_j \equiv \xi_j - \max_j \xi_j.}$ Note. $\mathcal{D}$ is a point process on $(-\infty, 0]$ with an atom at 0.

Theorem 2.1 (Main Theorem), [ABK13] Let $\mathcal{P}_Z$ be as in (7) and let $\{\mathcal{D}^{(i)} : i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (8). Then $\boxed{\mathcal{E} \equiv \lim_{t\to\infty} \mathcal{E}_t \stackrel{\mathcal{D}}{=} \sum_{i,j} \delta_{p_i+\Delta_j^{(i)}}.}$

Let $(\eta_i : i \in \mathbb{N})$ be the atoms of a PPP on $(-\infty, 0)$ with intensity measure

$$\sqrt{\frac{2}{\pi}}(-x)e^{-\sqrt{2}x}dx.$$

For each $i \in \mathbb{N}$ consider independent BBMs with drift $-\sqrt{2}$, i.e. $\{x_k^{(i)}(t) - \sqrt{2}t : k \le n^{(i)}(t)\}$. The auxiliary point process is defined as

$$\Pi_t \equiv \sum_{i,k} \delta_{\frac{1}{\sqrt{2}}\log(Z)+\eta_i+x_k^{(i)}(t)-\sqrt{2}t}.$$

Theorem (Theorem 3.6 (The auxiliary point process), [ABK13])

$$\mathcal{E} \stackrel{\mathcal{D}}{=} \lim_{t\to\infty} \Pi_t.$$

$\hookrightarrow_5$ Heuristic for appearance of the Poisson point measure

(Proposition 10.1 in [ABBS13])

Fix $k \geq 1$.

- Let $\mathcal{H}_k$ be the set of particles (at position k) that hit the spatial position k first in their line of descent.

  Note: Conditionally on $\mathcal{H}_k$, the subtrees rooted at the points of $\mathcal{H}_k$ are independent BBMs started at position k and at a random time (i.e. when the particle of $\mathcal{H}_k$ hit k).

- Define $H_k \equiv \#\mathcal{H}_k$. Note: finite a.s. (use $m(t) = +\frac{3}{2}\log(t)$ and that we consider the minimum of BBM)

- Now let

$$Z_k \equiv ke^{-k} H_k.$$

$Z = \lim_{t\to\infty} Z(t) = \lim_{t\to\infty} \sum_{k=1}^{n(t)} x_k(t) e^{-x_k(t)}$.

Neveu ([N88], (5.4)) shows that

$$\lim_{k\to\infty} Z_k = Z, \text{ a.s.}$$

- Let $\mathcal{H}_{k,t} \subset \mathcal{H}_k$ be the set of all particles that hit k before time t.
- For $u \in \mathcal{H}_{k,t}$, write $x_1^u(t)$ for the minimal position at time t of the particles which are descendants of u.
  If $u \in \mathcal{H}_k \backslash \mathcal{H}_{k,t}$, let $x_1^u(t) = 0$.

Now define the point measures

$$\mathcal{P}^*_{k,t} \equiv \sum_{u \in \mathcal{H}_k} \delta_{x_1^u(t) - m(t) + \log(CZ_k)}$$

and ($Z_k \equiv ke^{-k} H_k$ and $m(t) = \frac{3}{2}\log(t)$, i.e. $m(t+c) - m(t) \to 0$ for $t \to \infty$)

$$\mathcal{P}^*_{k,\infty} \equiv \sum_{u \in \mathcal{H}_k} \delta_{k + W^{(u)} + \log(CZ_k)},$$

where, conditionally on $\mathcal{F}_{\mathcal{H}_k}$ (sigma-algebra generated by the BBM when the particles are stopped upon hitting the position k), the $W^{(u)}$ are independent copies of the r.v. W.

Proposition (Proposition 10.1 of [ABBS13])

The following convergences hold in distribution.

$$\lim_{t\to\infty} \mathcal{P}^*_{k,t} = \mathcal{P}^*_{k,\infty} \quad \textit{and} \quad \lim_{k\to\infty} (\mathcal{P}^*_{k,\infty}, Z_k) = (\mathcal{P}, Z)$$

where $\mathcal{P}$ is as in Theorem 2.1 and $\mathcal{P}$ and Z are independent.