A generating function approach to duality

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August 8, 2017
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Searching dualities

Starting from a näive identity

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{n_1!}{(n_1-k_1)!}\cdot\frac{n_2!}{(n_2-k_2)!}\cdot\frac{\lambda^{n_1}}{n_1!}\cdot\frac{\lambda^{n_2}}{n_2!}=\lambda^{k_1+k_2}\cdot e^{-\lambda}\cdot e^{-\lambda}$$

Remark: In context of *interacting particle systems*

The first two terms in the summation

$\rightsquigarrow$ Self-duality functions for two-site system of (symmetric) random walkers (IRW)

The last two terms in the summation (up to normalizing factors)

$\rightsquigarrow$ Stationary measure (product of two Poisson(λ) distributions) for IRW

"Experimental" observation

The (discrete) integral in the n-variables of (self-)duality functions w.r.t. stationary (product) measure is (a function of) λ to the power k_1+k_2

Questions

- General fact, to some extent?
  - Other processes?
  - Other (self-)duality functions?
- Useful/characterizing?

Searching dualities

First queries

Questions

- General fact, to some extent?
  - Other processes?
  - Other (self-)duality functions?
- Useful/characterizing?

Answers

- Yes
  - Yes
  - Yes
- Yes

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Setting

Conservative and *factorized* interacting particle systems

- V a discrete set (*the graph*)
- $E = \mathbb{N}$ or $E = \{0, \ldots, N\}$ (*the spin space*)
- $\Omega = E^V$ (*the state space*)
- $\{\eta(t),\ t \geq 0\}$ the continuous-time Markov process with generator

$$L^{u,v} f(\eta) = \sum_{i,j \in V} p(i,j)\ u(\eta_i)\ v(\eta_j)\ \left(f(\eta^{i,j}) - f(\eta)\right)$$

where,

- ◇ $p : V \times V \to \mathbb{R}_+$ *symmetric* and *irreducible*
- ◇ $u(0) = 0$ and $u(n) > 0$ for $n \in E \setminus \{0\}$
- ◇ $v(0) \neq 0$ (and $v(N) = 0$ if $E = \{0, \ldots, N\}$)
- ◇ $\eta^{i,j} \in \Omega$ arises from $\eta \in \Omega$ by *removing* a particle at i and *putting* it at j

Setting

Stationary product measures

Full knowledge of stationary product measures

$\exists$ a one-parameter ($\lambda > 0$) family of *stationary product measures* with *marginals*

$$\nu_\lambda(n) = \left(\prod_{m=1}^{n} \frac{v(m-1)}{u(m)} \right) \frac{\lambda^n}{Z_\lambda} \ ,$$

for all $\lambda > 0$ for which $Z_\lambda < \infty$.

Examples

- ***Independent random walkers*** **(IRW)**

$$u(n) = n \ , \quad v(n) = 1 \ , \quad \nu_\lambda = \text{Poisson}(\lambda)$$

- ***Symmetric exclusion process*** **(SEP(γ), $\gamma \in \mathbb{N}$)**

$$u(n) = n \ , \quad v(n) = \gamma - n \ , \quad \nu_\lambda = \text{Binomial}\left(\gamma, \tfrac{\lambda}{1+\lambda}\right)$$

- ***Symmetric inclusion process*** **(SIP(α), $\alpha \in \mathbb{R}_+$)**

$$u(n) = n \ , \quad v(n) = \alpha + n \ , \quad \nu_\lambda = \text{DiscreteGamma}\left(\tfrac{\lambda}{1-\lambda}, \alpha\right)$$

Duality

Two Markov processes

$$\{\eta(t),\ t \geq 0\} \quad \rightsquigarrow \quad V\ , \quad E\ , \quad \Omega = E^V\ , \quad \mathbb{P}\ , \quad \mathbb{E}\ , \quad L$$

$$\{\xi(t),\ t \geq 0\} \quad \rightsquigarrow \quad V\ , \quad \hat{E}\ , \quad \hat{\Omega} = \hat{E}^V\ , \quad \hat{\mathbb{P}}\ , \quad \hat{\mathbb{E}}\ , \quad \hat{L}$$

Definition

If $\exists$ a (bounded) function $D : \hat{\Omega} \times \Omega \to \mathbb{R}$ such that for all $\xi \in \hat{\Omega}$, $\eta \in \Omega$ and $t \geq 0$,

$$\hat{\mathbb{E}}_\xi \left[D(\xi(t), \eta) \right] = \mathbb{E}_\eta \left[D(\xi, \eta(t)) \right] \ , \tag{1}$$

then the two processes $\{\xi(t),\ t \geq 0\}$ and $\{\eta(t),\ t \geq 0\}$ are *dual* and D is called *duality function* (*self-duality* if the distributions of the two processes coincide)

Other rewritings of (1):

- At the level of *generators*: $\hat{L}_{\text{left}} D(\xi, \eta) = L_{\text{right}} D(\xi, \eta)$
- Matrix interpretation (*finite* state space): $\hat{L} D = D L^T$

Duality

Hypothesis

[F] Duality functions *factorize* over sites.

$$D(\xi,\eta) = \prod_{i \in V} d(\xi_i, \eta_i) \ .$$

We refer to $d(k,n)$ as *single-site duality functions*.

[H] *Harmonic triviality* of the dual process $\{\xi(t),\ t \geq 0\}$.

If $H : \hat{\Omega} \to \mathbb{R}$ is *bounded* and *harmonic*, i.e. such that, for all $t \geq 0$,

$$\hat{\mathbb{E}}_\xi \left[H(\xi(t)) \right] = H(\xi) \ ,$$

then $H(\xi)$ is only a function of $|\xi| = \sum_{i \in V} \xi_i$.

(The total number of particles is the *only* conserved quantity)

[D] For all $i \in V$, $d(0, \cdot) \equiv 1$.

Relation between stationary product measure and duality functions

Theorem 1 (*V finite graph*)

Assume that $\{\eta(t),\ t \geq 0\}$ *and* $\{\xi(t),\ t \geq 0\}$ *are dual processes with duality function* $D(\xi, \eta)$, *with conditions* [F], [H] *and* [D] *in place. Moreover,* ν *is a probability measure on* Ω *for which the functions* $D(\xi, \cdot)$ *are* ν*-integrable for all* $\xi \in \hat{\Omega}$ *and uniformly over* $\{\xi : |\xi| = K\}$, *for all* $K \geq 0$.

Then,

(a) ν *is a stationary product measure for* $\{\eta(t),\ t \geq 0\}$

implies

(b) *For all finite configurations* $\xi \in \hat{\Omega}$ *and for all* $i \in V$, *we have*

$$\sum_{\eta \in \Omega} D(\xi, \eta)\ \nu(\eta) = \left(\sum_{\eta \in \Omega} D(\delta_i, \eta) \nu(\eta) \right)^{|\xi|} ,$$

where δ_i *denotes the configuration with a single particle at* i *and no particles elsewhere.*

Relation between stationary product measure and duality functions

Comments and generalizations

- If the functions $D(\xi, \cdot)$ are *measure-determining*, *(a)* and *(b)* are equivalent.

(Finite-configuration dual process $\{\xi(t),\ t \geq 0\}$)

- Theorem 1 extends to
  - V *infinite* graph
    - · $p : V \times V \to \mathbb{R}$ *finite-range* and *"bounded"*
    - · $\exists$ of a *successful coupling* for finite particles of $\{\xi(t),\ t \geq 0\}$ replaces [H]
  - $\{\eta(t),\ t \geq 0\}$ *interacting diffusions*
    - · $E = \mathbb{R}, [0, \infty)$
    - · $\mathscr{L}^\beta f(\eta) = \sum_{i,j} p(i,j) \mathscr{L}^\beta_{i,j} f(\eta)$, where

$$\mathscr{L}^\beta_{i,j} f(\eta) = \left(-\beta(\eta_i - \eta_j)(\partial_i - \partial_j) + \eta_i \eta_j (\partial_i - \partial_j)^2\right) f(\eta)$$

  - *Inhomogeneous systems*, i.e. the functions u and v depend on $i \in V$

From the stationary product measure to the duality functions

(Factorized duality, [F])

- Single-site duality functions $d(k,n)$ $\rightsquigarrow$ Full duality function $D(\xi,\eta)$

(Consequence *(b)* in Theorem 1 + [D])

- If $\xi = k\ \delta_i$ and ν_λ *marginal* of stationary product measure on Ω $\rightsquigarrow$

$$\sum_{n\in E} d(k,n)\ \nu_\lambda(n) = \left(\sum_{n\in E} d(1,n)\ \nu_\lambda(n)\right)^k \tag{2}$$

(Special form of $\nu_\lambda(n)$)

- If we denote $\theta(\lambda) = \sum_{n\in E} d(1,n)\nu_\lambda(n)$ and know that $\nu_\lambda(n) = \varphi(n)\frac{\lambda^n}{n!}\frac{1}{Z_\lambda}$ $\rightsquigarrow$

$$\sum_{n\in E} d(k,n)\ \varphi(n)\ \frac{\lambda^n}{n!} = \theta(\lambda)^k\ Z_\lambda \tag{3}$$

The l.h.s. is the Taylor series w.r.t. $\lambda > 0$ at $\lambda = 0$ of the r.h.s.

The series expansion method

The k-th single-site duality function

$$d(k,n) = \frac{1}{\varphi(n)} \left(\left[\frac{d^n}{d\lambda^n} \right]_{\lambda=0} \theta(\lambda)^k Z_\lambda \right)$$

Marginals $\nu_\lambda(n) = \varphi(n) \frac{\lambda^n}{n!} \frac{1}{Z_\lambda}$ of stationary product measures in:

- ***Independent random walkers*** (IRW)

$$E = \mathbb{N} \ , \quad \varphi(n) \equiv 1 \ , \quad Z_\lambda = e^\lambda$$

- ***Symmetric exclusion process*** (SEP(γ), $\gamma \in \mathbb{N}$)

$$E = \{0, \ldots, \gamma\} \ , \quad \varphi(n) = \frac{(\gamma)!}{(\gamma - n)!} \ , \quad Z_\lambda = (1+\lambda)^\gamma$$

- ***Symmetric inclusion process*** (SIP(α), $\alpha \in \mathbb{R}_+$)

$$E = \mathbb{N} \ , \quad \varphi(n) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \ , \quad Z_\lambda = (1-\lambda)^{-\alpha}$$

Missing item? $\theta(\lambda) = \sum_n d(1,n)\ \nu_\lambda(n) \quad \rightsquigarrow \quad d(1,\cdot)$

The series expansion method

The first single-site (self-)duality function $d(1, \cdot)$

We consider now *self-duality* for the generator $L^{u,v}$ with factorized $D(\xi, \eta)$

Proposition 1

Assume [F] *and* [D]. *Then*

(i) the first single-site self-duality function must be of the following form:

$$d(1, n) = a + b\ n\ ,$$

for some $a,\ b \in \mathbb{R}$.

(ii) If $b \neq 0$, *the process with generator* $L^{u,v}$ *is either* IRW, SEP(γ) *or* SIP(α).

The series expansion method

Full list of possible single-site self-duality functions

Strategy

$d(1,n) = a + b\, n \quad \rightsquigarrow \quad \theta(\lambda) \quad \rightsquigarrow \quad d(k,n)$ for all $k \in \hat{E}$

Case $a = 0$

- ***Independent random walkers*** **(IRW)**

$$d(k,n) = b^k \frac{n!}{(n-k)!}$$

- ***Symmetric exclusion process*** **(SEP(γ), $\gamma \in \mathbb{N}$)**

$$d(k,n) = (\gamma b)^k \frac{\binom{n}{k}}{\binom{\gamma}{k}}$$

- ***Symmetric inclusion process*** **(SIP(α), $\alpha \in \mathbb{R}_+$)**

$$d(k,n) = (\alpha b)^k \frac{n!}{(n-k)!} \, \frac{\Gamma(\alpha)}{\Gamma(\alpha+n)}$$

We recover the *classical* self-duality functions

The series expansion method

Full list of possible single-site self-duality functions (2)

Case $a \neq 0$

- ***Independent random walkers*** (IRW)

$$d(k,n) = a^k \ {}_2F_0 \left[\begin{matrix} -k & -n \\ & - \end{matrix} ; -\frac{b}{a} \right]$$

- ***Symmetric exclusion process*** (SEP(γ), $\gamma \in \mathbb{N}$)

$$d(k,n) = a^k \ {}_2F_1 \left[\begin{matrix} -k & -n \\ & -\gamma \end{matrix} ; -\gamma \frac{b}{a} \right]$$

- ***Symmetric inclusion process*** (SIP(α), $\alpha \in \mathbb{R}_+$)

$$d(k,n) = a^k \ {}_2F_1 \left[\begin{matrix} -k & -n \\ & \alpha \end{matrix} ; \alpha \frac{b}{a} \right]$$

Each of these hypergeometric functions is - up to a multiplicative factor and for suitable choices a and b - an *orthogonal polynomial* of order k in the n-variables

$$\begin{aligned} \text{IRW} &\rightsquigarrow C_k(n) \ \textit{Charlier polynomial} \\ \text{SEP}(\gamma) &\rightsquigarrow K_k(n) \ \textit{Kravchuk polynomial} \\ \text{SIP}(\alpha) &\rightsquigarrow M_k(n) \ \textit{Meixner polynomial} \end{aligned}$$

Generating functions of self-duality: the *generating function approach*

From coefficients *to* series expansion $\Leftrightarrow$ *from* series expansion *to* coefficients

(Exponential) generating function

A maps $f(n_1, n_2)$ to $Af(z_1, z_2)$, $z_1, z_2 \in \mathbb{R}_+$:

$$Af(z_1, z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} f(n_1, n_2) \frac{z_1^{n_1}}{n_1!} \frac{z_2^{n_2}}{n_2!}$$

Inversely,

$$f(n_1, n_2) = \left(\left[\frac{\partial^{n_1}}{\partial {z_1}^{n_1}} \right]_{z_1=0} \left[\frac{\partial^{n_2}}{\partial {z_2}^{n_2}} \right]_{z_2=0} Af(z_1, z_2) \right)$$

Generating function $\rightsquigarrow$ *self-duality relation*

- Self-duality function $\rightsquigarrow$ (exponential) generating function of self-duality function
- Difference operator (= jump process generator) $\rightsquigarrow$ Differential operator (= diffusion process generator)

The *generating function approach*

Self-duality functions

$d(k,n)$ single-site self-duality function

- $g(k,z) = \sum_{n=0}^{\infty} d(k,n)\frac{z^n}{n!}$
- $h(v,z) = \sum_{k=0}^{\infty} g(k,z)\frac{v^k}{k!}$

first-order (resp. *second-order*) generating function of $d(k,n)$

$D(k_1,k_2;n_1,n_2) = d(k_1,n_1)d(k_2,n_2)$ self-duality function

- $G(k_1,k_2;z_1,z_2) = g(k_1,z_1)g(k_2,z_2)$
- $H(v_1,v_2;z_1,z_2) = h(v_1,z_1)h(v_2,z_2)$

first-order (resp. *second-order*) generating function of $D(k_1,k_2;n_1,n_2)$

Remark

- $AD(k_1,k_2;z_1,z_2) = G(k_1,k_2;z_1,z_2)$
- $\hat{A}G(v_1,v_2;z_1,z_2) = H(v_1,v_2;z_1,z_2)$

The *generating function approach*

Generator

$L^{\beta,\sigma} = L^{u,v}$ difference operator (and generator) with $u(n) = n$ and $v(n) = \beta + \sigma n$

$$\begin{aligned} L^{\beta,\sigma} f(n_1, n_2) &= n_1(\beta + \sigma n_2)\left(f(n_1 - 1, n_2 + 1) - f(n_1, n_2)\right) \\ &\quad + n_2(\beta + \sigma n_1)\left(f(n_1 + 1, n_2 - 1) - f(n_1, n_2)\right) \end{aligned}$$

$\mathscr{L}^{\beta,\sigma}$ differential operator (and possible deterministic/diffusion generator)

$$\mathscr{L}^{\beta,\sigma} \hat{f}(z_1, z_2) = \left(-\beta(z_1 - z_2)\left(\partial_1 - \partial_2\right) + \sigma z_1 z_2 \left(\partial_1 - \partial_2\right)^2\right) \hat{f}(z_1, z_2)$$

Remark

- Particular choices of β and σ $\rightsquigarrow$ IRW, SEP(γ) and SIP(α)

$$\begin{aligned} \beta = 1,\ \sigma = 0 &\quad\rightsquigarrow\quad L^{\beta,\sigma} = L^{\text{IRW}} \\ \beta = \gamma,\ \sigma = -1 &\quad\rightsquigarrow\quad L^{\beta,\sigma} = L^{\text{SEP}(\gamma)} \\ \beta = \alpha,\ \sigma = +1 &\quad\rightsquigarrow\quad L^{\beta,\sigma} = L^{\text{SIP}(\alpha)} \end{aligned}$$

- Diffusive scaling limits of IRW and SIP(α) $\rightsquigarrow$ *deterministic process* and BEP(α)
- $\nexists$ limit process for SEP(γ)

The *generating function approach*

Lemma 1

For $f(n_1, n_2)$,

$$AL^{\beta,\sigma} f(z_1, z_2) = \mathscr{L}^{\beta,\sigma} Af(z_1, z_2) \ ,$$

i.e. by going to the generating function, the action of the difference operator $L^{\beta,\sigma}$ *transforms into the action of the differential operator* $\mathscr{L}^{\beta,\sigma}$ *acting on the generating function.*

Theorem 2 (Equivalent formulations of (self-)duality)

The following are equivalent:

(i) $d(k, n)$ *single-site self-duality function for* $L^{\beta,\sigma}$, *i.e.*

$$L^{\beta,\sigma}_{left} D(k_1, k_2; n_1, n_2) = L^{\beta,\sigma}_{right} D(k_1, k_2; n_1, n_2)$$

(ii) $g(k, z)$ *single-site duality function between* $L^{\beta,\sigma}$ *and* $\mathscr{L}^{\beta,\sigma}$, *i.e.*

$$L^{\beta,\sigma}_{left} G(k_1, k_2; z_1, z_2) = \mathscr{L}^{\beta,\sigma}_{right} G(k_1, k_2; z_1, z_2)$$

(iii) $h(v, z)$ *single-site self-duality function for* $\mathscr{L}^{\beta,\sigma}$, *i.e.*

$$\mathscr{L}^{\beta,\sigma}_{left} H(v_1, v_2; z_1, z_2) = \mathscr{L}^{\beta,\sigma}_{right} H(v_1, v_2; z_1, z_2)$$

Application to SIP(α)

- $\beta = \alpha, \ \sigma = +1$
- $d(1, n) = a + b\ n$

	$d(k,n)$	$g(k,z)$	$h(v,z)$
Classical $(a = 0)$	$(b\alpha)^k \frac{n!}{(n-k)!} \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}$	$(b\alpha z)^k e^z \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}$	$e^z {}_0F_1\left[{- \atop \alpha} ; b\alpha v z \right]$
Orthogonal $(a \neq 0)$	$a^k {}_2F_1\left[{-k \quad -n \atop \alpha} ; \alpha \frac{b}{a} \right]$	$e^z a^k {}_1F_1\left[{-k \atop \alpha} ; -\alpha \frac{b}{a} z \right]$	$e^{z+av} {}_0F_1\left[{- \atop \alpha} ; b\alpha v z \right]$

Note: here hypothesis [D] never used $\rightsquigarrow$ $d(k,n)$ *cheap* single-site self-duality function

Cheap	$b^k k! \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)} \mathbb{1}_{\{n=k\}}$	$(bz)^k \frac{\Gamma(\alpha)}{\Gamma(\alpha+k)}$	${}_0F_1\left[{- \atop \alpha} ; b v z \right]$

Final proof of (all) dualities

Remark

- The function ${}_0F_1\left[\begin{smallmatrix}-\\ \alpha\end{smallmatrix};\tau\right]$ is *simpler* than ${}_2F_1\left[\begin{smallmatrix}-k & -n\\ \alpha\end{smallmatrix};\hat{\tau}\right]$ and ${}_1F_1\left[\begin{smallmatrix}-k\\ \alpha\end{smallmatrix};\tilde{\tau}\right]$
- All candidate self-duality functions coincide, up to a multiplicative factor depending only on the conserved quantities $v_1 + v_2$ and $z_1 + z_2$, i.e.

$$H(v_1, v_2; z_1, z_2) = w(v_1 + v_2, z_1 + z_2)\ {}_0F_1\left[\begin{matrix}-\\ \alpha\end{matrix};b\alpha v_1 z_1\right]{}_0F_1\left[\begin{matrix}-\\ \alpha\end{matrix};b\alpha v_2 z_2\right]$$

Via simple relations on the first and second derivatives of ${}_0F_1\left[\begin{smallmatrix}-\\ \alpha\end{smallmatrix};\tau\right]$

Proposition 2

For any $c \in \mathbb{R}$, the function

$$f(v, z) = {}_0F_1\left[\begin{matrix}-\\ \alpha\end{matrix};cvz\right]$$

is a single-site self-duality function for the generator $\mathscr{L}^{\alpha,+1} = \mathscr{L}^{\mathrm{BEP}(\alpha)}$

Conclusion

Generating function approach

One-to-one correspondence between *duality relations* in *discrete* and *continuous* setting

$\Longrightarrow$

(a) *characterization* of dualities *from* discrete *to* continuum

(b) *single* and *simple* computation proving all duality relations (continuous + discrete),

(including *self-duality* for the continuous process)

... other dualities?

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