Stochastic domination in space-time for the supercritical contact process

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The contact process

- Let $G = (V, E)$ be a **connected** graph with **bounded degree**.
- Let $\lambda \in (0, \infty)$.

The contact process on G with **infection parameter** λ is the IPS on $\{0,1\}^V$ with local transition rates given by

$$\eta \to \eta_x \text{ at rate } \begin{cases} 1, & \text{if } \eta(x) = 1; \\ \lambda \sum_{\{y:\ \{x,y\} \in E\}} \eta(y), & \text{if } \eta(x) = 0, \end{cases}$$

where η_x is defined by $\eta_x(y) := \eta(y)$ for $y \neq x$, and $\eta_x(x) := 1 - \eta(x)$.

Some well-known properties

- The state $\bar{0}$ where $\bar{0}(x) = 0$ for all $x \in V$ is an **absorbing** state.
- Started from the state $\bar{1}$, where $\bar{1}(x) = 1$ for all $x \in V$, the contact process converges toward a stationary measure called the **upper invariant measure**. We denote this measure by $\bar{\nu}_\lambda$.
- For G infinite: there exists $\lambda_c \in (0, \infty)$ such that

$$\begin{cases} \bar{\nu}_\lambda = \delta_{\bar{0}}, & \text{if } \lambda < \lambda_c; \\ \bar{\nu}_\lambda \text{ is non-trivial} & \text{if } \lambda > \lambda_c. \end{cases}$$

The contact process is said to be **supercritical** when $\lambda > \lambda_c$.

Stochastic domination

Associate to $\{0,1\}^V$ the partial ordering, where $\eta \leq \xi$ if $\eta(x) \leq \xi(x)$ for all $x \in V$. An event B is **increasing** if $\eta \in B$ implies that $\xi \in B$ for all $\xi \geq \eta$.

A measure μ **stochastically dominates** ν if $\mu(B) \geq \nu(B)$ for all B increasing. Equivalently, if there exists a coupling $\widehat{\mathbb{P}}$ of μ and ν such that $\widehat{\mathbb{P}}(X^\mu \geq Y^\nu) = 1$.

Theorem (Liggett and Steif (2006))

Consider the contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$. Then **there exists** $\rho \in (0,1)$ *such that $\bar{\nu}_\lambda$ stochastically dominates a Bernoulli product measure with density ρ.*

Correlation inequalities

A measure μ on $\{0,1\}^V$ is said to be:

1. **positively associated** if $\mu(B_1 \cap B_2) \geq \mu(B_1)\mu(B_2)$ for any two increasing events B_1, B_2.
2. **downward FKG** if for every finite $\Lambda \subset V$, the measure $\mu(\cdot \mid \eta \equiv 0 \text{ on } \Lambda)$ is positively associated.
3. **FKG** if for every finite $\Lambda \subset V$ and $\sigma \in \{0,1\}^V$, the measure $\mu(\cdot \mid \eta \equiv \sigma \text{ on } \Lambda)$ is positively associated.

- Liggett (1994) showed that the contact process is not FKG (at least in some regime of λ for the process on $\mathbb{Z}$).

- Van den Berg, Häggström and Kahn (2006) proved that $\bar{\nu}_\lambda$ is downward FKG.

Stochastic domination for dFKG measures

Theorem (Liggett and Steif (2006))

Let μ be a translation invariant measure on $\{0,1\}^{\mathbb{Z}}$ which is dFKG. Then the following are equivalent:

1. *μ stochastically dominates a Bernoulli product measure with density ρ.*
2. *$\mu(\eta \equiv 0 \text{ on } \{1,2,\dots,n\}) \leq (1-\rho)^n$ for all n.*
3. *For all disjoint, finite subsets Λ and Δ of $\{1,2,3,\dots\}$, we have*

$$\mu\left(\eta(0) = 1 \mid \eta \equiv 0 \text{ on } \Lambda, \eta \equiv 1 \text{ on } \Delta\right) \geq \rho.$$

- Liggett and Steif also proved a generalization of this theorem for measures on $\{0,1\}^{\mathbb{Z}^d}$, $d \geq 2$.

Stochastic domination in space-time

For $\alpha \in [0,\infty)$, let $(\xi_t)_{t\geq 0}$ be the **independent spin-flip process** on $\{0,1\}^V$ with local transition rates given by

$$\eta \to \eta_x \text{ at rate } \begin{cases} 1, & \text{if } \eta(x) = 1; \\ \alpha, & \text{if } \eta(x) = 0. \end{cases}$$

This process is **uniquely ergodic** with the Bernoulli product measure with density $\rho = \frac{\alpha}{1+\alpha}$, denoted here by μ_ρ, as invariant measure.

Main question:
Does there, for some $\alpha > 0$, exists a coupling $\widehat{\mathbb{P}}$ of (ξ_t) and (η_t) such that,

$$\widehat{\mathbb{P}}\left(\eta_t(x) \geq \xi_t(x) \text{ for all } (x,t) \in V \times [0,\infty)\right) = 1? \qquad (1)$$

If not, does (1) possibly hold on certain subsets of V?

The answer depends on the graph

Let $d\colon V \times V \to \mathbb{N}$ be the graph distance.

1. We say that the graph $G = (V, E)$ is **amenable** if

$$\inf_{U \subset V, U \text{ finite}} \frac{|\partial_e U|}{|U|} = 0.$$

2. We say that G has **subexponential growth** if, for some $o \in V$,

$$\liminf_{n\to\infty} |\{x \in V\colon d(o, x) \leq n\}|^{1/n} = 1.$$

3. We say that $\Delta \subset V$ has **positive density** if, for some $o \in V$,

$$\liminf_{n\to\infty} \frac{|\{x \in \Delta\colon d(o, x) \leq n\}|}{|\{y \in V\colon d(o, y) \leq n\}|} > 0.$$

Negative answer #1: amenable graphs

Proposition (van den Berg, B. (2017))

Consider the contact process on an **amenable graph** G *with* $\lambda > 0$*. Then* **there does not exist** *a coupling* $\widehat{\mathbb{P}}$ *of* (ξ_t) *and* (η_t) *such that,*

$$\widehat{\mathbb{P}}\left(\eta_t(x) \geq \xi_t(x) \text{ for all } (x,t) \in V \times [0,\infty)\right) = 1$$

unless $\alpha = 0$*.*

- Liggett and Steif (2006) proved this for the case $V = \mathbb{Z}$.

Negative answer # 2: graphs of subexponential growth

Proposition (van den Berg, B. (2017))

Consider the contact process on a graph of **subexponential growth** *with* $\lambda > 0$. *Let* $\Delta \subset V$ *have positive density. Then* **there does not exist** *a coupling* $\widehat{\mathbb{P}}$ *of* (ξ_t) *and* (η_t) *such that,*

$$\widehat{\mathbb{P}}\left(\eta_t(x) \geq \xi_t(x) \text{ for all } (x,t) \in \Delta \times [0,\infty)\right) = 1$$

unless $\alpha = 0$.

- note that there are amenable graphs with exponential growth and which are not covered by the above statement.

Positive answer # 1: domination at a single site

For $x \in V$, denote by $\tau^x := \inf\{t \geq 0 \colon \eta_t^x \equiv \bar{0}\}$.

Theorem (van den Berg, B. (2017))

Consider the contact process on a connected graph $G = (V, E)$ having bounded degree with $\lambda > 0$. Let $x \in V$ for which $\mathbb{P}(\tau^x = \infty) > 0$ and such that, for some $C, c > 0$,

$$\mathbb{P}(s < \tau^x < \infty) \leq Ce^{-cs}, \qquad \text{for all } s \geq 0.$$

Then **there exists** $\alpha = \alpha(\lambda) > 0$ *and a coupling $\widehat{P}$ of (η_t) and (ξ_t) initialised from $\bar{\nu}_\lambda$ and $\mu_{\alpha/(1+\alpha)}$ respectively, such that*

$$\widehat{\mathbb{P}}(\eta_t(x) \geq \xi_t(x) \text{ for all } (x, t) \in \{o\} \times [0, \infty)) = 1,$$

Positive answer # 2: domination on trees

Theorem (van den Berg, B. (2017))

Consider the contact process on T_d, $d \geq 2$, and $\lambda > \lambda_c(\mathbb{Z})$. Then **there exists $\Delta \subset V$ of positive density**, *$\alpha > 0$, and a coupling $\widehat{\mathbb{P}}$ of (ξ_t) and (η_t) initialised from $\bar{\nu}_\lambda$ and $\mu_{\alpha/(1+\alpha)}$ respectively such that*

$$\widehat{\mathbb{P}}\left(\eta_t(x) \geq \xi_t(x) \text{ for all } (x,t) \in \Delta \times [0,\infty)\right) = 1. \tag{2}$$

- the theorem also holds for many other non-amenable graphs.

- **Question:** Does the theorem hold on the entire graph T_d?

Positive answer # 3: domination on finite sets

Theorem (van den Berg, B. (2017))

Consider the contact process on a connected graph $G = (V, E)$ *having bounded degree with* $\lambda > 0$*. Let* $\Delta \subset V$ **finite** *for which, for all* $x \in \Delta$, $\mathbb{P}(\tau^x = \infty) > 0$ *and such that, for some* $C, c > 0$,

$$\mathbb{P}(s < \tau^x < \infty) \leq Ce^{-cs}, \qquad \text{for all } s \geq 0.$$

Then **there exists** $\alpha = \alpha(\lambda) > 0$ *and a coupling* $\widehat{P}$ *of* (η_t) *and* (ξ_t) *initialised from* $\bar{\nu}_\lambda$ *and* $\mu_{\alpha/(1+\alpha)}$ *respectively, such that*

$$\widehat{\mathbb{P}}(\eta_t(x) \geq \xi_t(x) \text{ for all } (x, t) \in \Delta \times \mathbb{N}) = 1,$$

An application: cone-mixing on slabs

Theorem (van den Berg, B. (2017))

The contact process on $\mathbb{Z}^d$, $d \geq 1$, with $\lambda > \lambda_c$, projected onto $\mathbb{Z}^{d-1} \times \{0\} \times \mathbb{Z}$, is **cone-mixing**. *That is, for all $\theta \in (0, \frac{1}{2}\pi)$,*

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_{<0}, B\in\mathcal{F}_t^\theta \\ \mathbb{P}(A)>0}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| = 0.$$

where $\mathcal{F}_{<0}$ is the σ-algebra generated by the (discrete-time) lower half-space and $\mathcal{F}_t^\theta$ is the σ-algebra generated by the (discrete-time) forward cone with declination proportional to θ.

- Cone mixing is an important property in the study of random walks in (dynamic) random environment.

Thank you for your attention!

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