A review of C. Foucart (2012) Generalized Fleming-Viot processes with immigration via stochastic flows of partitions

by

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An distinguished exchangeable coalescent is a Markov process $(\Pi(t) : t \geq 0)$ valued in $\mathcal{P}_\infty^0$ such that given $\Pi(s)$

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$(\Pi(t) : t \geq 0)$ is characterized by a measure μ on $\mathcal{P}_\infty^0$, called the coagulation measure, which fulfills the following conditions:

i) μ is exchangeable,

ii) $\mu(\{0_{[\infty]}\}) = 0$,

iii) for all $n \geq 0$, $\mu(\pi \in \mathcal{P}_\infty^0 : \pi|_{[n]} \neq 0_{[n]}) < \infty$.

Let μ be a coagulation measure, then there exist c_0, c_1 non-negative real numbers and a measure ν on

$$\mathcal{P}_{\mathbf{m}} := \{s = (s_0, s_1, \ldots) : s_0 \geq 0, s_1 \geq s_2 \geq \cdots \geq 0, \sum_{i \geq 0} s_i \leq 1\}$$

that satisfies the condition

$$\int_{\mathcal{P}_{\mathbf{m}}} (s_0 + \sum_{i \geq 1} s_i^2) \nu(ds) < \infty,$$

such that:

$$\mu = c_0 \sum_{i \geq 1} \delta_{K(0,i)} + c_1 \sum_{1 \leq i < j} \delta_{K(i,j)} + \int_{s \in \mathcal{P}_{\mathbf{m}}} \rho_s(\cdot) \nu(ds),$$

where $K(i,j)$ is the simple partition with doubleton $\{i,j\}$ and ρ_s denotes the law of an s-distinguished paint box.

A Poisson construction

- Introduce a Poisson random measure $\mathcal{N}$ on $\mathbb{R} \times \mathcal{P}_\infty^0$ with intensity $dt \otimes \mu$.
- For each $n \in \mathbb{N}$, let $\mathcal{N}_n$ be the image of $\mathcal{N}$ by the map $\pi \mapsto \pi|_{[n]}$.
- Denote $\{(t_1, \pi^{(1)}), (t_2, \pi^{(2)}), \ldots, (t_K, \pi^{(K)})\}$, with $K := \mathcal{N}_n((s,t) \times (\mathcal{P}_n^0 \setminus \{0_{[n]}\}))$.

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- Define

$$\Pi^n(s,t) := \mathrm{coag}^K(\pi^{(1)}, \dots, \pi^{(K)}),$$

where for all $k \geq 2$,

$$\begin{aligned}\mathrm{coag}^k(\pi^{(1)}, \dots, \pi^{(k)}) &:= \mathrm{coag}(\mathrm{coag}^{k-1}(\pi^{(1)}, \dots, \pi^{(k-1)}), \pi^{(k)}) \\ &= \mathrm{coag}^{k-1}(\pi^{(1)}, \dots, \pi^{(k-2)}, \mathrm{coag}(\pi^{(k-1)}, \pi^{(k)})\end{aligned}$$

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$(\Pi(t) : t \geq 0) := (\Pi(0,t) : t \geq 0)$ is a distinguished exchangeable coalescent.

Definition

A flow of distinguished partitions is a collection of random variables $(\Pi(s,t), -\infty < s \leq t < \infty)$ valued in $\mathcal{P}_\infty^0$ such that:

i) For every $t \leq t'$, the distinguished partition $\Pi(t,t')$ is exchangeable with law depending only on $t'-t$.

ii) For evey $t < t' < t''$, $\Pi(t,t'') = \mathrm{coag}(\Pi(t,t'), \Pi(t',t''))$ almost surely,

iii) if $t_1' < t_2' < \cdots < t_n'$, the distinguished partitions $\Pi(t_1', t_2'), \ldots, \Pi(t_{n-1}', t_n')$ are independent.

iv) $\Pi(0,0) = 0_{[\infty]}$ and $\Pi(t,t') \to 0_{[\infty]}$ in probability when $t'-t \to 0$.

The dual flow is the process $(\hat{\Pi}(s,t) : -\infty < s \le t < \infty)$, defined by

$$\hat{\Pi}(s,t) = \Pi(-t,-s), \qquad \text{for all } s \le t.$$

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For any fixed $\pi \in \mathcal{P}_\infty^0$, define the map α_π by

$$\alpha_\pi(k) := \text{the index of the block of } \pi \text{ containing } k.$$

- The ancestor living at time s of any individual l at time t is given by $\alpha_{\hat{\Pi}(s,t)}(l)$.
- The process $(\hat{\Pi}(t) : t \ge 0) := (\hat{\Pi}(0,t) : t \ge 0)$ satisfies for all $0 \le s \le t$,

$$\hat{\Pi}(t) = \mathrm{coag}(\hat{\Pi}(s,t), \hat{\Pi}(s)).$$

  Thus, using the definition of the operator coag

$$\alpha_{\hat{\Pi}(t)}(l) = \alpha_{\hat{\Pi}(s)} \circ \alpha_{\hat{\Pi}(s,t)}(l).$$

Associate initially to each individual a type $U_i \in (0,1)$, by convention $U_0 = 0$ of the type of the immigrants.

- For any $k \in \mathbb{N}$ the type of the individual k at time t is $U_{\alpha_{\hat{\Pi}(t)}(k)}$.

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Lemma

Let $(U_i : i \geq 1)$ be an infinite exchangeable sequence taking values in $(0,1)$, with de Finetti measure ρ and fix $U_0 = 0$. Let π be an independent exchangeable distinguished partitition. Then the infinite sequence $(U_{\alpha_\pi(l)} : l \geq 1)$ is exchangeable. Furthermore, its de Finetti measure is

$$|\pi_0(t)|\delta_0 + \sum_{i=1}^{\infty} |\pi_i(t)|\delta_{U_i} + \left(1 - \sum_{i=0}^{\infty} |\pi_i(t)|\right)\rho$$

Definition

The process $(Z_t : t \geq 0)$ defined by

$$Z_t := |\hat{\Pi}_0(t)|\delta_0 + \sum_{i=1}^{\infty} |\hat{\Pi}_i(t)|\delta_{U_i} + (1 - \sum_{i=0}^{\infty} |\hat{\Pi}_i(t)|)\rho$$

starting from $Z_0 = \rho$, is called the generalized Fleming-Viot process with immigration.

Let f be a continuous function on $[0,1]^p$. Define a function from $\mathcal{M}_1 \times \mathcal{P}_p^0$ to $\mathbb{R}$ by

$$\Phi_f : (\rho, \pi) \in \times \mathcal{P}_p^0 \mapsto \int_{[0,1]^{p+1}} \delta_0(dx_0)\rho(dx_1)\dots\rho(dx_p) f(x_{\alpha_\pi(1)}, \dots, x_{\alpha_\pi(p)})$$

Lemma

Let $(\Pi(t) : t \geq 0)$ be a distinguished coalescent, then we have

$$\mathbb{E}^\rho[\Phi(Z_t, \pi)] = \mathbb{E}^\pi[\Phi(\rho, \Pi_{|[p]}(t))].$$

Let G_f be the map defined by

$$G_f(\rho) = \int_{[0,1]^p} f(x_1, \dots, x_p)\rho(dx_1)\dots\rho(dx_p).$$

Theorem

The law of the process $(Z_t : t \geq 0)$ is characterized by the following martingale problem. For every $p \geq 1$ and every continuous function f on $[0,1]^p$, the process

$$G_F(Z_t) - \int_0^t \mathcal{L} G_f(Z_s) ds$$

is a martingale in $(\hat{\mathcal{F}}_t : t \geq 0)$, the filtration of the dual flow.

Theorem

For every integer $p \geq 1$ and every continuous function f on $[0,1]^p$, the infinitesimal generator $\mathcal{L}$ of $(Z_t : t \geq 0)$ has the following property

$$\mathcal{L} G_f = \mathcal{L}^{c_0} G_f + \mathcal{L}^{c_1} G_f + \mathcal{L}^{\nu} G_f .$$