Convergence of branching-coalescing nonsimple random walks to the Brownian net

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Outline

1. Heuristics
2. Space of the Brownian web and net
3. The Brownian web $\mathcal{W}$
4. The Bronwian net $\mathcal{N}$
   - Lower bound
   - Upper bound
   - Tightness

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   - Lower bound
   - Upper bound
   - Tightness

Goal

Prove the weak convergence of branching-coalescing nonsimple random walks, and thus verify the universality class of the Brownian net.

Heuristics

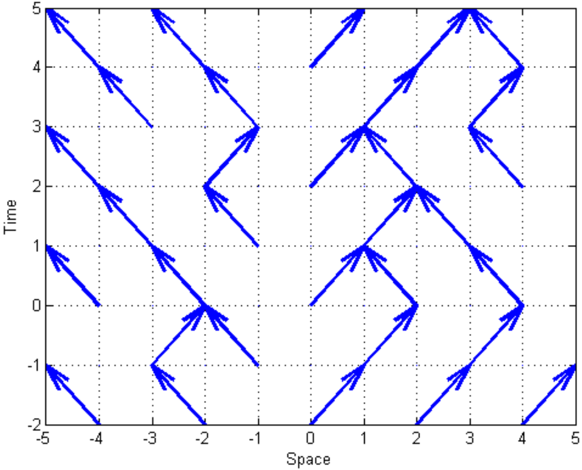


Figure: Simple arrows on $\mathbb{Z}^2_{\text{even}} := \{(x,t) \in \mathbb{Z}^2 : x + t \text{ is even }\}$

Heuristics

π : a path constructed from the arrows.
W : the collection of all coalescing paths.

Weak convergence (Fontes-Isopi-Newman-Ravishankar 04)

For the diffusively rescaled sequences (π_n) and (W_n),

$$\pi_n \overset{d}{\Longrightarrow} B,$$
$$W_n \overset{d}{\Longrightarrow} \mathcal{W}.$$

- $\mathcal{W}$: the Brownian web, which is a collection of coalescing Brownian motions.

Outline

Three spaces

Space-time plane

R_c^2 : completion of the space-time plane $\mathbb{R}^2$ w.r.t. the metric ρ

$$\rho\big((x_1, t_1), (x_2, t_2)\big) := \big|\tanh(t_1) - \tanh(t_2)\big| \vee \left|\frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|}\right|,$$ where $(x_1, t_1), (x_2, t_2) \in \mathbb{R}^2$.

- (R_c^2, ρ) is a compact space.
- The choice of the metric ρ is flexible.

Three spaces

A path π is a continuous function $\pi : [\sigma_\pi, \infty] \to [-\infty, \infty]$ with specified starting time σ_π. Then $(\pi(t), t)_{t \geq \sigma_\pi}$ is a subset of (R_c^2, ρ).

Space of paths in R_c^2

Π : the space of all paths in R_c^2 endowed with the metric d

$$d(\pi_1, \pi_2) := |\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \vee \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left| \frac{\tanh(\pi_1(t \vee \sigma_{\pi_1}))}{1+|t|} - \frac{\tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1+|t|} \right| .$$

- (Π, d) is a complete separable metric space (Polish space).

Three spaces

Space of compact subsets of (Π, d) (collection of paths)

$\mathcal{K}(\Pi)$: the space of compact subsets of (Π, d) endowed with the induced Hausdorff metric d_{H}

$$d_{\mathrm{H}}(K_1, K_2) = \sup_{\pi_1 \in K_1} \inf_{\pi_2 \in K_2} d(\pi_1, \pi_2) \vee \sup_{\pi_2 \in K_2} \inf_{\pi_1 \in K_1} d(\pi_1, \pi_2).$$

- $(\mathcal{K}(\Pi), d_{\mathrm{H}})$ is a complete separable metric space (Polish space).
- We study random variables (including the Brownian web and net) taking values in $(\mathcal{K}(\Pi), d_{\mathrm{H}})$.

Outline

Duality

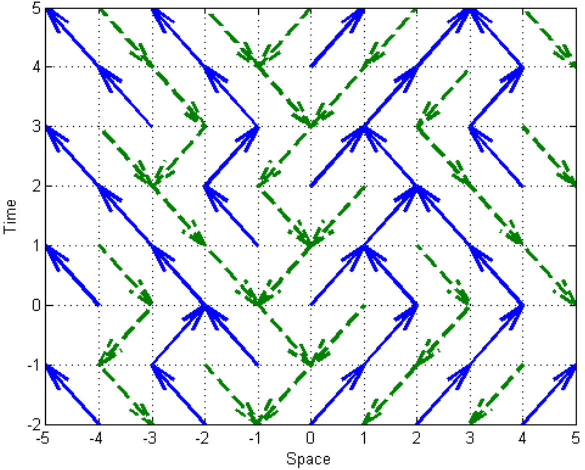


Figure: Discrete web W and its dual $\hat{W}$.

Classification of space-time poitns

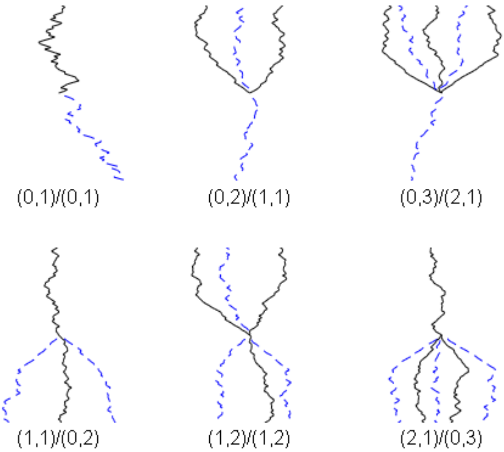


Figure: Point classification of the Brownian web

Outline

Set up

Discrete net

Let a be a probability distribution and $\varepsilon \in [0,1]$ be a constant. Define the distribution of arrows $a^{(2)}(x_1, x_2)$ as

$$a^{(2)}(x_1, x_2) := (1-\varepsilon)1_{\{x_1=x_2\}}a(x_1) + \varepsilon a(x_1)a(x_2).$$

Sample arrows independently.

- *Discrete net* N_ε : the collection of all paths constructed from the arrows together with all trivial paths.
- *Diffusively rescaled net* $\widetilde{N}_\varepsilon$: rescale the space by ε and time by $\sigma^2\varepsilon^2$, where σ^2 is the second moment of a.

Simple symmetric random walks

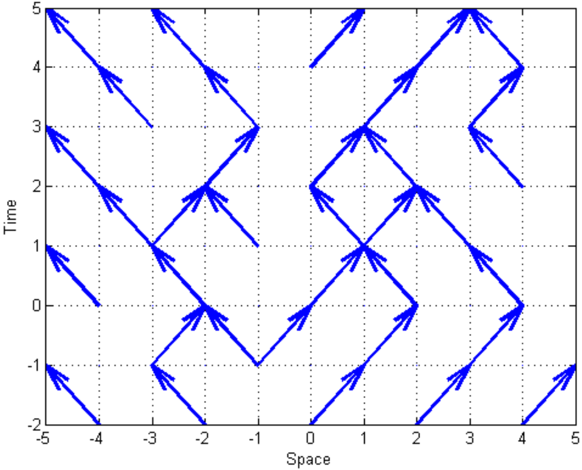


Figure: Simple discrete net

Simple symmetric random walks

Theorem (Sun-Swart 08)

$$\widetilde{N}_\varepsilon \overset{d}{\Longrightarrow} \mathcal{N}, \qquad \varepsilon \downarrow 0.$$

Proof sketch

- Tightness of $(\widetilde{N}_\varepsilon)_{\varepsilon>0}$.

  For any subsequential weak limit $\mathcal{N}^*$ of $(\widetilde{N}_\varepsilon)_{\varepsilon>0}$,
- Lower bound: $\mathcal{N}_{\text{hop}} \subset \mathcal{N}^*$.
- Upper bound: $\mathcal{N}^* \subset \mathcal{N}_{\text{wedge}}$.
- $\mathcal{N}_{\text{wedge}} \subset \mathcal{N}_{\text{hop}}$.

Lower bound: the left-right Brownian web $(\mathcal{W}^l, \mathcal{W}^r)$

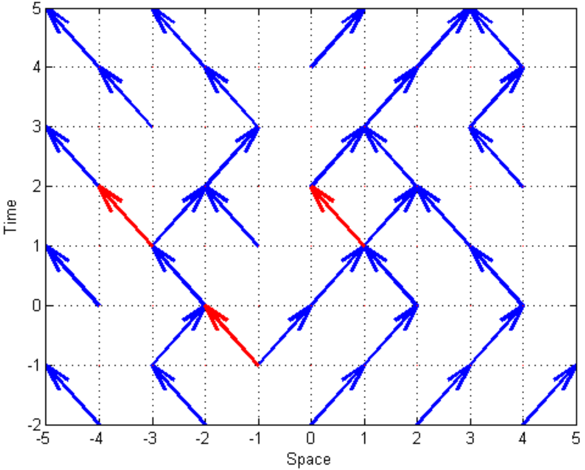


Figure: Rightmost discrete web

Lower bound: the left-right Brownian web $(\mathcal{W}^l, \mathcal{W}^r)$

- a is symmetric and simple, that is, $a(-1) = a(1) = 1/2$.
- Recall the distribution of arrows

$$a^{(2)}(x_1, x_2) := (1-\varepsilon)1_{\{x_1 = x_2\}} a(x_1) + \varepsilon a(x_1) a(x_2).$$

The drift β_+ of the rightmost discrete web W^r_ε is

$$\beta_+ = \varepsilon\big(1 \times 3/4 + (-1) \times 1/4\big) = \varepsilon/2.$$

Therefore as $\varepsilon \downarrow 0$, $\widetilde{W}^r_\varepsilon \xrightarrow{d} \mathcal{W}^r$, the rightmost Brownian web with drift $1/2$.
- Similarly, there exist the leftmost Brownian web $\mathcal{W}^l$ with drift $-1/2$ and the dual $(\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r)$.

Lower bound: the left-right Brownian web $(\mathcal{W}^l, \mathcal{W}^r)$

Construction of the Bronwian net

- The joint distribution of $(\mathcal{W}^l, \mathcal{W}^r)$ is uniquely determined by the following conditions. Each is a Brownian web, and their interaction is characterized by a set of well-posed SDEs

$$
\begin{aligned}
dl_t &= 1_{\{l_t \neq r_t\}} dB_t^1 + 1_{\{l_t = r_t\}} dB_t^3 - 1/2dt, \\
dr_t &= 1_{\{l_t \neq r_t\}} dB_t^2 + 1_{\{l_t = r_t\}} dB_t^3 + 1/2dt,
\end{aligned}
$$

with the constraint $l_t \leq r_t$ for all $t \geq T_0 := \inf\{t \geq 0 : l_t \leq r_t\}$.

Lower bound: the left-right Brownian web $(\mathcal{W}^l, \mathcal{W}^r)$

Construction of the Bronwian net

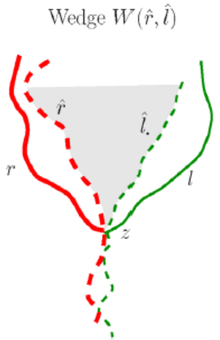
- $\mathcal{N}_{\text{hop}} := \overline{\mathcal{H}_{\text{hop}}(\mathcal{W}^l \cup \mathcal{W}^r)}$ is called the Brownian net, where $\mathcal{H}_{\text{hop}}(\mathcal{A})$ is the set of all paths $\pi \in \Pi$ of the form

$$\pi = \bigcup_{k=1}^{m} \big\{(\pi_k(s), s) : s \in [t_{k-1}, t_k]\big\},$$

where $\{\pi_k\} \subset \mathcal{A}$ and (t_k) are intersection times.

- Lower bound: $\mathcal{N}_{\text{hop}} \subset \mathcal{N}^*$.

Upper bound: wedge construction $\mathcal{N}_{\text{wedge}}$



- $W(\hat{r}, \hat{l}) := \{(y, s) : \hat{r} < y < \hat{l}, T < s < t\}$,
  where $t := \hat{\sigma}_{\hat{r}} \wedge \hat{\sigma}_{\hat{l}}$ and $z = (\hat{r}(T), T))$.
- $\mathcal{N}_{\text{wedge}} := \{\pi \in \mathcal{K}(\Pi) : \pi$ does not enter any wedge of $(\hat{\mathcal{W}}^l, \hat{\mathcal{W}}^r)$ from outside$\}$.

Upper bound: wedge construction $\mathcal{N}_{\text{wedge}}$

Upper bound: $\mathcal{N}^* \subset \mathcal{N}_{\text{wedge}}$

- π enters a wedge through the bottom point z:
  Consider three Brownian motions meeting at the same point. Impossible!
- π enters from the left/right boundary $\hat{l}, \hat{r}$:
  By discrete approximation, impossible!

Upper bound: wedge construction $\mathcal{N}_{\text{wedge}}$

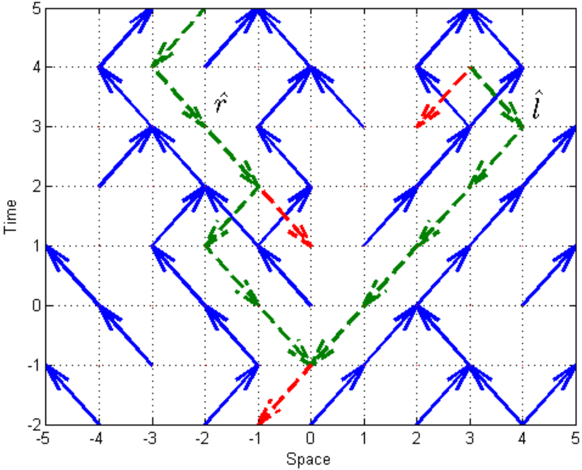


Figure: Discrete wedge

Conjecture

Let $\varepsilon_n \downarrow 0$, and let the fixed jump distribution a satisfy

$$\begin{array}{ll} \text{(i)} & a \text{ is irreducible and aperiodic,} \\ \text{(ii)} & \sum_{x\in\mathbb{Z}} xa(x) = 0, \\ \text{(iii)} & \sigma^2 = \sum_{x\in\mathbb{Z}} x^2 a(x), \\ \text{(iv)} & \sum_{x\in\mathbb{Z}} |x|^{3+\eta} a(x) < \infty \text{ for some } \eta > 0. \end{array}$$

If we denote $\widetilde{N}_n$ as the rescaled discrete net of N_{ε_n} by rescaling space and time ε_n and $\sigma^2\varepsilon_n^2$ respectively, then

$$\widetilde{N}_n \overset{d}{\Longrightarrow} \mathcal{N},$$

where $\mathcal{N}$ is the Brownian net with left and right speeds $\beta_{\mathrm{l}} = -1$ and $\beta_{\mathrm{r}} = +1$.

Nonsimple random walks

Idea of proof

- Tightness of $(\widetilde{N}_n)$.

  For any subsequential weak limit $\mathcal{N}^*$ of $(\widetilde{N}_n)$,
- Lower bound $\mathcal{N} \subset \mathcal{N}^*$.
- Upper bound $\mathcal{N}^* \subset \mathcal{N}$.

Lower bound

Lower bound: $\mathcal{N} \subset \mathcal{N}^*$

- Simple random walks (noncrossing paths):
  Leftmost/rightmost paths $\Rightarrow$ the left-right Brownian web.
- Nonsimple random walks (crossing paths):
  Sticky Brownian webs + marking construction of the Brownian net.

Lower bound: sticky Brownian webs + marking construction

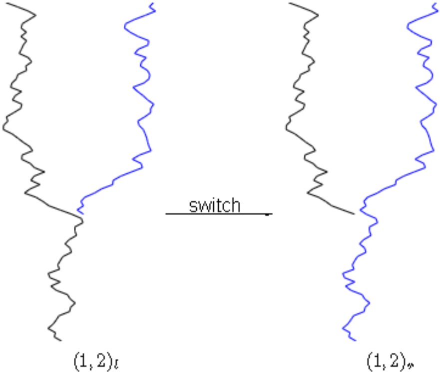


Figure: $(1,2)_l$ switched to $(1,2)_r$

Lower bound: sticky Brownian webs + marking construction

Theorem (sticky Brownian webs and the Brownian net) (Schertzer-Sun-Swart 14)

Let $\mathcal{W}$ be a Brownian web with drift β, let ℓ be the intersection local time of $\mathcal{W}$ and $\hat{\mathcal{W}}$ and let $\ell_{\rm l}$, $\ell_{\rm r}$ denote the restrictions of ℓ to the sets of points of type $(1,2)_{\rm l}$ and $(1,2)_{\rm r}$ in $\mathcal{W}$, respectively. Let $c_{\rm l}, c_{\rm r} \geq 0$ be constants and conditional on $\mathcal{W}$, let $S_{\rm l}$ and $S_{\rm r}$ be Poisson point sets with intensity $c_{\rm l}\ell_{\rm l}$ and $c_{\rm r}\ell_{\rm r}$, respectively. Then, for any sequence of finite sets $\Delta_n^{\rm l} \uparrow S_{\rm l}$ and $\Delta_n^{\rm r} \uparrow S_{\rm r}$, then

- $\mathcal{W}' := \lim_{n\to\infty} \mathrm{switch}_{\Delta_n^{\rm l}\cup\Delta_n^{\rm r}}(\mathcal{W})$ is the Brownian web with drift $\beta' := \beta + c_{\rm l} - c_{\rm r}$.
- $\mathcal{N} := \overline{\mathcal{H}_{\rm hop}(\mathcal{W}\cup\mathcal{W}')}$ is the Brownian net with left and right speeds $\beta_- = \beta - c_r$ and $\beta_+ = \beta + c_l$, respectively.

Lower bound: sticky Brownian webs + marking construction

Theorem (martingale characterization of sticky Brownian webs) (Howitt-Warren 09)

If $\mathcal{W}$ and $\mathcal{W}'$ are both Brownian webs with the same drift, and if for any pair of paths $(\pi, \pi') \in (\mathcal{W}, \mathcal{W}')$ starting at time 0,

$$\langle \pi, \pi' \rangle(t) = \int_0^t 1_{\{\pi(s)=\pi'(s)\}} ds,$$

and the following is a martingale

$$|\pi(t) - \pi'(t)| - \theta \int_0^t 1_{\{\pi(s)=\pi'(s)\}} ds,$$

then $(\mathcal{W}, \mathcal{W}')$ is a pair of sticky Brownian webs with stickiness θ. ($\theta = c_l + c_r$.)

Lower bound: sticky Brownian webs + marking construction

Theorem (Sun-Swart-Y. in progress) (Y. thesis)

$$\mathcal{N} \subset \mathcal{N}^*$$

Proof

- The distributions of arrows

$$a^{(2)}(x_1, x_2) := (1 - \varepsilon_n) 1_{\{x_1 = x_2\}} a(x_1) + \varepsilon_n a(x_1) a(x_2).$$

  provides a natural way to construct discrete webs (W_n, W_n'), which, after being diffusively rescaled, converges weakly to a pair of sticky Brownian webs $(\mathcal{W}, \mathcal{W}')$.
- Since $\mathcal{H}_{\rm hop}(W_n, W_n') \subset N_n$, we have $\mathcal{H}_{\rm hop}(\mathcal{W}, \mathcal{W}') \subset \mathcal{N}^*$. By the compactness of $\mathcal{N}^*$,

$$\mathcal{N} = \overline{\mathcal{H}_{\rm hop}(\mathcal{W}, \mathcal{W}')} \subset \mathcal{N}^*.$$

Upper bound

Upper bound: $\mathcal{N}^* \subset \mathcal{N}$

- Simple random walks (duality):
  Wedge construction $\mathcal{N}_{\text{wedge}}$.
- Nonsimple random walks:
  Density control + f.d.d. convergence + image set property + lower bound.

Upper bound: image set property

For a collection of paths $K \in \mathcal{K}(\Pi)$, its image set is

$$\cup K := \{z \in R_c^2 : \exists \pi \in K \text{ s.t. } z \in \pi\}.$$

The collection of all paths starting at t is

$$\Pi_t := \{\pi \in \mathcal{K}(\Pi) : \sigma_\pi = t\}.$$

Theorem (image set property) (Sun-Swart 08)

$$\mathcal{N} \cap \Pi_t = \{\pi \in \Pi_t : \pi \subset \cup(\mathcal{N} \cap \Pi_t)\}, \qquad t \in [-\infty, \infty].$$

To show $\mathcal{N}^* \subset \mathcal{N}$, it suffices to verify that a path $\pi \in \mathcal{N}^*$ starting at time t is a subset of the image set of the Brownian net $\mathcal{N}$.

Upper bound

Branching-coalescing point set and its counting random variable

$$\mathcal{P}_{\mathcal{N}}(t, h; a, b) := \{\pi(t+h) \cap (a, b) : \pi \in \mathcal{N}, \sigma_\pi = t\},$$
$$\eta_{\mathcal{N}}(t, h; a, b) := |\mathcal{P}_{\mathcal{N}}(t, h; a, b)|, \qquad t \in [-\infty, \infty], h > 0, a < b.$$

Finite dimensional distributions (f.d.d.)

The finite dimensional distribution of $\mathcal{N}$ is the joint distribution of all paths in $\mathcal{N}$ starting from finitely many points $(x_1, t_1), \cdots, (x_n, t_n)$.

Upper bound

Upper bound criterion

Weak convergence of f.d.d.s $+ \mathbb{E}[\eta_{\mathcal{N}^*}(t,h;a,b)] < \infty \Longrightarrow$

$$\mathcal{N}^* \subset \mathcal{N}.$$

Proof

- To show that $\mathcal{P}_{\mathcal{N}^*}(t,h;a,b)$ is contained in the image set of $\mathcal{N}$, only need to show the result for rational h, a, b.
- $\mathcal{P}_{\mathcal{N}^*}(t,\delta;a,b)$ is locally finite + f.d.d. convergence $\Longrightarrow$
  $\mathcal{P}_{\mathcal{N}^*}(t,\delta+s;a,b) \subset \mathcal{P}_{\mathcal{N}}(t+\delta,s;a,b), \qquad \forall s>0 \Longrightarrow$
  $\mathbb{E}[\eta_{\mathcal{N}^*}(t,h;a,b)] \le \mathbb{E}[\eta_{\mathcal{N}}(t,h;a,b)], \qquad (\delta \downarrow 0).$
- Combining with the lower bound $\mathcal{N} \subset \mathcal{N}^*$, we have the upper bound.

Upper bound

Theorem (Upper bound) (Sun-Swart-Y. in progress)

$$\mathcal{N}^* \subset \mathcal{N}$$

Proof

Verify the conditions of upper bound criterion for the so called Bernoulli net, which contains the branching-coalescing discrete net.

Tightness

Tightness of the discrete nets

- Simple random walks:
  Noncrossing paths + modulus of continuity of the left-right Brownian web.
- Nonsimple random walks (unsolved):
  Crossing paths!

Thank you!