Singular Yang-Mills connections on asymptotically cylindrical Kähler manifolds.

Adam Jacob
University of California at Davis

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Joint work with T. Walpuski and Henrique Sá Earp

Motivation

A G_2 manifold is a seven dimensional Riemannian manifold whose holonomy group is contained in the exceptional Lie group G_2.

- G_2 manifolds are Einstein.
- They have natural classes of calibrated submanifolds.
- G_2 manifolds play an important role in $M-$theory
- They admit natural classes of Yang-Mills bundles, called G_2 instantons. Donaldson and Thomas conjecture counting G_2 instantons can lead to a Casson-type invariant.

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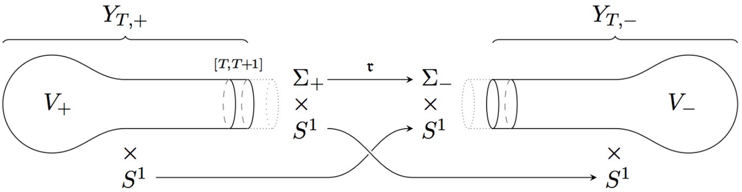
- The main motivation for this project is the construction of G_2 instantons.
- Idea is based off of the twisted connected sum construction, pioneered by Kovalev and later extended by Corti-Haskins-Nordström-Pacini.
- In short, one begins with two asymptotically cylindrical Calabi-Yau 3-folds, each equipped with a trivial S^1 bundle, and then glues the two pieces together in a prescribed fashion



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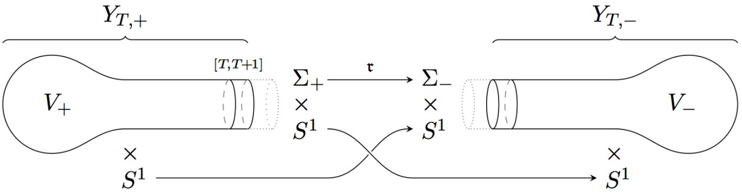

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This construction yields many millions of examples of G_2 manifolds. Can it be used to construct G_2 instantons?

The starting point for our work is the result of Sá Earp:

Theorem (Sá Earp)

Let $(E, \bar{\partial})$ be a holomorphic bundle over an asymptotically cylindrical Calabi-Yau 3-fold. If E is asymptotic to a degree zero stable bundle along the cylindrical end, then there exists a metric H on E satisfying the Hermitian Yang-Mills equations.

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From here, Sá Earp-Walpuski outlined a program and developed the perturbation theory needed to construct a G_2 instanton from the twisted connected sum.

- Exponential asymptotic decay of the connection on each building block needed.
- The perturbation theory places restrictions on the cohomology of the bundle.

As of yet no new example of G_2 instantons created by this method. However, reflexive sheaves are more abundant. Can they be used to construct singular G_2 instantons?

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Background

Let X be a complex manifold.

For a Kähler metric g on $T^{1,0}X$, we have the corresponding Kähler form

$$\omega = \frac{i}{2}\, g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k.$$

Let $(E, \bar{\partial})$ be a holomorphic vector bundle over X.

Given a Hermitian metric H on E, one can define the associated Chern connection d_H, compatible with H and the holomorphic structure $\bar{\partial}$. We write $d_H = \partial_H + \bar{\partial}$.

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The assumption that $\bar{\partial}$ is a holomorphic structure ($\bar{\partial}^2 = 0$) and metric compatibility imply the curvature F_H of d_H is a $(1,1)$ form.

- Locally, in a holomoprhic frame:

$$F_H = \bar{\partial}(H^{-1}\partial H).$$

Let Λ denote the adjoint in g of wedging with the Kähler form ω. For any $(1,1)$ form α, $i\Lambda\alpha = g^{j\bar{k}}\alpha_{j\bar{k}}$.

Definition

A metric H satisfying

$$i\Lambda F_H = \mu\, Id$$

is called a *Hermitian-Yang-Mills* metric (*Hermitian-Einstein*).

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If H solves the Hermitian-Yang-Mills equations, then d_H solves the Yang-Mills equations.

The Kähler identities imply

$$d_H^* F_H = -\partial_H(i\Lambda F_H) + \bar{\partial}(i\Lambda F_H) = 0.$$

d_H is a critical point of the Yang-Mills functional

$$YM(d_A) = ||F_A||^2_{L^2(X)}.$$

Question: When does $(E, \bar{\partial})$ admit a Hermitian-Yang-Mills metric?

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Previous results

If X is compact, solution is given by the following beautiful result:

Theorem (Donaldson, Uhlenbeck-Yau)

E admits a Hermitian-Yang-Mills metric if and only if it is stable in the sense of Mumford-Takemoto:

$$\frac{deg(\mathcal{F})}{rk(\mathcal{F})} < \frac{deg(E)}{rk(E)}$$

for all proper, reflexive subsheaves $\mathcal{F} \subset E$.

The degree is given by

$$deg(E) = i \int_X tr(F_H) \wedge \omega^{n-1},$$

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Many interesting generalizations of the above Theorem. Most notably for us, the result was extended to the case where E is a reflexive sheaf by Bando-Siu.

Here, metrics are only defined away from the singular set (of complex codimension at least 3), where E is a holomorphic bundle.

The solution satisfies

$$||i\Lambda F_H||_{L^\infty(X)} \leq C \qquad \text{and} \qquad ||F_H||_{L^2(X)} \leq C.$$

Such metrics are called admissible.

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Let H_0 be a fixed metric and H a metric along either method. Right away one sees $|i\Lambda F_H|$ is controlled.

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Question: What happens if X is complete, non-compact?

In some cases you can make the structure of X work for you.

Theorem (Ni-Ren)

If X admits a spectral gap ($\lambda_1(X) > 0$), and E admits a metric H_0 such that $|i\Lambda F_{H_0} - \mu Id| \in L^p(X)$ for some $p > 1$, then there exists a metric H such that

$$i\Lambda F_H = \mu\, Id.$$

This uses an argument similar to Donaldson's solution of the Dirichlet problem, since we have along the flow

$$\left(\frac{d}{dt} + \Delta\right) |i\Lambda F_H - \mu\, Id|^2 \leq 0.$$

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Previous results

(In this talk I use the Geometer's Laplacian, $\Delta = d^*d$ on functions)

Ni also showed that the same conclusion holds, for example, if X satisfies a L^2 Sobolev inequality and $p \in [1, \frac{n}{2})$, or if it is non-parabolic (i.e., admits a positive Greens function) and p = 1.

In this case, for a fixed initial metric H_0, one can solve

$$\Delta u = |i\Lambda F_{H_0}|,$$

and use u as a barrier to control s, since

$$\Delta \log \operatorname{tr}(e^s) \leq 4|i\Lambda F_{H_0}|.$$

For asymptotically cylindrical manifolds we have linear volume growth, so the above results can not be used.

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Main result

Building off of Sá Earp's result, we prove the following

Theorem (J. - Walpuski)

Let V be an asymptotically cylindrical Kähler manifold with asymptotic cross-section D. Let E_D be a stable vector bundle over D, and E a reflexive sheaf asymptotic to E_D. There exists an asymptotically translation-invariant Hermitian metric H on E which satisfies the projective Hermitian Yang-Mills (PHYM) equation

$$K_H := i\Lambda F_H - \frac{tr(i\Lambda F_H)}{rk(E)} Id = 0.$$

Furthermore $|F_H| \in L^2_{loc}(V)$.

Rmk: Every PHYM metric can be converted to a HYM metric via a conformal change. However, this metric will typically not be asymptotically translation invariant.

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ACyl Kähler manifolds

Definition: Let (D, g_D, I_D) be a compact Kähler manifold. A Kähler manifold (V, g, I) is called *asymptotically cylindrical* (ACyl) with asymptotic cross-section (D, I_g, I_D) if there exists a constant $\delta_V > 0$, a compact subset $K \subset V$, and a diffeomorphism $\pi : V \setminus K \to (0, \infty) \times S^1 \times D$, such that

$$|\nabla^k(\pi_* g - g_\infty)| + |\nabla^k(\pi_* I - I_\infty)| = O(e^{-\delta_V \ell})$$

for all $k \geq 0$. Here (ℓ, θ) are the canonical coordinates on $(0, \infty) \times S^1$ and

$$g_\infty := d\ell^2 \oplus d\theta^2 \oplus g_D \qquad I_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_D$$

ACyl Kähler manifolds

By a slight abuse of notation denote $\ell : V \to [0,\infty)$. Given $L > 0$, define the truncated manifold

$$V_L := \ell^{-1}([0,L]).$$

Let E be a reflexive sheaf over V. Let S be the singular set of E and *assume* $S \subset V_{L_0}$ for some L_0. Then we have the following:

Definition: Let $(E_D, \bar\partial_D)$ be a holomorphic vector bundle over D. Let $(E_\infty, \bar\partial_\infty)$ denote the pullback to $(L_0,\infty) \times S^1 \times D$. We say E is *asymptotic* to E_D if there exists a bundle isomorphism $\bar\pi : E|_{V \setminus V_{L_0}} \to E_\infty$ and a constant δ_E such that

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Linear analysis

We define the following weighted Hölder spaces. For $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\delta \in \mathbb{R}$, define:

$$C^{k,\alpha}_\delta(V) := \{f \in C^{k,\alpha}(V) \,|\, ||f||_{C^{k,\alpha}_\delta} < \infty\}$$

with

$$||\cdot||_{C^{k,\alpha}_\delta} := ||e^{\delta\ell}\cdot||_{C^{k,\alpha}}.$$

Proposition

For $0 < \delta << 1$, the linear map $C^{k+2,\alpha}_\delta(V) \oplus \mathbb{R} \to C^{k,\alpha}_\delta(V)$ defined by

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Proof

As a first step, we prove the result when E is smooth.

Construct a background Hermitian metric H_0 on E which is asymptotically translation-invariant and satisfies

$$K_{H_0} \in C^\infty_\delta(V, isu(E, H_0)).$$

Given such an H_0, we define a map

$$\mathcal{L} : C^\infty_\delta(V, isu(E, H_0)) \times [0,1] \to C^\infty_\delta(V, isu(E, H_0))$$

by

$$\mathcal{L}(s, t) : Ad(e^{s/2})K_{H_0 e^s} + t \cdot s.$$

A solution s to the equation $\mathcal{L}(s, 0) = 0$ proves the theorem.

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For estimates, it is helpful to think of the the equation $\mathcal{L}(s,t) = 0$ as

$$\left(\frac{1}{2}\nabla^*_{H_0}\nabla_{H_0} + t\right)s + B(\nabla_{H_0}s \otimes \nabla_{H_0}s) = C(K_{H_0}),$$

where B and C are linear with coefficients depending on s, but not on its derivatives.

We now follow the method of continuity. Set

$$I := \{t \in [0,1] : \mathcal{L}(s,t) = 0 \text{ for some } s\}.$$

We prove I is open, closed, and nonempty.

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- First, to show $1 \in I$, we use a trick discovered by Lübke-Teleman.
- To show I is open, we demonstrate that the Linearization of $\mathcal{L}$ is invertible.
- Key step is to show that $|s|$ is bounded uniformly, from which all other estimates follow.
- Use barriers to show that if $|s|$ is large, it must be large far down the tube, where we can take advantage of the stability assumption.

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C^0 bound

Fix L_0 large, and denote $N := ||s||_{L^\infty(V)}$ and $M := ||s||_{L^\infty(V\setminus V_{L_0})}$.

Using the equation $\mathcal{L}(s,t) = 0$, derive the inequality

$$\Delta|s|^2 \leq 4N|K_{H_0}|.$$

Let $f \in C^\infty_\delta(V)$ and $A > 0$ be the unique solution to

$$\Delta(f - A\ell) = 4|K_{H_0}|.$$

Apply the maximum principle to $|s|^2 - N(f - A\ell)$ on V_{L_0} to conclude

$$N^2 \leq M^2 + N(AL_0 + 2||f||_{L^\infty})$$

so

$$N \leq M + C(L_0 + 1).$$

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Thus if N approaches infinity so does M. As a result it suffices to bound the supremum of $|s|$ on the tubular end.

Using the barrier function, we show that if $|s|$ gets large at some point down the tube, it must be large on a portion of the tube with length proportional to M. Integrating along this portion of tube and applying bounds from stability yields the result.

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A few more details. Let $x_0 \in \overline{V \backslash V_{L_0}}$ be such that $|s|(x_0) = M$.

Apply maximum principle to

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on V_L, for $L \geq \ell(x_0)$, to see

$$M \leq C(||s||_{L^\infty(\partial V_L)} + L - \ell(x_0)).$$

Thus, for a length of tube $L - \ell(x_0) = M/2C$, we have $||s||_{L^\infty}$ is larger than $M/2C$ on transverse slices.

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On a transversal slice D_z, by a Theorem of Donaldson

$$||s||_{L^\infty(D_z)} \leq C(\mathcal{M}(H_0, H_0 e^s|_{D_z}) - 1).$$

This relays on stability

In fact, one can argue further that

$$||s||_{L^\infty(D_z)} \leq C(M \int_{D_z} |K_{H_0 e^s|_{D_z}}| - 1).$$

An energy bound shows we can control the curvature term, even when integrated along the tube of length $L - \ell(x_0)$. Thus integrating we achieve our desired bound:

$$M^2 \leq CM.$$

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Asymptotic decay

This establishes uniform C^0 control of $|s|$. By openess along the method of continuity, it follows that $|s| \in C^\infty_\delta$ for each time t, although this bound may depend on t.

We need a uniform bound of the form

$$|s| \leq Ce^{-\delta \ell}.$$

To accomplish this, we use the following inequality derived from $\mathcal{L}(s,t) = 0$ and our C^0 bound:

$$|\nabla^{H_0} s|^2 \leq C(|K_{H_0}| - \Delta |s|^2).$$

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Integrating the above inequality over V gives:

$$\int_V |\nabla^{H_0} s|^2 \le C.$$

This is not good enough to give us decay. Instead we integrate over $V \backslash V_L$ to see

$$\int_{V \backslash V_L} |\nabla^{H_0} s|^2 \le C(e^{-\delta L} + \int_{\partial V_L} |\nabla^{H_0} s||s|).$$

Because the bundle E_D is stable, it follows that ∇^{H_0} has trivial kernel on trace free endomorphisms. This yields

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As a result we conclude

$$g(L) := \int_{V\setminus V_L} |\nabla^{H_0} s|^2 \leq C(e^{-\delta L} + \int_{\partial V_L} |\nabla^{H_0} s|^2).$$

We can now follow an ODE argument.

Proposition

If $g : [0,\infty) \to [0,\infty)$ satisfies $g(L) \leq Ae^{-\delta L} - Bg'(L)$, with $A, B > 0$, then

$$g(L) \leq (2A + g(0))e^{-\epsilon L}$$

with $\epsilon := min\{\delta, \frac{1}{B}\}$.

This gives the correct decay for g, and by elliptic regularity we can bootstrap this up to exponential control of $|s|$ and all its derivatives.

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The singular case

To prove our main Theorem for reflexive sheaves E we use a regularization scheme based on ideas of Bando and Siu.

Blow up V along $S :=$sing(E)

$$\pi : \tilde{V} \to V,$$

and equip $\tilde{V}$ with a family of Kähler metrics ω_ϵ that degenerate to $\pi^*\omega$ as $\epsilon \to 0$.

$\tilde{V}$ carries a holomorphic vector bundle $\tilde{E}$, which agrees with the reflexive sheaf E outside S, and to which the smooth case can be applied to construct a PHYM metric H_ϵ.

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$$\tilde{E}|_{\tilde{V} \backslash \pi^{-1}(S)} \cong \pi^*(E|_{V \backslash S}).$$

Moreover, there exists a one-parameter family of Kähler metrics ω_ϵ on $\tilde{V}$ such that

- *on $\pi^{-1}(V \backslash B_\epsilon(S))$, we have $\omega_\epsilon = \pi^* \omega$.*
- *for $L \geq L_0$, the Neumann-Poincaré constant of $(\pi^{-1}(V_L), g_\epsilon)$ is bounded above by a constant independent of ϵ.*
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We can not solve the equation with the metric $\pi^*\omega$ on $\tilde{V}$ directly, since this metric is singular and the associated operators fail to be uniformly elliptic.

Instead, for each $\epsilon \in (0,1]$, we can construct a PHYM metric $\tilde{H}_\epsilon$ on $\tilde{E}$. Define

$$\tilde{s}_\epsilon := \log \tilde{H}_1^{-1}\tilde{H}_\epsilon.$$

The desired PHYM metric on E will be constructed by taking the limit as ϵ tends to zero.

Fix and arbitrary neighborhood U of $S \subset V$. Need estimates independent of ϵ, specifically

$$||\tilde{s}_\epsilon||_{C^k_\delta(\tilde{V}_\epsilon \setminus \tilde{U})} \leq C_{k,U}.$$

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$$K_\epsilon := i\Lambda_\epsilon F_{\tilde{H}_1} - \frac{\mathrm{tr}(i\Lambda_\epsilon F_{\tilde{H}_1})}{rk(E)} Id$$

and let $f_\epsilon \in C^0_\delta(\tilde{V}_\epsilon)$ and $A_\epsilon > 0$ b the unique solutions to

$$\Delta_\epsilon(f_\epsilon - A_\epsilon \ell) = 4|K_\epsilon|.$$

The key is to show that these barriers are independent of ϵ, i.e.

$$||f_\epsilon||_{L^\infty(\tilde{V}_\epsilon \setminus \tilde{U})} \leq c_U, \qquad A_\epsilon < c, \qquad ||F_{\tilde{H}_1}||_{L^2(\tilde{V}_\epsilon, L_0)} \leq c.$$

If so we can use the same argument as before to achieve convergence.

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Control of $||F_{\tilde{H}_1}||_{L^2(\tilde{V}_\epsilon, L_0)}$ follows by scaling

$$|F_{\tilde{H}_1}|^p_{\omega_\epsilon} vol_\epsilon \leq C(1 + \epsilon^{2n-2p})|F_{\tilde{H}_1}|^p_{\omega_1} vol_1.$$

Also note that ω_ϵ is independent of ϵ on the tubular end, so

$$||K_\epsilon||_{C^k_\delta(V \setminus V_{L_0})} \leq c_k.$$

Both of these estimates yield control of A_ϵ, since

$$A_\epsilon \leq C||K_\epsilon||_{L^1(\tilde{V}_\epsilon)}.$$

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Control of $||F_{\tilde{H}_1}||_{L^2(\tilde{V}_\epsilon, L_0)}$ follows by scaling

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Thus, all that is left is control for f_ϵ. Here we need our proposition which gave a uniform weighted Poincaré inequality

$$||e^{-\frac{\delta\ell}{2}}(f_\epsilon - \bar{f}_\epsilon)||^2_{L^2(\tilde{V}_\epsilon)} \leq C||\nabla_\epsilon f_\epsilon||^2_{L^2(\tilde{V}_\epsilon)}.$$

Using control of A_ϵ and that f_ϵ solves the Poisson equation, we have

$$||\nabla_\epsilon f_\epsilon||^2_{L^2(\tilde{V}_\epsilon)} = \int_{\tilde{V}_\epsilon} \langle \Delta_\epsilon (f_\epsilon - \bar{f}_\epsilon), f_\epsilon - \bar{f}_\epsilon \rangle$$

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Combining with the Poincaré inequality gives L^2 control of both $e^{-\frac{\delta\ell}{2}}(f_\epsilon - \bar{f}_\epsilon)$ and $\nabla_\epsilon f_\epsilon$.

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Now, working on the tube, set

$$F(L) := \int_{V \setminus V_{L_0}} |\nabla_\epsilon f_\epsilon|^2.$$

We just saw that $F(L) \leq c$.

As before, an integration by parts estimate shows

$$F(L) \leq C(e^{-\frac{\delta L}{2}} - F'(L)),$$

which implies on $V \setminus V_{L_0}$

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This completes the existence theorem. However, it says nothing about the behavior of the connection near the singular set (where E is not locally free).

In the case of constructing G_2 instantons, for the perturbation theory to work we need to know the structure of the singularity. Recall the construction uses Calabi-Yau 3-folds as building blocks.

In complex dimension three, reflexive sheaves only have point singularities.

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We consider the case of a reflexive sheaf E with an isolated singularity at the origin in $B_1(0) \subset \mathbb{C}^n$. Equip $B_1(0)$ with a metric

$$\omega = \frac{i}{2}\partial\bar{\partial}|z|^2 + O(|z|^2).$$

Let H solve $i\Lambda F_H = 0$ on $B_1(0)$, and denote by A the Chern connection.

For $\lambda_i \to 0$, let $\tau_i : B_1(0) \to B_{\lambda_i}(0)$ be defined by $z \mapsto \lambda_i z$. Set $A_i := \tau_i^* A$.

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A_i is a sequence of Yang-Mills connections, using Price monotonicity A_i converges (away from a singular set) to a limit connection A_∞, satisfying $\iota_{\frac{\partial}{\partial r}} A_\infty = 0$. (Tian)

Questions:

- Does A_∞ depend on the choice of sequence λ_i?
- Can A_∞ be identified?

In general this type of question is very hard. B. Yang demonstrated an affirmative answer to the first question assuming that $|F_A| \leq \frac{C}{r^2}$. Can we use the complex structure to our advantage?

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Let $\pi : \mathbb{C}^n \backslash \{0\} \to \mathbb{P}^{n-1}$ be the natural projection, and $\iota : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}^n$ the inclusion. Let $\Theta = \sum_{j=1}^n \frac{\bar{z}^j \, dz^j - z^j \, d\bar{z}^j}{2i \, |z|^2}$ be the pullback of the standard contact structure on S^{2n-1}.

We assume that $E \cong \iota_* \pi^* \mathcal{F}$, where $\mathcal{F} = \oplus \mathcal{F}_p$ is a direct sum of stable bundles on $\mathbb{P}^{n-1}$.

If B_p is the unique HYM connection on $\mathcal{F}_p$ with HYM constant μ_p, then

$$A_0 := \bigoplus \pi^* B_p + i \, \mu_p \, Id_{\mathcal{F}_p} \, \Theta$$

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Theorem (J. - Sá Earp-Walpuski)

In $B_1(0)$, assume $E \cong \iota_* \pi^* \mathcal{F}$, and let A be a HYM connection on E. Then there exists a unique connection A_0 satisfying

$$|z|^{k+1}|\nabla_0^k(A - A_0)| \leq \frac{C_k}{(-log|z|)^{1/2}}.$$

The constants C_k depend on $\omega, \mathcal{F}, A|_{B_R(0)\setminus B_{\frac{R}{2}}(0)}$ and $||F_A||^2_{L^2(B_R(0))}$.

We do have some examples satisfying the assumptions on $(E, \bar{\partial})$. In the future, we hope to improve our assumptions to include more general complex structures.

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Thank You!