Lelong numbers of singular metrics on vector bundles.

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It has proved useful to allow ϕ to be singular, $\phi \in L^1_{loc}$. We can still defined the curvature form (or current)

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What if h is singular? What is a singular metric? What is its curvature?

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But, what is the curvature?

Example (Raufi): Let $E = D \times \mathbb{C}^2$ and define h by $\|u\|^2 = |u_1 + zu_2|^2 + |zu_1|^2$. Then h is negatively curved, but Θ is a current of order 1. So the curvature is not measure valued. □

Let now $Y = D$, the unit disk. E is trivial and we *fix* a trivialization, $E = \mathcal{D} \times V$ where V is a vector space of dimension n. For subharmonic functions ϕ, $i\partial\bar{\partial}\phi(\{0\})$ is closely related to the Lelong number of ϕ at 0. So we look at the Lelong numbers of h.

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We also look at

$$\alpha_h(u, 0) = -\gamma_h(u, 0) = \limsup_{z \to 0} \frac{\log \|u\|_z^2}{\log(\frac{1}{|z|})} (\leq 0).$$

Let $V_\alpha := \{u \in V; \alpha(u) \leq \alpha\}$. It is a linear subsapce of V (by the triangle inequality). The family V_α is increasing and can jump at, at most, n places.

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where, if $\alpha_j \leq \alpha < \alpha_{j+1}$, $V_\alpha = V_{\alpha_j}$ and $\dim V_{\alpha_j} = j$. (We also put $\alpha_0 = -\infty$, so $V_{\alpha_0} = \{0\}$.)

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How can we describe F_α?

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Now $\log \|u\|_t^2$ is convex, and for u in V

$$\alpha(u) = \lim_{t\to\infty} \frac{1}{t} \log \|u\|_t^2.$$

If the matric comes from a negative curved metric on the disk, we have $\|u\| \leq \|u\|_0$, and $\alpha(u) \leq 0$. It is no essential difference to allow

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for some C, A.
Note also that if $\|u\|_0 = 1$, then $u \in V_\alpha$ if and only if $\|u\|_t^2 \leq e^{\alpha t}$ for all $t > 0$.

Theorem

Assume $h(t)$ *has negative curvature;* $\|u\|_t = \|u\|_{h(t)}$*. Let* $\|v\|^2_{-t}$ *be the induced norms on* $F = V^*$*. (A positively curved metric on* E^**).*

Let $\alpha_j \leq \alpha < \alpha_{j+1}$ *(so* α_j *is a jumping number). Then TFAE:*

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In particular, the set of α such that the integral in 2. is finite, is open.

During the proof we get the following statement :

Proposition

In the same setting as before ($h(t)$ negatively curved), there is a flat metric h_∞ such that $h_\infty \geq h$ and h_∞ defines the same filtration and has the same jumping numbers as h.

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In our world, we get a filtration instead. When h satisfies our extra condition (it depends only on $|z|$ or $\mathrm{Re}\,\zeta$) we can describe the dual filtration in terms of integrability.

What is this supposed to be good for?

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Positivity of direct images:

Assume also that $\mathcal{X}$ is Kähler. Then there is a vector bundle E of positive curvature over Y with fibers

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There is also a version of the theorem for non proper fibrations. Then we assume instead that $\mathcal{X}$ is Stein. The simplest case is the following:

Theorem

Let D be a pseudoconvex domain in $\mathbb{C}^n$ and let U be the unit disk. Let $\psi(t, z)$ be psh in $\mathcal{D} = U \times D$. Then the vector bundle E with fibers $E_t = H^2(D, e^{-\psi(t,z)})$ and the natural L^2-norm has positive curvature.

Here is another application (of the Stein case). It is a special case of a theorem of Guan-Zhou:

Theorem

Let $\psi \leq 0$ be psh in the unit ball of $\mathbb{C}^n$, with an isolated singularity at the origin. Assume h is holomorphic in the ball and

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Then there is an $\epsilon > 0$ such that

$$\int_B |h|^2 e^{-(1+\epsilon)\psi} < \infty.$$

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As we have seen the set of p such that the RHS is finite is open. □

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It can be framed as estimating

$$\|1\|_0^2$$

where $\|\cdot\|_0$ is the quotient norm in $F := H^2(D, e^{-\psi})/\mathcal{J}_0$, where $\mathcal{J}_0$ is the ideal of functions vanishing at the origin.

Let G be a negative psh-function in D with a logarithmic pole at 0, and let $D_t = \{z \in D; G(z) < -t\}$. Replacing D by D_t we get a scale of problems, and as shown by Blocki-Lempert one gets the good estimate by using that the corresponding family of quotient norms

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$$\|1\|_0^2 \leq \lim_{t \to \infty} \|1\|_{-t} e^{2nt},$$

which is easy to estimate.

One can also look at a more general problem, where $\mathcal{J}_0$ is replaced by another ideal, $\mathcal{J}$, at the origin. (Popovici, Demailly, Cao-Demailly-Matsumura). Then

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In general, one decomposes v as a sum in $F_\alpha \ominus F_{\alpha_{j+1}}$ in $\| \cdot \|_0$. This makes the result non explicit.

Thank you!