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Let $\mathcal{X}$ and Y be complex manifolds and assume that $\mathcal{X}$ is Kähler. Let (L, ϕ) be a hermitian holomorphic line bundle over $\mathcal{X}$, and let $p : \mathcal{X} \to Y$ be a smooth proper fibration.

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Theorem

Assume $i\partial\bar{\partial}\phi \geq 0$. Then there is a holomorphic vector bundle E over Y with fibers

$$E_y = H^{n,0}(X_y, L|_{X_y}).$$

Give E the natural L^2-metric

$$\|u\|_y^2 = c_n \int u \wedge \bar{u} e^{-\phi_L}$$

Then $(E, L^2\text{-metric})$ has positive curvature.

There is a similar result when p is non-proper and $\mathcal{X}$ is Stein, e g the natural map

$$p : \mathcal{D} \to \mathbb{C},$$

where $\mathcal{D}$ is a pseudoconvex domain in $\mathbb{C}^{n+1}$. (cf Hössjer, Maitani-Yamaguchi).

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Definition (Schumacher, Siu): Let V be a complex $(1, 0)$ vector field on $\mathcal{X}$. V is *horizontal* if

$$\Omega(V, \bar{V}') = 0$$

for any vertical field. □

If W is a field on Y, we say that V is a lift of W if $dp(V) = W$ everywhere. Lifts always exist.

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If V is any lift of a holomorphic field W on Y, we let $\kappa_V = \bar{\partial} V|_{X_y} \in \mathcal{Z}^{0,1}(X_y, T^{1,0}(X_y))$. The cohomology class of κ in $H^{0,1}(X_y, T^{1,0}(X_y))$ does not depend on the lift.

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Similarly, $[\kappa] \cup u \in H^{n-1,1}$.

Theorem

(Griffiths) Take $L = 0$. *Then*

$$\langle \Theta^{E}_{W,\bar{W}} u, u \rangle_y = \| \, [\kappa_W \cup u] \, \|^2_y .$$

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We next look at the case when L is not trivial and assume that $i\partial\bar{\partial}\phi_L > 0$ on each fiber. To simplify the writing we assume the base dimension is 1, and let t be a local coordinate on the base. Let $\Omega = i\partial\bar{\partial}\phi$.

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We then define

$$c(\phi) = \frac{1}{n+1} \frac{\Omega^{n+1}}{\Omega^n \wedge i dt \wedge d\bar{t}}.$$

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$$c(\phi) = \frac{1}{n+1} \frac{\Omega^{n+1}}{\Omega^n \wedge idt \wedge d\bar{t}} .$$

The function $c(\phi)$ has an interesting interpretation when $\mathcal{X} = X \times Y$ is a trivial fibration, and L is the pullback to $\mathcal{X}$ of a line bundle on X. Then all the fibers X_y are the same, and the line bundles $L|_{X_y}$ are also all the same. The only thing that changes is the metric $\phi = \phi_t$.

When the base is one dimensional, say a domain in $\mathbb{C}$, we can think of a metric on L over $\mathcal{X}$ as a complex curve ϕ_t of metrics on L over X. If ϕ_t only depends on the real part of t it is a real curve.

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Let $\mathcal{H}_L$ be the space of positively curved metrics on L. This *Mabuchi space* has a natural Riemannian structure:

$$\|\dot{\phi}\|^2 = \int_X |\dot{\phi}|^2 (i\partial\bar{\partial}\phi)^n/n!$$

If $c(\phi) = 0$, then ϕ_t is a geodesic in the Mabuchi space. In general, $c(\phi)$ is the geodesic curvature of the curve $t \to \phi_t$ in the Mabuchi space. (Semmes, Donaldson)

Theorem

$$\langle \Theta u, u\rangle_y = c_n \int_{X_y} c(\phi) u \wedge \bar{u} e^{-\phi} + \langle (1+\Box)^{-1} \kappa_{\partial/\partial t} \cup u, \kappa_{\partial/\partial t} \cup u\rangle_y \geq$$

$$\|[\kappa_{\partial/\partial t} \cup u]\|_y^2.$$

Example: Let $\mathcal{X} = X \times Y$ and let $L = -K_X > 0$. (So X is Fano.) Then $H^n(X, -K_X) = \mathbb{C}$ and E is a trivial bundle with a nontrivial metric. We have a trivializing section, 1. The positivity of E means that

$$i\partial\bar{\partial}_Y \log \int_X e^{-\phi} \leq 0$$

(a complex Brunn-Minkowski theorem). Let $\mathcal{E}(\phi)$ be the Aubin-Yau energy, so that

$$(d/dt)\mathcal{E}(\phi_t) = \frac{1}{V} \int_X \dot{\phi}_t (i\partial\bar{\partial}\phi_t)^n / n!.$$

Then

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It follows that the Ding functional is convex along geodesics, and the explicit formula tells us when it is linear:

Theorem

Assume that ϕ_t only depends on t, and that

$$\log \int_X e^{-\phi_t}$$

is linear. Then there is a holomorphic vector field V on X with flow F_t such that $i\partial\bar{\partial}\phi_t = F_t^*(i\partial\bar{\partial}\phi_0)$.

This can be used to prove uniqueness; the (generalized) Bando-Mabuchi theorem.

The Ding functional plays a role in the recent theorem of CDS on existence of KE-metrics.

This leads to a philosophical problem:

Donaldson's program was based on the Mabuchi K-energy, whose derivative is interpreted as a moment map, μ, for the symplectic group of X acting on an infinite dimensional manifold, $\mathcal{J}$. The KE-equation was then the equation $\mu = 0$.

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He then showed that the Ding functional can also be seen as a moment map, but for a different symplectic structure on $\mathcal{J}$:

$\mathcal{J}$ is the space of all complex structures on X that are biholomorphic with the given one and compatible with a fixed Kähler form on X, ω. This space can be indentified with the space of all diffeomorphisms, f, such that $f^*(\omega)$ is $(1,1)$ and positive for the given structure.

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If *f* lies in $\mathcal{J}$ and *g* is a symplectic map for ω, then *gf* lies in $\mathcal{J}$, so the symplectic group acts on $\mathcal{J}$. We get a map from $\mathcal{J}$ to a space of Kähler forms on *X* by $p(f) = f^*(\omega)$.

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If f lies in $\mathcal{J}$ and g is a symplectic map for ω, then gf lies in $\mathcal{J}$, so the symplectic group acts on $\mathcal{J}$. We get a map from $\mathcal{J}$ to a space of Kähler forms on X by $p(f) = f^*(\omega)$.
Any function on the Mabuchi space of Kähler forms, F, therefore induces a function on $\mathcal{J}$, $F \circ p$, which is invariant under the action of the symplectic group on $\mathcal{J}$.

This applies to both the K-energy and the Ding functional. Since these functions are convex, they become plurisubharmonic on $\mathcal{J}$ and define Kähler forms on $\mathcal{J}$.

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One can check that

$$\langle (1+\Box)^{-1}\kappa_\phi \cup 1, \kappa_\phi \cup 1 \rangle$$

is the new Kähler form found by Donaldson.

Example: The fibration is nontrivial and such that $K_{X_y} > 0$ for all y (a canonically polarized family). Take $L = K_{X/Y}$, so $L|_{X_y} = K_{X_y}$ for all y.

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Then the bundle with fibers

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is positive. When $n = 1$, this is the bundle of quadratic differentials; the cotangent space of Teichmüller space. So we get a negatively curved metric on Teichmüller space.

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When ϕ_L is a Kähler-Einstein potential, we get the classical Weil-Peterson metric (Ahlfors, Royden, Wolpert), and our curvature formula reduces to Wolpert's explict formula for the curvature of the WP-metric.

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Look at a bundle with fibers

$$H^{p,q}(X_y, L|_{X_y}),$$

where $p + q = n$.
We assume that $i\partial\bar{\partial}\phi_L > 0$ or $i\partial\bar{\partial}\phi_L < 0$ on fibers. Let $\Omega = \pm i\partial\bar{\partial}\phi_L$. (And assume the base is onedimensional.)

Theorem

If $I\partial\bar\partial\phi_L < 0$ *on fibers*

$$\langle \Theta u, u\rangle_y =$$

$$-\langle (1+\Box)^{-1}\mu_\perp, \mu_\perp\rangle_y - \langle (1+\Box)^{-1}\xi, \xi\rangle_y - \langle c(\phi)u, u\rangle_y + \|\eta_h\|_y.$$

If $i\partial\bar\partial\phi_L > 0$ *on fibers*

$$\langle \Theta u, u\rangle_y = \langle (1+\Box)^{-1}\eta, \eta\rangle_y + \langle (1+\Box)^{-1}\nu, \nu\rangle_y + \langle c(\phi)u, u\rangle_y - \|\xi_h\|_y.$$

(This formula was found independently with a different method by Ph Naumann.)

Here $\eta = \kappa \cup u$, $\xi = \bar\kappa \cup u$.

Let us focus on the negative case, $i\partial\bar{\partial}\phi_L < 0$ on fibers and take $L = -K_{X/Y}$. Look first at $H^{n,0}(X_y, -K_{X_y})$. This is a trivial line bundle as before, and $\mu = \xi = 0$. Let $u^0 = 1$ be the trivializing section. Here $\eta = \eta^0 = \kappa \cup u^0$ and the last term can be interpreted as a norm of κ.

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Then look at $H^{n-1,1}(X_y, -K_{X_y})$ which can be identified with $H^{0,1}(X_y, T^{1,0}(X_y))$, the tangent space to Teichmüller space. If the metric is given by a KE-potential our L^2-norm is the Weil-Peterson norm. In this case our formula coincides with a classical formula of Siu, which was generalized to all $H^{p,q}$ by Schumacher. It contains $u^1 := \kappa \cup u^0$, and our new η is $\kappa \cup u^1 = (\kappa\cup)^2 u^0$. Continuing in this way we eventually get $(\kappa\cup)^n u^0$, whose $n:th$ root defines a negatively curved Finsler metric (the n:th η vanishes for bidegree reasons).

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The advantage in allowing other metrics ϕ than KE-potentials comes when we allow singularities in the fibration. Then we construct an ad hoc metric, which allows to continue our metrics over singularities.

Thank you!