On the first order asymptotics of partial Bergman kernels

joint work with George Marinescu (University of Cologne)

Dan Coman

Department of Mathematics, Syracuse University
Syracuse, NY 13244-1150, USA

Conference on Complex Geometry, Dynamical Systems and Foliation Theory

Institute for Mathematical Sciences
National University of Singapore
May 15-19, 2017

Plan of the talk:

1. Singular Hermitian holomorphic line bundles
2. Exponential decay of the partial Bergman kernel near the zero locus
3. Bergman kernel of a singular metric with logarithmic poles
4. Estimates for the partial Bergman kernel

1. Singular Hermitian holomorphic line bundles

Let (X, ω) be a compact Hermitian manifold of dimension n and L be a holomorphic line bundle on X.

Then $X = \bigcup U_\alpha$, such that there exist $e_\alpha : U_\alpha \longrightarrow L$ *local holomorphic frames*, $e_\beta = g_{\alpha\beta} e_\alpha$, $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ are the *transition functions*.

$H^0(X, L)$ denotes the space of global holomorphic sections $S : X \longrightarrow L$: $S|_{U_\alpha} = s_\alpha e_\alpha$, $s_\alpha \in \mathcal{O}(U_\alpha)$, $s_\alpha = s_\beta \, g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.

Singular Hermitian metric h on L: collection $\{\varphi_\alpha \in L^1_{loc}(U_\alpha, \omega^n)\}$, with $h(e_\alpha, e_\alpha) = |e_\alpha|_h^2 = e^{-2\varphi_\alpha}$, $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ on $U_\alpha \cap U_\beta$.
If φ_α are locally bounded we say that h has *locally bounded weights*.

The *curvature current* $c_1(L, h)$ of h is defined by $c_1(L, h)|_{U_\alpha} = dd^c \varphi_\alpha$.
Note $c_1(L, h) \geq 0$ if and only if each φ_α is psh on U_α.

Suppose that h, $\widetilde{h}$ are singular Hermitian metrics on L such that *h has locally bounded weights*. Set $L^p := L^{\otimes p}$, $h_p := h^{\otimes p}$, $\widetilde{h}_p := \widetilde{h}^{\otimes p}$.

Bergman spaces of L^2-holomorphic sections:

$$
\begin{aligned}
H^0_{(2)}(X, L^p, h_p) &= \left\{ S \in H^0(X, L^p) : \ \|S\|_p^2 := \int_X |S|^2_{h_p} \frac{\omega^n}{n!} < \infty \right\}, \\
H^0_{(2)}(X, L^p, \widetilde{h}_p) &\subseteq H^0_{(2)}(X, L^p, h_p) = H^0(X, L^p).
\end{aligned}
$$

$P_p :=$ (full) Bergman kernel function of $H^0(X, L^p) = H^0_{(2)}(X, L^p, h_p)$
$\widetilde{P}_p :=$ Bergman kernel function of $H^0_{(2)}(X, L^p, \widetilde{h}_p)$

Note that $d_p := \dim H^0(X, L^p) < \infty$, since X is compact.
If $S_1, \ldots, S_{d_p}$ is an o.n. basis of $H^0(X, L^p)$ then P_p is defined by

$$
P_p(x) = \sum_{j=1}^{d_p} |S_j(x)|^2_{h_p}, \ \ x \in X.
$$

2. Exponential decay of the partial Bergman kernel near the zero locus

We consider the following setting:

(A) (X, ω) *is a compact Hermitian manifold,* $\dim X = n$, Σ *is a smooth complex hypersurface of* X, *and* $t > 0$ *is a fixed real number.*

(B) (L, h) *is a singular Hermitian holomorphic line bundle on* X *such that* h *has locally bounded weights.*

Consider the subspace of holomorphic sections of L^p vanishing to order at least $\lfloor tp \rfloor$ along Σ:

$$H^0_0(X, L^p) := H^0\big(X, L^p \otimes \mathcal{O}\big(-\lfloor tp \rfloor \Sigma\big)\big) \leqslant H^0(X, L^p),$$

$$d_{0,p} := \dim H^0_0(X, L^p) \leq d_p = \dim H^0(X, L^p).$$

Fix an o.n. basis $\{S_j^p : 1 \le j \le d_p\}$ of $H^0(X, L^p)$ such that $\{S_j^p : 1 \le j \le d_{0,p}\}$ is an o.n. basis of $H_0^0(X, L^p)$.

The *partial Bergman kernel function* $P_{0,p}$ of $H_0^0(X, L^p)$ is

$$P_{0,p}(x) = \sum_{j=1}^{d_{0,p}} |S_j^p(x)|^2_{h_p}\,,\ x \in X.$$

Then $P_{0,p}(x) \le P_p(x)$ on X and

$$P_{0,p}(x) = \max\Big\{|S(x)|^2_{h_p} :\ S \in H_0^0(X, L^p),\ \|S\|_p^2 = \int_X |S|^2_{h_p} \frac{\omega^n}{n!} = 1\Big\}.$$

Theorem 1

Assume that conditions (A)-(B) are fulfilled. Then there exist a neighborhood U_t of Σ and a constant $a_t \in (0,1)$ such that

$$P_{0,p}(x) \le a_t^p\,,\ \textit{for } x \in U_t \textit{ and } p > 2/t.$$

Let $S_\Sigma \in H^0(X, \mathcal{O}(\Sigma))$ be a canonical section vanishing to first order on Σ, and h_Σ be a smooth Hermitian metric on $\mathcal{O}(\Sigma)$ such that

$$\varrho := \log \left| S_\Sigma \right|_{h_\Sigma} < 0 \text{ on } X.$$

Fix an open cover $\mathcal{W} = \{W_j\}_{1 \leq j \leq N}$ of Σ (compact), such that W_j are Stein contractible coordinate neighborhoods centered at $y_j \in \Sigma$, and

$$\Delta^n(y_j, 2) \subset W_j \,,\ \ \Sigma \subset W := \bigcup_{j=1}^{N} \operatorname{int} \Delta^n(y_j, 1),$$

$$\Sigma \cap W_j = \left\{ z \in W_j : z_1 = 0 \right\}, \text{ for } j = 1 \ldots, N.$$

Here $z = (z_1, \ldots, z_n)$ are the coordinates on W_j and

$$\Delta^n(y, r) := \{ z \in W_j : |z_\ell - y_\ell| \leq r,\ \ell = 1, \ldots, n \}$$

is the (closed) polydisk of radius $r > 0$ centered at $y \in W_j$.

Since $L|_{W_j}$ is trivial, let e_j be a holomorphic frame of $L|_{W_j}$ and let $|e_j|_h = e^{-\varphi_j}$. Set

$$\|\varphi_j\|_{\infty,W_j} = \sup\left\{|\varphi_j(w)| : \, w \in \Delta^n(y_j, 2)\right\},$$

$$\|h\|_\infty = \|h\|_{\infty,\mathcal{W}} := \max\left\{\|\varphi_j\|_{\infty,W_j} : \, 1 \leq j \leq N\right\}.$$

Theorem 2

In the setting of Theorem 1, there exists $A = A(\rho, \mathcal{W}) \geq 1$ such that

$$P_{0,p}(x) \leq (Ae^{\rho(x)})^{2\lfloor tp \rfloor} e^{4p\|h\|_\infty}, \quad \forall\, x \in W, \; p \geq 1.$$

In particular, if $U_t := \left\{x \in W : \, (Ae^{\rho(x)})^t \, e^{4\|h\|_\infty} < 1\right\}$ *then*

$$P_{0,p}(x) \leq \left[(Ae^{\rho(x)})^t \, e^{4\|h\|_\infty}\right]^p, \quad \forall\, x \in U_t, \; p > 2/t.$$

Lemma 3

If $k \geq 0$ and $f \in \mathcal{O}(\Delta(0,2))$, where $\Delta(0,2) \subset \mathbb{C}$ is the closed disk centered at 0 and of radius 2, then

$$\int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) \leq \frac{k+1}{2^{2k}} \int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \, dm(\zeta) \, .$$

Proof. Let $f(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$ in $\Delta(0,2)$, so $\zeta^k f(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^{j+k}$. Then

$$\int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) = 2\pi \sum_{j=0}^{\infty} |a_j|^2 \int_0^2 r^{2j+1} \, dr = 2\pi \sum_{j=0}^{\infty} \frac{2^{2j+2}}{2j+2} |a_j|^2 \, ,$$

$$\int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \, dm(\zeta) = 2\pi \sum_{j=0}^{\infty} \frac{2^{2j+2+2k}}{2j+2+2k} |a_j|^2$$

$$\geq \frac{2^{2k}}{k+1} 2\pi \sum_{j=0}^{\infty} \frac{2^{2j+2}}{2j+2} |a_j|^2 = \frac{2^{2k}}{k+1} \int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) \, . \qquad \square$$

Proof of Theorem 2. Let $x \in \operatorname{int} \Delta^n(y_j, 1) \subset W$, for some $j \in \{1, \ldots, N\}$.

If $S \in H^0_0(X, L^p)$ we write $S = s e_j^{\otimes p}$, so $s(z) = z_1^{\lfloor tp \rfloor} \widetilde{s}(z)$, with $\widetilde{s} \in \mathcal{O}(W_j)$. Then by the sub-averaging inequality

$$\begin{aligned} |S(x)|^2_{h_p} &= |x_1|^{2\lfloor tp \rfloor} |\widetilde{s}(x)|^2 e^{-2p\varphi_j(x)} \\ &\leq |x_1|^{2\lfloor tp \rfloor} e^{-2p\varphi_j(x)} \frac{1}{\pi^n} \int_{\Delta^n(x,1)} |\widetilde{s}(z)|^2 \, dm(z) \\ &\leq |x_1|^{2\lfloor tp \rfloor} e^{-2p\varphi_j(x)} \int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \, dm(z) \, . \end{aligned}$$

We estimate

$$\int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \, dm(z)$$

by using Fubini's theorem for the splitting $z = (z_1, z')$ and Lemma 3 for the variable z_1.

$$
\begin{aligned}
\int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \, dm(z) &= \int_{\Delta^{n-1}(0,2)} \int_{\Delta(0,2)} |\widetilde{s}(z_1, z')|^2 \, dm(z_1) dm(z') \\
&\leq \frac{\lfloor tp \rfloor + 1}{2^{2\lfloor tp \rfloor}} \int_{\Delta^n(0,2)} |z_1|^{2\lfloor tp \rfloor} |\widetilde{s}(z)|^2 \, dm(z) \\
&\leq C \exp\left(2p \sup_{\Delta^n(0,2)} \varphi_j\right) \int_{\Delta^n(0,2)} |s(z)|^2 e^{-2p\varphi_j(z)} \, \frac{\omega^n}{n!} \, ,
\end{aligned}
$$

where $C = C(\mathcal{W}) \geq 1$ is such that $dm(z) \leq C\omega^n/n!$ on each $\Delta^n(y_j, 2)$.

Since $|x_1| \leq A' e^{\rho(x)}$ on W, with $A' = A'(\rho, W) > 1$, we conclude that

$$
\begin{aligned}
|S(x)|^2_{h_p} &\leq C \, |x_1|^{2\lfloor tp \rfloor} \exp\left(2p \sup_{\Delta^n(0,2)} \varphi_j - 2p\varphi_j(x)\right) \|S\|_p^2 \\
&\leq (Ae^{\rho(x)})^{2\lfloor tp \rfloor} e^{4p\|h\|_\infty} \|S\|_p^2 \, ,
\end{aligned}
$$

where $A := A'C = A(\rho, W) > 1$.

Since $Ae^{\rho(x)} < 1$ for $x \in U_t$, and $2\lfloor tp \rfloor > 2tp - 2 > tp$ for $p > 2/t$, we get

$$|S(x)|^2_{h_p} \leq \left[(Ae^{\rho(x)})^t e^{4\|h\|_\infty}\right]^p \|S\|^2_p . \qquad \square$$

A version of Theorem 2 in the case when X is not compact:

Theorem 4

Let (X, ω) be a Hermitian manifold of dimension n, Σ be a smooth complex hypersurface of X, $t > 0$ a fixed real number, and (L, h) a singular Hermitian holomorphic line bundle on X such that h has locally bounded weights. Then for any compact set $E \subset X$ there exists a neighborhood W of $\Sigma \cap E$ and a constant $A = A(\rho, W) \geq 1$, where $\varrho := \log \left|S_\Sigma\right|_{h_\Sigma} < 0$ on E, such that

$$P_{0,p}(x) \leq (Ae^{\rho(x)})^{2\lfloor tp \rfloor} e^{4p\|h\|_\infty} , \quad \forall\, x \in W , \; p \geq 1.$$

3. Bergman kernel of a singular metric with logarithmic poles

Recall that $\varrho := \log \left|S_\Sigma\right|_{h_\Sigma} < 0$ on X, and $t > 0$ is a fixed real number.

Suppose $\xi : X \to \mathbb{R} \cup \{-\infty\}$ is smooth on $X \setminus \Sigma$ and $\xi = t\rho$ in a neighborhood of Σ. Let $\operatorname{dist}(\cdot\,,\cdot)$ be the distance on X induced by ω.

Theorem 5

Let $(X,\omega), (L,h), \Sigma$ be as in (A)-(B), and assume ω is Kähler and h is smooth. Consider the singular metric $\widetilde{h} = he^{-2\xi}$ on L and assume that $c_1(L,\widetilde{h}) \geq \varepsilon\omega$ for some constant $\varepsilon > 0$. Let $\widetilde{P}_p$ be the Bergman kernel function of $H^0_{(2)}(X, L^p, \widetilde{h}_p)$. Then there exists a constant $C > 1$ such that for every $x \in X \setminus \Sigma$ and every $p \in \mathbb{N}$ with $p \operatorname{dist}(x,\Sigma)^{8/3} > C$ we have

$$(1) \qquad \left| \frac{\widetilde{P}_p(x)}{p^n} \, \frac{\omega_x^n}{c_1(L,\widetilde{h})_x^n} - 1 \right| \leq Cp^{-1/8}\,.$$

In the case of a *positive* line bundle (L, h) (i.e. h is smooth and $c_1(L, h)$ is a Kähler form) on a compact Kähler manifold (X, ω), $\dim X = n$, the first order asymptotics of the Bergman kernel P_p of $H^0(X, L^p, h_p, \omega^n/n!)$,

$$\left\| \frac{1}{p^n} P_p - \frac{c_1(L,h)^n}{\omega^n} \right\|_{\mathscr{C}^2(X)} \leq \frac{C}{p} \,,$$

was first proved by Tian (1990).

Later generalizations by Catlin (1999), Zelditch (1998), Dai-Liu-Ma (2004), Ma-Marinescu (2004), ...

Interpretation of Theorem 5:

1. If $x \in K$, where $K \subset X \setminus \Sigma$ is compact, we have a concrete bound $p_0 = C\,\mathrm{dist}(K,\Sigma)^{-8/3}$ such that for $p > p_0$ the estimate (1) holds. By Hsiao-Marinescu (2014) $\widetilde{P}_p$ has an asymptotic expansion

$$\widetilde{P}_p(x) = \sum_{r=0}^{\infty} \mathbf{b}_r(x) p^{n-r} + O(p^{-\infty}), \text{ locally uniformly on } X \setminus \Sigma.$$

In particular, there exist $p_0(K) \in \mathbb{N}$ and C_K such that for $p > p_0(K)$,

$$\left| \frac{\widetilde{P}_p(x)}{p^n} \, \frac{\omega_x^n}{c_1(L,\widetilde{h})_x^n} - 1 \right| \leq C_K p^{-1} \text{ on } K.$$

2. Theorem 5 gives a uniform estimate in p for $\widetilde{P}_p$ (C is independent of K) on compact sets whose distance to Σ decreases as $p^{-3/8}$. Indeed, the estimate (1) holds on $K_p := \{x \in X : \mathrm{dist}(x,\Sigma) \geq (C/p)^{3/8}\}$, for every p.

About the proof of Theorem 5. We use ideas of Berndtsson (2003) who gave a simple proof for the first order asymptotics of the Bergman kernel function in the case of powers of a positive line bundle.

Fix $x \in X \setminus \Sigma$, $0 < r_p < c\,\mathrm{dist}(x, \Sigma)$, such that

$$r_p^2 \to 0\,,\ pr_p^3 \to 0\,,\ r_p\sqrt{p} \to \infty.$$

If $S \in H^0_{(2)}(X, L^p, \widetilde{h}_p)$ by using the sub-averaging inequality one shows that

$$|S(x)|^2_{\widetilde{h}_p} \leq \frac{p^n c_1(L, \widetilde{h})^n_x}{\omega^n_x}\,\big(1 + Cr_p^2\big)\big(1 + Cpr_p^3\big)\big(1 + Ce^{-2\varepsilon pr_p^2}\big)\|S\|^2_p.$$

So

$$\frac{\widetilde{P}_p(x)}{p^n}\,\frac{\omega^n_x}{c_1(L, \widetilde{h})^n_x} \leq \big(1 + Cr_p^2\big)\big(1 + Cpr_p^3\big)\big(1 + Ce^{-2\varepsilon pr_p^2}\big).$$

By solving $\overline{\partial}$ one constructs a section $S \in H^0_{(2)}(X, L^p, \widetilde{h}_p)$ such that

$$\begin{aligned} |S(x)|^2_{\widetilde{h}_p} &\geq 1 - \frac{C}{r_p\sqrt{p}}\,, \\ \|S\|^2_p &\leq \frac{\omega_x^n}{p^n c_1(L, \widetilde{h})^n_x} \left(1 + Cr_p^2\right)\left(1 + Cpr_p^3\right)\left(1 + \frac{C}{r_p\sqrt{p}}\right). \end{aligned}$$

So

$$\frac{\widetilde{P}_p(x)}{p^n}\,\frac{\omega_x^n}{c_1(L, \widetilde{h})^n_x} \geq \left(1 - Cr_p^2\right)\left(1 - Cpr_p^3\right)\left(1 - \frac{C}{r_p\sqrt{p}}\right).$$

Choosing $r_p = p^{-3/8}$ we have $pr_p^3 = \dfrac{1}{r_p\sqrt{p}} = p^{-1/8}$, $r_p^2 = p^{-3/4}$, so

$$1 - Cp^{-1/8} \leq \frac{\widetilde{P}_p(x)}{p^n}\,\frac{\omega_x^n}{c_1(L, \widetilde{h})^n_x} \leq 1 + Cp^{-1/8},$$

provided that $p^{-3/8} < c\,\mathrm{dist}(x, \Sigma)$. $\square$

4. Estimates for the partial Bergman kernel

Let $(X, \omega), (L, h), \Sigma$ be as in (A)-(B), and assume ω is Kähler, h is smooth, and $c_1(L, h) \geq \varepsilon\omega$ for some constant $\varepsilon > 0$.

Recall that P_p denotes the (full) Bergman kernel function of $H^0(X, L^p)$.

Given a compact set $K \subset X \setminus \Sigma$ we let

$$t_0(K) := \sup \Big\{ t > 0 : \exists\, \eta \in \mathscr{C}^\infty(X, [0,1]),\ \operatorname{supp} \eta \subset X \setminus K,\ \eta = 1 \text{ near } \Sigma,\\ c_1(L, h) + t\, dd^c(\eta\varrho) \text{ is a Kähler current on } X \Big\},$$

where $\varrho := \log \big|S_\Sigma\big|_{h_\Sigma} < 0$ on X. Note ρ is a qpsh function.

We fix now a compact set $K \subset X \setminus \Sigma$ and let $t \in (0, t_0(K))$.

Using Theorems 1 and 5 we show the following asymptotics of $P_{0,p}$:

Theorem 6

In the above setting, there exist constants $C > 1$, $M > 1$ and a neighborhood U_t of Σ, all depending on t, such that:

$$(2)\qquad Me^{t\varrho(x)} < 1 \text{ and } P_{0,p}(x) \leq (Me^{t\varrho(x)})^p, \text{ for } x \in U_t \text{ and } p > 2/t,$$

$$(3)\qquad P_{0,p}(x) \geq \frac{p^n}{C}\, e^{2tp\varrho(x)}, \text{ for } x \in U_t \text{ and } p\,\mathrm{dist}(x,\Sigma)^{8/3} > C,$$

$$(4)\quad P_{0,p}(x) = P_p(x) + O(p^{-\infty}), \text{ as } p \to \infty, \text{ in any } \mathscr{C}^\ell \text{ topology on } K.$$

In particular, $P_{0,p}(x) = \mathbf{b}_0(x)p^n + \mathbf{b}_1(x)p^{n-1} + O(p^{n-2})$, as $p \to \infty$, uniformly on K, where

$$\mathbf{b}_0 = \frac{c_1(L,h)^n}{\omega^n}, \quad \mathbf{b}_1 = \frac{\mathbf{b}_0}{8\pi}\,(r^X - 2\Delta \log \mathbf{b}_0),$$

and r^X, Δ, are the scalar curvature, respectively the Laplacian, of the Riemannian metric associated to $c_1(L,h)$.

Recall that $P_{0,p}(x) = P_p(x) + O(p^{-\infty})$ means that for every $\ell, N \in \mathbb{N}$ there exists $C_{\ell,N} > 0$ such that $\|P_{0,p} - P_p\|_{\mathscr{C}^\ell(K)} \leq C_{\ell,N} p^{-N}$ for all $p \geq 1$.

Remarks about Theorem 6:

(2) and (3) show that on U_t the exponential decay estimate for $P_{0,p}$ is sharp. By (4) $P_{0,p}$ has the same asymptotics on K as the full Bergman kernel P_p. However, in Theorem 6 we do not obtain a *partition* of X in two sets with different regimes since $U_t \cup K \neq X$.

Such a partition (*a neighborhood* $U(\Sigma)$ *of* Σ *such that* $P_{0,p} = O(p^{-\infty})$ *on* $U(\Sigma)$ *and* $P_{0,p}(x) = P_p(x) + O(p^{-\infty})$ *on* $X \setminus \overline{U}(\Sigma)$) was exhibited under further hypotheses by: Berman (2007), Pokorny-Singer (2014) for a toric variety X and a toric Σ, Ross-Singer (2013) and Zelditch-Zhou (2016) under the assumption that Σ is invariant under an S^1-action. Moreover they study the asymptotics of the partial Bergman kernel on the interface region $\partial U(\Sigma)$ as well.

Proof of Theorem 6. There exist $\eta \in \mathscr{C}^\infty(X,[0,1])$ and $\delta > 0$ such that $\eta = 0$ near K, $\eta = 1$ near Σ, $c_1(L,h) + tdd^c(\eta\varrho) \geq \delta\omega$. Define

$$\widetilde{h}_t = h\exp(-2t\eta\varrho)\,,\ \ \widetilde{h}_{t,p} = \widetilde{h}_t^{\otimes p}\,.$$

Thus: $\widetilde{h}_t = h$ in a neighborhood of K, $\widetilde{h}_t \geq h$ on X, $c_1(L,\widetilde{h}_t) \geq \delta\omega$.

As Σ is smooth, $H^0_0(X,L^p) = H^0_{(2)}(X,L^p,\widetilde{h}_{t,p})$. Norm on $H^0_{(2)}(X,L^p,\widetilde{h}_{t,p})$:

$$\|S\|^2_{t,p} = \int_X |S|^2_{\widetilde{h}_{t,p}}\,\frac{\omega^n}{n!} = \int_X |S|^2_{h_p}\exp(-2tp\eta\varrho)\,\frac{\omega^n}{n!} \geq \|S\|^2_p\,,\ \text{since } \varrho < 0.$$

$\widetilde{P}_{t,p} :=$ Bergman kernel function of $H^0_{(2)}(X,L^p,\widetilde{h}_{t,p})$.

$P_{0,p} \leq P_p =$ Bergman kernel function of $H^0(X,L^p) \geqslant H^0_0(X,L^p)$.

If $S \in H^0_0(X, L^p)$ and $\|S\|^2_{t,p} \leq 1$, then $\|S\|^2_p \leq 1$, so

$$|S|^2_{\widetilde{h}_{t,p}} = |S|^2_{h_p} \exp(-2tp\eta\varrho) \leq P_{0,p} \exp(-2tp\eta\varrho) ,$$
$$\widetilde{P}_{t,p} \leq P_{0,p} \exp(-2tp\eta\varrho) .$$

Consequently we have shown:

$$\widetilde{P}_{t,p} \exp(2tp\eta\varrho) \leq P_{0,p} \leq P_p \text{ on } X , \quad \widetilde{P}_{t,p} \leq P_{0,p} \leq P_p \text{ near } K .$$

By Theorem 2, if $U_t := \left\{ x \in W : (Ae^{\rho(x)})^t \, e^{4\|h\|_\infty} < 1 \right\}$ then

$$P_{0,p}(x) \leq \left[(Ae^{\rho(x)})^t \, e^{4\|h\|_\infty} \right]^p , \quad \forall \, x \in U_t \, , \; p > 2/t.$$

Setting $M := e^{4\|h\|_\infty} A^t$ we obtain that

$$Me^{t\varrho(x)} < 1 \, , \quad P_{0,p}(x) \leq (Me^{t\varrho(x)})^p \, , \quad \forall \, x \in U_t \, , \; p > 2/t.$$

Shrinking U_t we can assume that $\eta = 1$ on U_t. By Theorem 5,

$$\widetilde{P}_{t,p}(x) \geq (1 - Cp^{-1/8})p^n \, \frac{c_1(L, \widetilde{h}_t)^n_x}{\omega^n_x}\,, \;\text{ if } p \in \mathbb{N},\; p\,\mathrm{dist}(x, \Sigma)^{8/3} > C.$$

Note that $c_1(L, \widetilde{h}_t)$ is smooth on $X \setminus \Sigma$, so $\dfrac{c_1(L, \widetilde{h}_t)^n}{\omega^n} \geq \delta^n$. Hence

$$P_{0,p}(x) \geq \widetilde{P}_{t,p}(x)\, e^{2tp\eta(x)\varrho(x)} \geq \frac{p^n}{C}\, e^{2tp\varrho(x)},$$

for $x \in U_t$ and $p > C\,\mathrm{dist}(x, \Sigma)^{-8/3}$.

Since $\widetilde{h}_t = h$ near K we have by the localization Theorem 7 which follows that $\widetilde{P}_{t,p} - P_p = O(p^{-\infty})$ as $p \to \infty$, in any $\mathscr{C}^\ell$ topology on K.

As $\widetilde{P}_{t,p} \leq P_{0,p} \leq P_p$ near K we conclude that

$$P_{0,p}(x) = P_p(x) + O(p^{-\infty}),\; \text{ as } p \to \infty, \text{ in any } \mathscr{C}^\ell \text{ topology on } K. \quad \square$$

Theorem 7 (Hsiao-Marinescu)

Let (X, ω) be a compact Hermitian manifold and L be a holomorphic line bundle on X. Let h_1 and h_2 be singular Hermitian metrics on L which are smooth outside an analytic set $\Sigma \subset X$ and such that $c_1(L, h_1)$, $c_1(L, h_2)$ are Kähler currents. Assume that $h_1 = h_2$ on an open set $U \Subset X \setminus \Sigma$. If $P_p^{(j)}$ is the Bergman kernel function of $H^0(X, L^p, h_j^{\otimes p}, \omega^n/n!)$, $j = 1, 2$, then $P_p^{(1)} - P_p^{(2)} = O(p^{-\infty})$ on U in any $\mathscr{C}^\ell$-topology, $\ell \in \mathbb{N}$, as $p \to \infty$.