Complexification of real manifolds and complex Hamiltonians

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Outline

We present two problems in the complexification of real manifolds where complex Hamiltonians play an important role.

1. A geometric construction for the Exponential map of the formal complexification of the Hamiltonian diffeomorphism group proposed by Donaldson.

- Motivated by semi-classical analysis
- Related to recent work on non-unitary quantization
- Joint work with Ernesto Lupercio and Alejandro Uribe.

2. Determination of new invariants for the global properties of a Grauert tube complexification beyond the known curvature conditions.

- Left-invariant metrics on $SU(2)$ admit an invariant holomorphic extension to the complex group $SL(2, C)$.
- We study the geodesic flow on $SL(2, C)$ for this holomorphic metric.
- Key role played by complete integrability, coming from classical mechanics.
- Joint work with Vaqaas Aslam and Daniel Irvine.

Part 1. gives a geometric interpretation of Donaldson's geodesic flow in the space of Kähler metrics, introduced to facilitate a conjectured proof method for proving the existence of cscK metrics. Our construction works until now only in the $\mathcal{C}^\omega$ case. We speculate about possible connections with the original purpose of Donaldson's construction.

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Part 2. studies real analytic manifolds with "large" complexifications, such as entire Grauert tubes. $SU(2)$ has an obvious large complexification, $SL(2, \mathbb{C})$. We compare tubes for left-invariant metrics on$SU(2)$ to $SL(2, \mathbb{C})$, and comment on a larger framework.

1. The Exponential map for complex Hamiltonians

Exp for Ham: Donaldson's Proposal

Let (M, ω, J) be a (compact) Kähler manifold, and let *Ham* denote the group of hamiltonian symplectomorphisms of (M, ω), with Lie algebra $C^\infty(M, \mathbb{R})/\mathbb{R}$.

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From a physical point of view this would correspond to finding a sensible way to associate a dynamical system to a complex-valued hamiltonian, $h : M \to \mathbb{C}$, in a manner that extends the notion of Hamilton flow in case h is real-valued.

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From our point of view, the interest in finding a complexification of *Ham* is based on the fact, discovered independently by Atiyah and Guillemin and Sternberg, that if a compact Lie group acts in a Hamiltonian fashion on a Kähler manifold then the action extends to the complexified group in an interesting way that can be understood.

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The appeal of extending the finite-dimensional picture to this infinite-dimensional setting of Kähler metrics is that it was hoped to lead to a way of constructing extremal metrics.
We will not enter further into the connections with Kähler geometry, other than to point out here that exponentiating purely-imaginary hamiltonians leads to geodesics in the space of Kähler potentials.

Our focus is on the following issue:

Donaldson has put forward a notion of "formal Lie group" to conceptualize his notion of a formal complexification of *Ham*. Briefly, a formal Lie group with Lie algebra $\mathfrak{G}$ is a "manifold" (the notion is of interest only in infinite dimensions), $\mathcal{G}$, together with a trivialization of its tangent bundle of the form

$$T\mathcal{G} \cong \mathcal{G} \times \mathfrak{G},$$

such that the map

$$\mathfrak{G} \ni h \mapsto \text{corresponding vector field } h^\sharp \text{ on } \mathcal{G}$$

is a Lie algebra homomorphism (with respect to the commutator of vector fields). The vector fields $h^\sharp$ should be thought of as "left-invariant", though no group structure on $\mathcal{G}$ exists.

In the present case

$$\mathfrak{G} = C^{\omega}(M, \mathbb{C})/\mathbb{C},$$

and the exponential map in our title refers to the problem of constructing the flow of the fields $h^{\sharp}$. The main Theorem 2 states that the family of diffeomorhpisms $\{f_t\}$ there "exponentiate" the complex-valued hamiltonian h, in one of Donaldson's models for the complexification of $\mathcal{G}$.[1]

[1] S. Donaldson, Symmetric spaces, Kähler geometry and Hamiltonian dynamics. *Northern California Symplectic Geometry Seminar*, 13-33. Amer. Math. Soc. Transl. Ser. 2, **196** (1999).

Local existence and uniqueness

Our approach is based on the following simple existence result:

Proposition

Let (M, J, ω) be a real analytic Kähler manifold of real dimension $2n$. There exists a holomorphic complex symplectic manifold (X, I, Ω) of complex dimension $2n$ and an inclusion $\iota : M \hookrightarrow X$ such that $\iota^*\Omega = \omega$, and with the following additional structure:

- *An anti-holomorphic involution $\tau : X \to X$ whose fixed point set is the image of ι and such that $\tau^*\Omega = \overline{\Omega}$.*

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- *A holomorphic projection $\Pi : X \to M$, $\Pi \circ \iota = Id_M$, whose fibers are holomorphic lagrangian submanifolds.*

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- *A holomorphic projection $\Pi : X \to M$, $\Pi \circ \iota = Id_M$, whose fibers are holomorphic lagrangian submanifolds.*
- *The **germ** of the structure above is unique.*

The local existence is simple:

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However there exist *natural* complexifications that make our results below much more global in some cases. (The *uniqueness* cannot be global, as is obvious from topological considerations in the examples below.)

Some examples

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2. Donaldson's canonical complexification[2] which is even compact, with Π, X all global projective algebraic, Ω global and rational, but τ not global, in general.

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2. Donaldson's canonical complexification[2] which is even compact, with Π, X all global projective algebraic, Ω global and rational, but τ not global, in general.

3. For M a co-adjoint orbit, all elements are global, and can be compactified canonically algebraically in several ways (see below).

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The Definition

To describe our results we need some notation. Given a function $h : M \to \mathbb{C}$ whose real and imaginary parts are real analytic, there is a holomorphic extension $H : X \to \mathbb{C}$ perhaps only defined near $\iota(M)$, but we will not make a notational distinction between X and such a neighborhood, as our results are local in time.

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Let h and H be as above. The fibers of Π are the leaves of a holomorphic foliation, $\mathcal{F}$, of X. Denote by $\Phi_t : X \to X$ the Hamilton flow of $\Re H$ (where $\Re H$ is the real part of H) with respect to the real part of Ω. We denote by $\mathcal{F}_t$ the image of the foliation $\mathcal{F}$ under Φ_t (so that $\mathcal{F}_0 = \mathcal{F}$).

The Definition *(cont)*

We assume that there exists $E \subset \mathbb{R}$ an open interval containing the origin such that $\forall t \in E$ the leaves of $\mathcal{F}_t$ are the fibers of a projection $\Pi_t : X \to M$ (always true for M compact).

We will set

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Let $\phi_t : M \to M$ be defined by

$$\phi_t := \Pi_t \circ \Phi_t \circ \iota. \tag{1}$$

The Picture

We now explain the geometry behind the construction of ϕ_t summarized by (1).

To find the image of $x \in M$ under ϕ_t one flows the leaf $\mathcal{F}_0^x = \Pi^{-1}(x)$ of the foliation $\mathcal{F} = \mathcal{F}_0$ by Φ_t, and intersects the image leaf with M.

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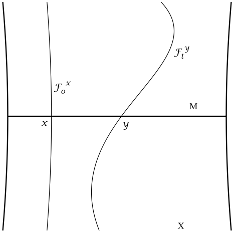
In other words, (1) can be stated equivalently as:

$$\{\phi_t(x)\} = \Phi_t\left(\Pi^{-1}(x)\right) \cap M. \tag{2}$$

The definition of ϕ is summarized in the following figure, where $\mathcal{F}_t^y := \Pi_t^{-1}(y)$.



Motivation/Interpretation

The present construction is motivated by semiclassical analysis.

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Ignoring domain issues, the notion of the exponential of a non-hermitian *quantum* hamiltonian is clear, if M is compact and Planck's constant is fixed: this amounts to exponentiating a matrix. Therefore a very natural approach to exponentiating a non-hermitian classical hamiltonian is to first quantize it, exponentiate it on the quantum side, and then take the semiclassical limit. This approach has been developed by Rubinstein and Zelditch, and raises a number of interesting but difficult analytical questions.

In our construction we bypass these analytic difficulties by considering the following geometric remnants of quantization:

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1. Each leaf of the foliation $\mathcal{F}_0$ corresponds to a quantum state (element of the projectivization of the quantum Hilbert space) represented by a *coherent state* centered at a point on the leaf, that is, an element in the Hilbert space that semiclassically concentrates at the intersection of the leaf with M.

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2. The fact that Π is holomorphic says the the coherent states are associated to the metric of (M, J, ω).

3. On the quantum side the evolution of a coherent state remains a coherent state, whose lagrangian is simply the image of the one at time $t = 0$ by the complexified classical flow.

Theorem

Let $\phi_t, E \subset \mathbb{R}$ be as above. Then, $\forall t \in E$:

1. *There is a complex structure $J_t : TM \to TM$ such that $J_t \circ d\Pi_t = d\Pi_t \circ I$ (and $J_0 = J$).*

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2. *$\phi_t : (M, J) \to (M, J_t)$ is holomorphic ($J_t \circ d\phi_t = d\phi_t \circ J$).*
3. *The infinitesimal generator of ϕ_t is the (time dependent) vectorfield*
$$\dot{\phi}_t \circ \phi_t^{-1} = \Xi^{\omega}_{\Re h} + J_t\left(\Xi^{\omega}_{\Im h}\right) \tag{3}$$
where $\Xi^{\omega}_{\Re h}$, denotes the Hamilton vector field of $\Re h$ with respect to ω, etc.

Remarks on the Main Theorem

1. Graefe and Schubert have given a very clear and detailed account of the case when M is equal to $\mathbb{R}^{2n} \cong \mathbb{C}^n$, h is a quadratic complex hamiltonian, and the lagrangian foliations are by complex-linear positive subspaces, corresponding to standard Gaussian coherent states with possibly complex centers. (See below.)

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By explicit calculations on both quantum and classical sides, they show that the evolution of a coherent state centered at x is another coherent state whose center may be complex, but that represents the same quantum state as a suitable Gaussian coherent state centered at $\phi_t(x)$.

Remarks *(cont.)*

2. Conditions (2) and (3) (together with the initial condition $\phi_0 = Id_M$) *characterize* the family $\{\phi_t\}$ (Cauchy-Kowalewski).

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3. Suppose G is a compact Lie group acting on M in a Hamiltonian fashion and preserving J. Then the action extends as a holomorphic action to the complexification $G_{\mathbb{C}}$. The extended action is as follows:
If $a, b : C^\infty(M) \to \mathbb{R}$ are two components of the moment map of the G action, then the infinitesimal action corresponding to $a + ib$ is the vector field

$$\Xi_a^\omega + J(\Xi_b^\omega).$$

Remarks *(cont.)*

The corresponding 1-p.g. of diffeomorphisms, $\varphi_t : M \to M$, satisfies (2) and (3) of Theorem 2, with $J_t = J_0$ for all t, so we must have $\varphi_t = \phi_t$. In other words, our construction is an extension of the process of complexifying the action of a compact group of symmetries of (M, ω, J).

Sketch of Proof

We have $h = f + ig$ the complex Hamiltonian on M, and $H = F + iG$ its holomorphic extension to X. ω extends holomorphically to $\Omega = \omega_1 + i\omega_2$. ξ is the ω_1 Hamilton field of F, and Φ_t its flow.

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Lemma 1: Φ_t is holomorphic.

This is because Ω, H are holomorphic.

Lemma 2: If h is real, then ξ is tangent to M along M.

You check how τ operates on H, Ω and hence ξ to get $d\tau_* \xi = \xi$.

Lemma 3. $\dot{\phi}_t(x) = d\Pi_{y,*}(\xi_y)$, where $\Pi_t(y) = x$.

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For the main theorem:

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3. To show:

$$\dot{\phi}_t \circ \phi_t^{-1} = \xi_f^\omega + J_t\left(\xi_g^\omega\right),$$

which is linear in h. For h real, this is what has been shown above. For h imaginary, one checks that $I(\xi)$ is the ω_1-Hamilton field of -G, which is tangent to M. This gives $\dot{\phi}_t \circ \phi_t^{-1} = J_t \xi_t^\omega$.

The space of potentials

Consider the space

$$\mathcal{H} := \{a : M \to \mathbb{R}\ ;\ \omega_a = \omega + i\bar{\partial}\partial a > 0\}/\mathbb{R}$$

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of Kähler potentials for Kähler forms in the same cohomology class as ω.

$\mathcal{H}$ has a natural Riemannian metric:

$$\|\delta a\|^2 = \int_M |\delta a|^2 d\mu_a.$$

How the "exponential" interacts with $\mathcal{H}$:

Given $h = f + ig : M \to \mathbb{C}$, let f_t be its exponential. Notice that ω is of type (1,1) for J_t – this is because $\mathcal{F}_t$ is Ω-Lagrangian.

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Let ω_t be the symplectic form defined by $f_t^* \omega = \omega_t$, and write $\omega_t = \omega + i\bar{\partial}\partial a_t$, where the a_t are taken modulo constants. Then

$$\dot{a}_t = 2 f_t^* G. \tag{4}$$

The geodesic equations

One can check that the geodesic equation for $\mathcal{H}$ is given by

$$\ddot{a} = -\frac{1}{2}|\nabla^a \dot{a}|_a^2,$$

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Theorem

Let $h = ig$ be purely imaginary. Then the exponential of ig is the geodesic with initial conditions $a = 0$ and $\dot{a}_t(0) = 2g$.

Coadjoint orbits

Construction Example:

For G compact, and $\lambda \in \mathfrak{g}^*$ the orbit $\mathcal{O}_\lambda$ is a symplectic manifold with form $\omega_\lambda(\xi, \eta) = \lambda([\xi, \eta])$.

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The $G_{\mathbb{C}}$-orbit of λ in $\mathfrak{g}^*_{\mathbb{C}}$ is a complex symplectic manifold $\mathcal{O}_{\lambda,\mathbb{C}}$. One can define a complex structure on $\mathcal{O}_\lambda$ using the roots of $\mathfrak{g}$, and this is a Kähler manifold. τ above is just the usual conjugation fixing $\mathfrak{g}^*$. The leaves of $\mathcal{F}$ are given by unipotent subgroups of $G_{\mathbb{C}}$.

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Think of the group $G = SO(3)$, $M = \mathcal{O}_\lambda = S^2$ and $\mathcal{O}_{\lambda,\mathbb{C}}$ is an affine quadric, and $\mathcal{F}$ and $\tau(\mathcal{F})$ are given by the two rulings of $\mathcal{O}_{\lambda,\mathbb{C}}$.

A global example: incomplete geodesics in $\mathcal{H}(S^2)$

Any linear function ($\mathbb{C}$-valued) on $\mathcal{O}_{\lambda,\mathbb{C}} \subset \mathfrak{g}^*_{\mathbb{C}}$ is the Hamiltonian on $\mathcal{O}_{\lambda,\mathbb{C}}$ for a 1-p.g. from $G_{\mathbb{C}}$, and in particular is *complete*.

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Let h be such a function. Then any function $F(h)$ also generates a complete flow on $\mathcal{O}_{\lambda,\mathbb{C}}$, since it has the same flow curves but reparametrizes by a constant factor on each flow line.

If we take $M = S^2$ and then $\lambda = \mathbf{N} \in \mathbb{R}^3$ then $h(x,y,z) = \sqrt{-1}z^2$ has a complete flow Φ_t on $\mathcal{O}_{\lambda,\mathbb{C}}$. Note that since the leaves of the foliation are closed and proper, by topology, if the leaves $\mathcal{F}_t$ are transverse to M, then ϕ_t is globally defined on M. Thus, this transversality is the only thing obstructing the existence of the ϕ_t for all t.

Incomplete geodesics in $\mathcal{H}(S^2)$

However, this fails in finite (imaginary) time. (Easy to solve ODE.) This gives $\mathcal{C}^{\omega}$ geodesics which are incomplete in the space $\mathcal{H}$.

Earlier such examples were given by Lempert-Vivas (local, via HCMA formulation), others in non-smooth cases.

Open Questions:

- Can one use, say, microlocal analysis to describe the map ϕ_t above?
- Does the solution of the cscK metric problem offer any insights about quantization of complex Hamiltonians?

2. Grauert tubes and large complexifications

Complexifications and tubes: definitions

If M is a real manifold, then a complexification is a complex manifold $M_{\mathbb{C}} \supset M$, usually with an anti-holomorphic involution σ such $M = \text{Fix}(\sigma)$. Abstractly, the germ of $M_{\mathbb{C}}$ around M is uniquely defined if M is $\mathcal{C}^{\omega}$ (Bruhat-Whitney).

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- If the complexification is "big" is it special: affine algebraic, homogeneous, etc.?
- Leads to function theory questions and restricted Levi problems (functions with growth conditions).

Both Lempert-Szőke and Guillemin-Stenzel gave a canonical method for complexifying a real analytic, Riemannian manifold M. This is, in general, defined on an open subset of the tangent bundle containing the 0-section M, and we call the resultant manifold the Grauert tube associated to the metric g. If the complex structure is defined on all of TM, we call the tube **entire**. Entire tubes posses s.p.s.h. exhaustions τ for which $\tau := e^u$ on $TM \setminus M$, where $u = |v|$ is the length function on TM.

These are considered “big” as loosely stated earlier, and one natural question is whether these are affine algebraic.

Entire tubes appear to be rare, and almost all constructions of them so far are closely related to homogeneous examples. Lempert and Szőke have shown that a necessary condition for the tube of M, g to be entire is that the sectional curvatures of g be non-negative. This is far from sufficient, however.

Examples of entire tubes

Examples

- If M is a Riemannian submersion of a compact symmetric space, the tube of M is entire, and $\Omega \neq 0$. It is known by other means that $X =$ the entire tube of M is affine algebraic. This uses Peter-Weyl theory.

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- Szőke showed there is a one parameter family of distinct surfaces of revolution which have entire tubes. None but the round metric has $\Omega^{\otimes 2}$ holomorphic. Explicitly, these are

$$ds_\delta^2 = d\psi^2 + \frac{\sin^2\psi}{1 + \delta\sin^2\psi}d\theta^2,$$

where $\delta \in [0, +\infty)$ is a real parameter. $\delta = 0$ is the round sphere.

Aguilar's examples

Aguilar found a method based on symplectic reduction to generate twisted forms of a metric with infinite tube in such a way as to create a new entire tube metric. The Szőke spheres are special cases. In fact, all known entire tubes arise in this fashion from symmetric spaces (i.e., by quotients and twists).

For M of two dimensions, Aguilar has found a sequence of conditions generalizing that of Lempert-Szőke to higher order invariants which give necessary and sufficient conditions that a tube be entire. Unfortunately, these are very difficult to interpret in particular examples. In particular, it is not known whether the conditions are effectively finite, i.e., are redundant after a certain degree.

Left invariant metrics on $SU(2)$

One of the problems in estimating whether a given metric has an entire tube is that the complex structure on the tube viewed as a subset of the tangent bundle TM is not very explicit, and while the original real manifold may have an obvious “large” complexification which presents itself as a natural guess for an entire tube complexification, it is hard to say how far out in such a complex manifold the tube's structures extend: for example, the solution u of HCMA. For example, any metric on S^n has a tube locally inside $\mathcal{Q}^n$, the affine quadric in $\mathbb{C}^{n+1}$, but it is not known how many extend to all of the quadric.

Recently, Vaqaas Aslam , Daniel Irvine and I have examined closely the situation of left-invariant (but not necessarily bi-invariant) metrics on $SU(2) \cong S^3$. Here the obvious "large" complexification would be $\mathcal{Q}^3 \cong SL(2, \mathbb{C})$. For such metrics, the natural geometric tensors have $SL(2, \mathbb{C})$-invariant, holomorphic extensions to $SL(2, \mathbb{C})$. The question of whether the tube structure extends to all of $SL(2, \mathbb{C})$ concerns whether the MA foliation and the HCMA solution extends to all of $\mathcal{Q}^3$.

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We convert this to a geometric problem. Its solution, however, depends on classical mechanics, namely, the complete integrability of the spinning top equations in three dimension.

Another reason for studying these examples is that the "entire" property of the Grauert tube construction depends, in principle, on two elements: behavior of the complexified geodesics on M in the complex domain, but also whether the metric, etc., converge rapidly enough to admit an entire extension. In order to answer this, one has to have an *a priori* notion of how to measure extension in the complex manifold, other than by the HCMA solution.

Lemma

If g is a left-invariant metric on a compact Lie group G, and its Grauert tube is entire, then that tube is (equivariantly) biholomorphic to $G_{\mathbb{C}}$.

Given the lemma, we can convert our problem to a question of focal points for the holomorphic exponential map for $g_{\mathbb{C}}$ on $G_{\mathbb{C}}$.

Lemma

Consider $M = G \subset G_{\mathbb{C}}$, and identify $TG = N_G \subset T_{\mathbb{C}}G_{\mathbb{C}}$. Then the tube of the left invariant metric g is entire iff $Exp_{g_{\mathbb{C}}} : N_G \to G_{\mathbb{C}}$ is a diffeomorphism.

Given these lemmas, we can prove that the tube of g is not entire by showing that a complexified geodesic with initial condition in N_G is not an entire function of one complex variable. To see this, we restrict to the case of $G = SU(2)$.

The role of classical mechanics

Fix the bi-invariant metric $(\cdot,\cdot)$ on $\mathfrak{su}(2) = T_eSU(2)$. For any positive definite, symmetric linear transformation A of $\mathfrak{su}(2)$ to itself, let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be its eigenvalues. For our question, we can ignore an overall scale factor. If exactly two of the e.v.'s are equal, these are Berger spheres: this is the case when the isometry group is of dimension 4. For certain λ's, the metric g has negative sectional (even scalar) curvature, so those do not have entire tubes (Lempert-Szőke). Any complexified geodesic $\gamma_{\mathbb{C}}$ of g (holomorphic geodesic of $g_{\mathbb{C}}$)) gives a holomorphic map of $\mathbb{C} \supset U \to G_{\mathbb{C}} \times \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$. This map must be an entire function for the tube of g to be entire. But this is what integrability obstructs.

The geodesic flow on $SU(2)$ is completely integrable. If ξ_1, ξ_2, ξ_3 is an o.n.basis for $\mathfrak{su}(2)$, which we consider a frame for $TSU(2)$, then the left invariant integrals are

$$I_1(\xi) = \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2$$

and

$$I_2(\xi) = \lambda_1^2 a^2 + \lambda_2^2 b^2 + \lambda_3^2 c^2,$$

where $\xi = a\xi_1 + b\xi_2 + c\xi_3$. These are the total energy and angular momentum for a spinning top. The third integral is any Hamiltonian for the action of any $\eta \in \mathfrak{su}(2)$.

We write $\dot{\gamma}(t) \in \mathfrak{su}(2)$ expressed in the invariant frame. Then $I_1(\dot{\gamma}), I_2(\dot{\gamma})$ are *constant*, as are $I_1(\dot{\gamma}_{\mathbb{C}}), I_2(\dot{\gamma}_{\mathbb{C}})$ along the holomorphic geodesic. The constants are determined by the (real) initial conditions of γ.

Given the initial conditions, we have a curve

$$\{I_1(z) = s_1, I_2(z) = s_2\} \subset \mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3 \subset \mathbb{P}^3.$$

Let C_{s_1,s_2} denote its closure in $\mathbb{P}^3$.

Lemma

If the initial conditions are generic, and the λ's distinct, then $C_{s_1,s_2} \subset \mathbb{P}^3$ is non-singular, of genus 1.

From this we get an immediate corollary.

Corollary

$\dot{\gamma}_{\mathbb{C}}$ cannot be entire, if the λ's are distinct. As a result, if the λ's are distinct, the tube of g cannot be entire.

This is because $\dot{\gamma}_{\mathbb{C}}$ maps into the affine part of C_{s_1,s_2}, which is a hyperbolic Riemann surface, and $\dot{\gamma}_{\mathbb{C}}$ must be constant. But not all geodesics have constant $\dot{\gamma}_{\mathbb{C}}$.

Remarks:

- For λ_i distinct, all close to 1, the metric g has positive curvature, so this is a condition distinct from the Lempert-Szőke condition.
- For any invariant metric, this gives implicitly a bound on the $|\mathrm{Im}\,\zeta|$ for which $\dot{\gamma}_{\mathbb{C}}$ may be defined (Schottky).

Now we are reduced to considering the Berger spheres, and we will describe the metrics in the form $g = ds^2 = \omega_1^2 + \omega_2^2 + \lambda\omega_3^2$, where $\lambda > 0$. Note: we have normalized $(\cdot, \cdot)$, i.e., the case $\lambda = 1$, so that this metric is round with $K \equiv +1$. There are three relevant intervals of λ values for the question at hand.

- $\lambda \in (0, 1]$: in this case, we have that g is a twisted version of the round metric on S^3. By a result of Aguilar, it has an entire tube, via a quotient construction.

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- $\lambda > \frac{4}{3}$: here the Lempert-Szőke result applies and gives a bound on the complexification radius.
- $\lambda \in (1, \frac{4}{3}]$: here we compute the Jacobi fields explicitly to show the complex *Exp* has singularities.

Scarcity of entire tubes

There are very few examples of entire tubes, though we cannot state simply that they are all affine algebraic. The underlying complex manifolds and affine varieties seem very rigid, and one suspects that they might depend on only a finite number of parameters. To focus on the case of $M = S^2$, we first have the following theorem:

Theorem

(Totaro; Aguilar-B.) Let X be an affine, surface over $\mathbb{C}$, with X diffeomorphic to TS^2. (Such would be the case for any algebraicized entire tube over S^2.) Then X is algebraically equivalent to the standard affine quadric

$$\mathcal{Q} = \{z_1^2 + z_2^2 + z_3^2 = 1\} \subset \mathbb{C}^3.$$

Remarks on uniqueness

Remarks

- If all entire tubes on S^2 were algebraic, this says these affine varieties are all just $\mathcal{Q}$.
- All canonical Riemannian tensors, or at least powers of them, are extendible to the entire tube as meromorphic tensors, and if the tube is algebraic, these tensors are rational. If these all live on the same quadric, the real metric might be determined by some tensors living in a finite dimensional space. This would require showing something like a **bound** on the order of poles of these rational tensors.

Thanks for your attention!

Thanks to the organizers!