SUPERSYMMETRIC STRING VACUA AND COMPLEX GEOMETRY

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Vacua and supersymmetry

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- This gives rise to a "principle of least action" and Lagrangian mechanics. Let M be the configuration space of points q. Then the physical path is a stationary point of an action functional defined in the space of paths $t \to q(t)$ in M,

$$I = \int (\frac{1}{2} m\dot{q}(t)^2 - V(q(t)))dt$$

  where $V(q)$ is the potential, and the force F given by $F = -\partial V/\partial q$. The expression $\mathcal{L}(q, \dot{q}) = \frac{1}{2} m\dot{q}^2 - V(q)$ is a function on the cotangent bundle $T(M) = \{(q, \dot{q})\}$ called the Lagrangian, and its integral I is the action of the path $t \to q(t)$.

Gauge theories and general relativity (19th-20th centuries)

- In electromagnetism, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be unified in a single field F, which is the curvature $F = dA$ of a $U(1)$ connection. The action is $I = \int |F|^2$, and its critical points, combined with the equation $dF = 0$, reproduce the system of Maxwell's equations for the electric and magnetic fields.

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- The weak and strong interactions of particle physics are described similarly by a connection A on a vector bundle E, which is valued in a suitable non-abelian gauge group depending on the interaction. The field is described again by the curvature $F = dA + A \wedge A$, and the action is the Yang-Mills action

  $$I(A) = -\int \mathrm{Tr}(F \wedge \star F)$$

  Maxwell's equations generalize to the Yang-Mills equation and the Bianchi identity

  $$d_A^\dagger F = 0, \qquad d_A F = 0$$

  When the vector bundle E is a holomorphic vector bundle over a Kähler manifold (X, ω) and the curvature F is a $(1,1)$-form, the first equation is equivalent to $g^{j\bar{k}} F_{\bar{k}j}$ be covariantly constant, or for some constant μ,

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- In general relativity, the field is a metric $g_{ij}(x)$, the action is the Einstein-Hilbert action, $I(g_{ij}) = \int R\sqrt{g}$ and the critical points are given by Einstein's equation,

  $$R_{ij} - \frac{1}{2} g_{ij} R = 0$$

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- A supersymmetric theory will contain pairs of bosonic fields and their fermionic partners. Supergravity theories are supersymmetric theories which contain a graviton, whose field is a metric g_{MN}, and its partner, which is called the gravitino, and whose field ${\chi_M}^{\alpha}$ is a one-form valued in the bundle of spinors.

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- Some basic facts about spinors: the Clifford algebra in n dimensions is the associative algebra generated by elements γ^M, $1 \leq M \leq n$, satisfying the relation

  $$\gamma^M \gamma^N + \gamma^N \gamma^M = 2\delta^{MN} I$$

  If the Clifford algebra can be realized as a space of endomorphisms of a vector space S, then the elements of S are called spinors.
  The point of the above relations is that the Dirac operator D defined as $D\psi = \gamma^M \partial_M \psi$ will be a square root of the Laplacian

  $$D^2\psi = \frac{1}{2}(\gamma^M \gamma^N + \gamma^N \gamma^M)\partial_M \partial_N \psi = \Delta\psi$$

  More generally, on curved Riemannian manifolds, we obtain spin bundles by choosing an orthonormal frame and letting S be the fiber at each point. A spinor field is a section of the corresponding bundle S, and the Levi-Civita connection extends naturally to a covariant derivative on spinor fields,

  $$\nabla_J \psi = \partial_J \psi + \frac{1}{2}\omega_{JMN}\gamma^M \gamma^N \psi$$

- The infinitesimal generator of supersymmetry transformations is a spinor field ψ^α. Its action on the gravitino field is of the form

$$\delta\chi_M = \nabla_M\psi + H_{MN_1\cdots N_p}\gamma^{[N_1}\cdots\gamma^{N_p]}\psi$$

where γ^N are Dirac γ-matrices, and H is another field in the theory which is a $(p+1)$-form. It is for example a 3-form called the Kalb-Ramond field in the case of the heterotic string, and a 4-form in the case of 11-dimensional supergravity.

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- In the mathematics literature, the existence of a covariant constant spinor is characteristic of reduced holonomy and special geometry. Such problems go back to works of Berger, Lichnerowicz, et al. Here physics has provided supersymmetry as motivation, and also introduced the possibility of other connections than the Levi-Civita connection, which are characterized by an additional field, such as the Kalb-Ramond field.

- For compactifications, we assume that the space-time of the vacuum state is of the form, e.g., $M^{3,1} \times K$, where $M^{3,1}$ is 4-dimensional Minkowski space-time, and K is an intermediate space. We are then interested in the additional geometric structure on K which follows from the existence of a covariantly constant spinor ψ.

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- A famous example is obtained in the heterotic string, with the Kalb-Ramond field taken to be 0, and the covariantly constant spinor implying that the internal manifold K has $SU(3)$ holonomy. More concretely, it leads to two conjugate covariantly constant spinors $\eta_\pm$ on K, and an integrable complex structure can be constructed out of bilinears in $\eta_\pm$,

$$J_m{}^n = i\eta_+^\dagger \gamma_m{}^n \eta_+ = -i\eta_-^\dagger \gamma_m{}^n \eta_-$$

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- Similarly, for 11-dimensional supergravity, the existence of a covariantly constant spinor implies that the holonomy of the internal space must be G_2. By work of Bär (1993), such a structure is closely related to that of an almost Kähler structure.

The Hull-Strominger System

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Let Y be a 3-dimensional compact complex manifold Y, equipped with a nowhere vanishing holomorphic 3-form Ω, and let $E \to Y$ be a holomorphic vector bundle over Y.

The Hull-Strominger system is the following system of equations for a Hermitian metric ω on Y and a Hermitian metric $H_{\bar{\alpha}\beta}$ on E,

$$F^{2,0} = F^{0,2} = 0, \qquad \omega^2 \wedge F^{1,1} = 0$$

$$i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F)) = 0$$

$$d^\dagger \omega = i(\bar{\partial} - \partial) \log \|\Omega\|_\omega$$

Here α' is the slope parameter. The expressions Rm and F are the curvatures of the metrics ω and $H_{\bar{\alpha}\beta}$, viewed as a $(1,1)$-forms valued in $End(T^{1,0}(Y))$ and in $End(E)$ respectively. The norm $\|\Omega\|_\omega$ is defined by

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- The first equation is just the usual Hermitian-Einstein equation.
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- The third condition is a torsion constraint, more explicitly, if we set $T = i\partial\omega$, then

$$g^{j\bar{k}}\,T_{\bar{k}jm} = \partial_m \log \|\Omega\|_{\omega}$$

.

- Because ω is not necessarily Kähler, there are many natural unitary connections associated to it. The most familiar one is the Chern unitary connection, defined by

$$\nabla_{\bar{j}} V^k = \partial_{\bar{j}} V^k, \qquad \nabla_j V^k = g^{j\bar{p}} \partial_j (g_{\bar{p}m} V^m).$$

In this case, the Riemann curvature tensor is given by,

$$Rm = R_{\bar{k}j}{}^p{}_q dz^j \wedge d\bar{z}^k, \qquad R_{\bar{k}j}{}^p{}_q = -\partial_{\bar{k}} (g^{p\bar{m}} \partial_j g_{\bar{m}q}),$$

with a similar expression for the curvature F of $H_{\bar{\alpha}\beta}$, $F = F_{\bar{k}j}{}^\alpha{}_\beta dz^j \wedge d\bar{z}^k$, $F_{\bar{k}j}{}^\alpha{}_\beta = -\partial_{\bar{k}} (H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta})$. But one can define another line of unitary connections $\nabla^{(\kappa)}$, called the Gauduchon line, by

$$\nabla_j^{(\kappa)} V^k = \nabla_j V^k - \kappa T^k{}_{jm} V^m$$

The connection with $\kappa = 1$ seems to have been introduced first by Yano, and was subsequently rediscovered by Bismut. That connection with $\kappa = 1/2$ is the Lichnerowicz connection.

- As shown by C. Hull, the anomaly cancellation mechanism does not require a specific unitary connection for ω. In this work, we shall mostly restrict ourselves to the choice of the Chern unitary connection, and discuss only one example with Yano-Bismut connections, namely the case of unimodular Lie groups suggested by Biswas and Mukherjee, Andreas and Garcia-Fernandez, and Fei and Yau.

- The Hull-Strominger system is a generalization of a system of equations proposed by Candelas, Horowitz, Strominger, and Witten for compactifications of the heterotic string to 4-d spacetime which preserve supersymmetry.

  A well-known solution is to take Y Kähler, set $E = T^{1,0}(Y)$, $H_{\bar{\alpha}\beta} = \omega$. Then $d\omega = 0$, $Rm = F$, and the second equation is automatically satisfied. Next,

$$\omega \wedge F = \omega^2 \wedge Rm = 3\, Ric(\omega)$$

  (viewed as an endomorphism of $T^{1,0}(M)$), and thus the first equation reduces to the condition of vanishing Ricci curvature

$$Ric(\omega) = 0.$$

  As conjectured by Calabi, and proved by Yau, metrics satisfying this condition can be found in any given Kähler class, as long as $c_1(Y) = 0$, which is the case here, because we have assumed the existence of a nowhere vanishing holomorphic 3-form Ω.

  Finally, we have, directly from the definition of the Ricci curvature,

$$Ric(\omega) = \partial\bar{\partial} \log \|\Omega\|^2$$

  and thus $\|\Omega\|$ is constant. The torsion condition follows then from the Kähler condition,

$$d^\dagger \omega = - \star d \star \omega = - \star d\omega^2 = 0.$$

  These "Calabi-Yau solutions" have had an enormous influence on both geometry and physics for the last 30 years or so.

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- *Canonical metrics in non-Kähler geometry:* in Kähler geometry, a canonical metric is usually defined by a cohomological condition (e.g. $d\omega = 0$), and by a curvature condition (e.g. ω has constant scalar curvature). As pointed out by J. Li and S.T. Yau, the third equation in Strominger systems is equivalent to the following "conformally balanced" condition

$$d(\|\Omega\|_\omega \omega^2) = 0$$

  so the last two equations in Strominger systems can be viewed (for given F) as a generalization of the notion of canonical metric on X to the non-Kähler setting. The notion of balanced metric, i.e. $d(\omega^2) = 0$, was introduced in mathematics by Michelsohn (1981). The existence of a balanced metric is a property invariant under birational transformations (Alessandrini-Bassanelli).

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- *Geometric flows:* We shall see shortly that there are compelling reasons for studying Hull-Strominger systems as the fixed points of a geometric flow. Remarkably, even though this flow is a flow of $(2,2)$-forms, it will turn out to have some strong resemblance with the Ricci flow (or RG flow for sigma models), although it will of course be more complicated. As such, it should provide a good model for the development of new flows, in particular of $(n-1, n-1)$-forms.

Some special solutions of Strominger systems

- By now, a very large number of solutions of the Hull-Strominger system have been found, either by perturbations from Calabi-Yau solutions, or by duality arguments from string theory, or in highly symmetric situations such as complex Lie groups, or by geometric constructions building on earlier constructions of Calabi-Eckmann and Calabi-Gray.

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- In this last context of geometric constructions, a very recent and noteworthy development has been the construction by T. Fei, S. Picard, and Z. Huang (2017) of an infinite number of solutions, which are non-Kähler fibrations over Riemann surfaces, and which are of an infinite number of distinct topological types.

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- But the first non-perturbative, non-Kähler solutions was found by PDE methods by Fu and Yau (2006) on certain toric fibrations $\pi : Y \to X$ over K3 surfaces constructed by Goldstein and Prokushkin, building on earlier ideas of Calabi and Eckmann. It turned out that, in this case, the Hull-Strominger system can be reduced to a single non-linear PDE of Monge-Ampère type on the two-dimensional base X,

$$i\partial\bar\partial(e^u\omega - \alpha' e^{-u}\rho) + \alpha' i\partial\bar\partial u \wedge i\partial\bar\partial u + \mu = 0.$$

  Here ρ and μ are given smooth $(1,1)$ and $(2,2)$ forms respectively, with μ satisfying the integrability condition

$$\int_X \mu = 0.$$

  The existence of solutions to equations of this type was shown by Fu and Yau in two separate papers, for $\alpha' > 0$ and $\alpha' < 0$ respectively. Analytically, the two cases are quite different.

- By now, a very large number of solutions of the Hull-Strominger system have been found, either by perturbations from Calabi-Yau solutions, or by duality arguments from string theory, or in highly symmetric situations such as complex Lie groups, or by geometric constructions building on earlier constructions of Calabi-Eckmann and Calabi-Gray.
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- But the first non-perturbative, non-Kähler solutions was found by PDE methods by Fu and Yau (2006) on certain toric fibrations $\pi : Y \to X$ over K3 surfaces constructed by Goldstein and Prokushkin, building on earlier ideas of Calabi and Eckmann. It turned out that, in this case, the Hull-Strominger system can be reduced to a single non-linear PDE of Monge-Ampère type on the two-dimensional base X,

$$i\partial\bar{\partial}(e^u\omega - \alpha' e^{-u}\rho) + \alpha' i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u + \mu = 0.$$

Here ρ and μ are given smooth $(1,1)$ and $(2,2)$ forms respectively, with μ satisfying the integrability condition

$$\int_X \mu = 0.$$

The existence of solutions to equations of this type was shown by Fu and Yau in two separate papers, for $\alpha' > 0$ and $\alpha' < 0$ respectively. Analytically, the two cases are quite different.

Difficulties with the balanced condition

- Even if the holomorphic vector bundle $E \to Y$ and the Hermitian-Einstein metric $H_{\bar{\alpha}\beta}$ are known (say by the Donaldson-Uhlenbeck-Yau theorem, if the conformal class of ω is known), the Hull-Strominger system is still difficult because the metric ω has to satisfy two simultaneous conditions, namely the anomaly cancellation condition and the conformally balanced condition.

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- Balanced metrics have been constructed by many authors, e.g. Gauduchon, Tosatti-Weinkove, Teng Fei et al. One possible strategy is to select some way of deforming a given one while preserving the balanced condition, and then try to find a solution of the anomaly equation among the deformed metrics.

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- For Kähler metrics, this works well because if ω_0 is Kähler, then any Kähler metric ω in the same class must be of the form

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- However, there is no such simple characterization known for balanced metrics. For example, if ω_0 is balanced ($d\omega_0^2 = 0$), then an analogue of the above deformation may be a metric ω defined by

$$\omega^2 = \omega_0^2 + i\partial\bar{\partial}(\varphi\tilde{\omega})$$

which is automatically balanced for any scalar function φ and $\tilde{\omega}$ any $(1,1)$-form which keeps ω^2 positive. The drawback is that no particular deformation seems more compelling than the others, and the resulting equations all seem very complicated and unnatural.

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The key idea is to address the conformally balanced condition by introducing the following flow of $(2,2)$-forms, whose stationary points are solutions of the Hull-Strominger system,

$$\partial_t(||\Omega||_\omega \omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F))$$

$$H^{-1}\,\partial_t H = -3\frac{\omega^2 \wedge F}{\omega^3}$$

with $\omega = \omega_0$ when $t = 0$, where ω_0 is a balanced metric. We can also consider the flow of ω alone, for a given $(2,2)$-form $\mathrm{Tr}(F \wedge F)$. We call all these flows "Anomaly flows", in reference to the Green-Schwarz anomaly cancellation mechanism.

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Theorem 1 The above flow of positive $(2,2)$-forms defines a vector field on the space of positive $(1,1)$-forms.
(a) The corresponding flow preserves the balanced property of the metric $\omega(t)$.
(b) Clearly its stationary points are solutions of the Hull-Strominger system.
(c) The flow exists at least for a short time, assuming that $|\alpha' Rm(\omega)|$ is small enough.

- It had been shown a while ago by Michelsohn that, given a positive $(n-1,n-1)$-form Ψ, there is a unique positive $(1,1)$-form ω so that $\omega^{n-1}=\Psi$. It turns out that ω can be expressed algebraically in Ψ. In fact, $\star\omega=\Psi$.

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After this most basic issue of short-time existence has been settled, the next concern is that Anomaly flows may turn out to be prohibitively messy, due to the fact that we have to deduce $\partial_t\omega$ from $\partial_t(\|\Omega\|\omega^2)$, and to the profusion of notions of Ricci curvature for general Hermitian metrics

$$R_{\bar kj}=R_{\bar kj}{}^p{}_p,\quad \tilde R_{\bar kj}=R^p{}_{p\bar kj},\quad R'_{\bar kj}=R_{\bar k}{}^p{}_{pj},\quad R''_{\bar kj}=R^p{}_{j\bar kp}$$

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The saving feature is that the conformally balanced condition $d(\|\Omega\|\omega^2)=0$, which is by design preserved by the Anomaly flows, turns out to be surprisingly powerful, as it implies that

$$R'_{\bar kj}=R''_{\bar kj}=\frac{1}{2}R_{\bar kj},\quad \tilde R_{\bar kj}=\frac{1}{2}R_{\bar kj}+\nabla^mT_{\bar kjm}$$

$$R=\tilde R,\quad R'=R''=\frac{1}{2}R$$

Theorem 2 Consider the anomaly flow with a *conformally balanced initial metric*. Then the flow is given by

$$\partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega}\Big\{ - \tilde{R}_{\bar{k}j} + g^{s\bar{r}} g^{p\bar{q}}\, T_{\bar{q}sj}\, \bar{T}_{p\bar{r}\bar{k}} - \alpha' g^{s\bar{r}} (R_{[\bar{k}s}{}^{\alpha}{}_{\beta} R_{\bar{r}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{k}s\bar{r}j}) \Big\}$$

Here $\tilde{R}_{\bar{k}j} = g^{p\bar{q}} R_{\bar{q}p\bar{k}j}$ is the Chern-Ricci tensor, $i\partial\omega = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k$ is the torsion tensor, and we have set $\Phi = \mathrm{Tr}(F \wedge F)$. The bracket $[,]$ denote anti-symmetrization in each of the two sets of barred and unbarred indices.

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Thus, although its original motivation is rather different, the Anomaly flow appears to be a higher order version of the well-known Kähler-Ricci flow defined by

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- The appearance of $\|\Omega\|^{-1}$, torsions, and quadratic terms in the curvature tensor;
- The resulting complication in the flow for the curvature. For example, the flow of the curvatures in the Ricci flow is given by

$$\partial_t R = \Delta R + R_{\bar{k}j} R^{j\bar{k}}, \qquad \partial_t R_{\bar{k}j} = \Delta R_{\bar{k}j} + R_{\bar{k}m\bar{p}q} R^{q\bar{p}m}{}_j$$

Thus the diffusion operator is $\Delta = g^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$. For the Anomaly flow, we find

$$\partial_t R_{\bar{k}j}{}^{\rho}{}_{\lambda} = \frac{1}{2\|\Omega\|_\omega} (\Delta R_{\bar{k}j}{}^{\rho}{}_{\lambda} - \frac{\alpha'}{2} g^{\rho\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} R_{\bar{k}j}{}^{\alpha}{}_{\beta}) + \cdots$$

and hence the diffusion operator is now

$$\delta R_{\bar{k}j}{}^{\rho}{}_{\lambda} \to \frac{1}{2\|\Omega\|} (\Delta(\delta R_{\bar{k}j}{}^{\rho}{}_{\delta}) + \frac{\alpha'}{2} g^{\rho\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} \delta R_{\bar{k}j}{}^{\alpha}{}_{\beta})$$

- The Kähler-Ricci flow preserves the Kähler property, so that $T_{\bar{k}jm} = 0$ for all time. The torsion does evolve in the Anomaly flow, and we find

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$$[\omega(t)] = [\omega(0)] - t\, c_1(X)$$

  as long as the $(1,1)$-class on the right hand side is > 0. The analogous statement in the Anomaly flow is

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But the positivity of a $(2,2)$-form appears to provide a lot less information as the positivity of a $(1,1)$-form.

- At this moment, we do not know of a canonical way of choosing a representative in the $(2,2)$-class $[\|\Omega\|\omega^2]$ whose evolution would be the analogue of the Kähler-Ricci flow on potentials, which is a parabolic Monge-Ampère equation and a very powerful tool.

Some Immediate Questions for Anomaly Flows

- A first major concern is, is the Anomaly flow the right parabolic flow for the Hull-Strominger system ?

  In general, for a given elliptic equation, say $F(D^2u) = e^\psi$, there are an infinite number of possible parabolic equations, for example

  $$\partial_t u = F(D^2u) - e^\psi \qquad \text{or} \qquad \partial_t u = \log F(D^2u) - \psi.$$

  However, they can behave quite differently. A well-known example is the Monge-Ampère equation, with $F(D^2u) = \det D^2 u$, where the equation with $\log F(D^2u)$ is much better behaved, because of concavity properties.

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- In the present case of Hull-Strominger systems, our choice of parabolic equation is dictated by the need to preserve the conformally balanced condition. Thus it is a particularly important issue to find out whether the anomaly flow is well-behaved.

- For example, there should be simple criteria for the development of singularities in a well-behaved parabolic flow. The solutions of the elliptic equation should be stationary points of the flow with a good basin of attraction.

Illustrative Special Cases of the Anomaly Flow

- The case $\alpha' = 0$: The most difficult quadratic terms in the curvature tensor won't occur. But the flow still presents the difficulties involving $\|\Omega\|_\omega$ and non-vanishing torsion, and it appears still at least as complicated as the Ricci flow. Interestingly, its fixed points satisfy the combined equations

$$i\partial\bar{\partial}\omega = 0, \qquad d(\|\Omega\|\omega^2) = 0$$

which can be easily seen to imply that ω is Kähler. Thus this flow could answer the question of whether a given conformally balanced metric can be deformed to a Kähler metric.

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- The case of Calabi-Eckmann-Goldstein-Prokushkin fibrations: this is the case where the elliptic equation was solved by Fu and Yau. So it is important to find out whether the anomaly flow can at least recapture this case. We shall see that it can, and even though it requires a different set of techniques, it will prove in a way to be even more robust than the Monge-Ampère equations techniques used by Fu and Yau.

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- The case of $SL(2, \mathbf{C})$: This case allows a detailed analysis of dependence on initial data, as the invariance reduces the Anomaly flow to a system of ODE's. In particular, it is instructive to determine for which initial data will the flow converge to the solution found by Fei and Yau.

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Theorem 3 Assume that the flow exists for $t \in [0, \frac{1}{A}]$ and that

$$|Rm| + |DT| + |T|^2 \leq A, \quad z \in X.$$

Then for any $k \in \mathbf{N}$, there exists a constant C_k depending on a uniform lower bound for $\|\Omega\|_\omega$ so that

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The case $\alpha' = 0$

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This implies that the flow exists for all time $t \geq 0$, unless there is a finite time T and a sequence (z_j, t_j) with $t_j \to T$, and either $\|\Omega(z_j, t_j)\|_{\omega_j} \to 0$, or

$$(|Rm| + |DT| + |T|^2)(z_j, t_j) \to \infty.$$

The case of CEGP fibrations

Basic facts about Calabi-Eckmann-Goldstein-Prokushkin fibrations

Let $(X, \hat{\omega})$ be a Calabi-Yau surface, with Ricci-flat metric $\hat{\omega}$, and holomorphic form Ω normalized so that $\|\Omega\|^2_{\hat{\omega}} = 1$. Given any two forms $\omega_1, \omega_2 \in 2\pi H^2(X, \mathbf{Z})$ with $\omega_1 \wedge \hat{\omega} = \omega_2 \wedge \hat{\omega} = 0$, Goldstein and Prokushkin (2004) construct a toric fibration $\pi : Y \to X$, equipped with a $(1,0)$-form θ on Y satisfying $\partial\theta = 0$, $\bar{\partial}\theta = \pi^*(\omega_1 + i\omega_2)$. Furthermore, the form

$$\Omega_Y = \sqrt{3}\,\Omega \wedge \theta$$

is a holomorphic nowhere vanishing $(3,0)$-form on Y, and for any scalar function u on X, the $(1,1)$-form

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The Fu-Yau equation

Look now for a solution of the Strominger system on $Y, \pi^*(E)$ under the form $(\omega_u, \pi^*(H))$, where H is a Hermitian-Einstein metric on a stable vector bundle $E \to (X, \hat{\omega})$. Then the only equation left to solve is the anomaly equation,

$$i\partial\partial\omega_u - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\omega_u) \wedge Rm(\omega_u) - F \wedge F) = 0.$$

In a key calculation, Fu and Yau (2006) showed that this equation descends to an equation on X, which they showed can be solved if and only if $\int_X \mu = 0$,

$$i\partial\bar{\partial}(e^u \hat{\omega} - e^{-u}\rho) + \frac{\alpha'}{4} i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u + \mu = 0.$$

Theorem 4 Consider the anomaly flow

$$\partial_t(\|\Omega\|_\chi \chi^2) = i\partial\bar{\partial}\chi - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\chi) \wedge Rm(\chi) - F \wedge F)$$

on a Calabi-Eckmann-Goldstein-Prokushkin fibration $\pi : Y \to X$, with initial data $\chi(0) = \pi^*(M\hat{\omega}) + i\theta\bar{\theta}$, where M is a positive constant. Assume the integrability condition on μ (which depends only on the Calabi-Eckmann-Goldstein-Prokushkin data). Then there exists $M_0 > 0$, so that for all $M \geq M_0$, the flow exists for all time, and converges to a metric ω_∞ with $(\omega_\infty, \pi^*(H))$ satisfying the Strominger system.

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This theorem holds for $\alpha' > 0$ and $\alpha' < 0$. We formulated it in terms of flows on the 3-fold Y. But of course the advantage of Goldstein-Prokushkin fibrations is that it descends to a flow on the surface X, and the theorem which is equivalent to Theorem 5 and that we shall actually prove is the following:

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Theorem 4' Let $(X\,\hat\omega)$ be a Calabi-Yau surface, with a Ricci-flat metric $\hat\omega$ and a holomorphic $(2,0)$-form Ω normalized to $\|\Omega\|_{\hat\omega} = 1$. Consider the flow

$$\partial_t\omega = -\frac{1}{2\|\Omega\|_\omega}\left(\frac{R}{2} - |T|^2 - \frac{\alpha'}{4}\sigma_2(iRic_\omega) + 2\alpha'\frac{i\partial\bar\partial(\|\Omega\|_\omega\rho)}{\omega^2} - 2\frac{\mu}{\omega^2}\right)\omega$$

with an initial metric of the form $\omega(0) = M\hat\omega$. Assume the integrability condition on μ. Then there exists a constant M_0 so that, for all $M \geq M_0$, the flow exists for all time and converges exponentially fast to a metric ω_∞ satisfying the Fu-Yau equation

$$i\partial\bar\partial(\omega_\infty - \frac{\alpha'}{4}\|\Omega\|_{\omega_\infty}\rho) - \frac{\alpha'}{8}Ric_{\omega_\infty}\wedge Ric_{\omega_\infty} + \mu = 0.$$

General remarks about the proof

- We assume that $|\alpha' Ric_\omega| << 1$, so that the diffusion operator

$$\Delta_F = F^{p\bar{q}} \nabla_p \nabla_{\bar{q}}, \qquad F^{p\bar{q}} = g^{p\bar{q}} + \alpha' \|\Omega\|_\omega^3 \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2} (R g^{p\bar{q}} - R^{p\bar{q}})$$

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- The estimates for the curvature, torsion, and their higher order derivatives are obtained by applying the maximum principle to suitable barrier functions. The computations are necessarily very complicated, but here the key feature of CEGP fibrations is that they are all governed by the same leading diffusion operator, which is the operator Δ_F defined above.

The Anomaly Flow on Unimodular Lie Groups

For simplicity, we consider now $Y = SL(2, \mathbf{C})$, with e_a a basis for its Lie algebra, and $c^d{}_{ab}$ the structure constants

$$[e_a, e_b] = c^d{}_{ab} e_d$$

If e^a is the dual basis of left-invariant holomorphic forms on Y, then $\Omega = e^1 \wedge e^2 \wedge e^3$ is a nowhere vanishing holomorphic 3-form. Furthermore, if $\omega = \sum_a i g_{\bar{a}b} e^b \wedge \overline{e^a}$, then $\|\Omega\|$ is constant, and $d\omega^2 = 0$. Thus if we let E be a flat holomorphic vector bundle over Y, then the Hull-Strominger system reduces to the single equation

$$i\partial\bar{\partial}\omega - \frac{\alpha'}{4}\mathrm{Tr}(Rm \wedge Rm) = 0.$$

Using an ansatz of Biswas-Mukherjee and Andreas-Garcia-Fernandez, Fei and Yau were able to find a solution of this equation under the form

$$g_{\bar{a}b} = 2\beta\,\delta_{ab}, \quad \beta = \frac{1}{2}\alpha'\kappa^2(2\kappa - 1)$$

if the connection is taken to be any connection $\nabla^{(\kappa)}$ on the Gauduchon line, but the Chern and the Lichnerowicz connections. A key step in their arguments is to show that, even though Rm is not an $End(T^{1,0}(Y))$-valued $(1,1)$-form, nevertheless the expression $\mathrm{Tr}(Rm \wedge Rm)$ is a $(2,2)$-form.

We would like to examine the behavior of the Anomaly flow

$$\partial_t(\|\Omega\|\omega^2) = \pm(i\partial\bar{\partial}\omega - \frac{\alpha'}{4}\mathrm{Tr}(Rm \wedge Rm))$$

in this setting, especially the dependence of the flow on the initial data. (Both signs are allowed, because this is an ODE system, and parabolicity is not relevant. We shall actually state our results with the sign $-$, as they are simpler to state.) Consider an initial data in the diagonal form $g_{\bar{a}b} = \lambda_a \delta_{ab}$, $\lambda_a > 0$. Then this form is preserved along the flow, and the flow can be expressed as

$$\partial_t \lambda_1 = \frac{(\lambda_1\lambda_2\lambda_3)^{\frac{1}{2}}}{2}(\beta(\frac{2}{\lambda_1} + \frac{\lambda_1}{\lambda_2^2} + \frac{\lambda_1}{\lambda_3^2}) - \frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2})$$

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Theorem 5 (a) The flow admits a unique stationary point, which is the one found by Fei and Yau, $\lambda_1 = \lambda_2 = \lambda_3 = 2\beta$.
(b) The stationary point is hyperbolic, with eigenvalues $+1, +1, -2$. In particular, the flow is not asymptotically stable.
(c) Initial data of the form $\lambda_1(0) = \lambda_2(0) = \lambda_3(0)$ are preserved, and the flow converges then to the fixed stationary point.
(d) If say, $\lambda_1 > \lambda_2$ and $\lambda_1 > \beta$, then the ratio λ_1/λ_2 is actually monotone increasing, and the flow cannot converge.
(e) If $\beta < 0$, then the flow terminates in finite time $T < (2|\beta|)^{-1}\lambda_1(0)^{\frac{1}{2}}$.