The Squeezing Function

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Some background

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Theorem

(Liu-Sun-Yau and Yeung) Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of compact Riemann surfaces of genus g with n punctures. Then g_C, g_K and g_B are all quasi-isometric on $\mathcal{T}_{g,n}$.

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Proof.

Bers' embedding theorem: there are constants $0 < a < b < \infty$ such that given a point $p \in \Omega := \mathcal{T}_{g,n}$, there exists an embedding $\phi : \mathcal{T}_{g,n} \to \mathbb{C}^n$ such that

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(ii) $\mathbb{B}^n_a \subset \phi(\Omega) \subset \mathbb{B}^n_b$.

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Definition
(Lin-Sun-Yau) Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. If for any $p \in \Omega$ there exists an embedding $\phi : \Omega \to \mathbb{C}^n$ satisfying (i) and (ii), then Ω is said to be *holomorphic homogenous regular*.

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Problem

Which (pseudoconvex) domains in $\mathbb{C}^n$ are uniformly squeezing?

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(iii) strongly pseudoconvex domains (F. Deng, Q. Guan, L. Zhang and K. Diederich, J. E. Fornæss, E. F. Wold)

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$$\phi(z) = 0 \text{ and } \mathbb{B}_a^n \subset \phi(\Omega) \subset \mathbb{B}_b^n. \tag{1}$$

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Theorem

(Deng-Guan-Zhang and Fornæss-Wold) Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain. Then

$$\lim_{z \to b\Omega} S_\Omega(z) = 1. \tag{2}$$

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Theorem

(Zimmer) For all $n \in \mathbb{N}$ *and* $\alpha > 0$ *there exists a constant* $\epsilon = \epsilon(n, \alpha)$ *such that the following holds: if* $\Omega \subset \mathbb{C}^n$ *is a bounded* $C^{2,\alpha}$*-smooth convex domain and if* $S_\Omega(z) \geq 1 - \epsilon$ *for* $z \in \Omega \setminus K$*, with* $K \subset \Omega$ *compact, then* Ω *is strictly psedoconvex.*

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Theorem

(Zimmer) Let $\Omega_1 \subset \mathbb{C}^n$ be strictly pseudoconvex, and let Ω_2 be a convex domain with $C^{2,\alpha}$-smooth boundary. Then if Ω_1 is biholomorphic to Ω_2 we have that Ω_2 is strictly pseudoconvex.

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Theorem

(Kim-Joo) If $\Omega \subset \mathbb{C}^2$ is a C^∞-smooth boundary of finite type, and if $\lim_{z \to b\Omega} S_\Omega(z) = 1$, then Ω is strictly pseudoconvex.

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Let $\Omega_1 \subset \mathbb{C}^n$ be a strictly pseudoconvex domain, and let $\Omega_2 \subset \mathbb{C}^n$ be a domain of class $C^{2,\alpha}$ for $\alpha \geq 0$. If Ω_2 is biholomorphic to Ω_1, is Ω_2 strictly pseudoconvex?

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We now consider again the Kobayashi and Carathéodory metrics g_K and g_C on a bounded domain Ω. Then

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Corollary

(Fornæss-Wold)

$$\frac{g_C(z,v)}{g_K(z,v)} \ge 1 - C \cdot \mathrm{dist}(z, b\Omega). \tag{4}$$

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Theorem

(Wold) In the normal direction we have that

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Theorem

(Diederich-Fornæss-Wold) Let $\Omega \subset \mathbb{C}^n$ be a C^2-smooth domain. Then if there is a sequence of points $z_j \to b\Omega$ and a sequence $\epsilon_j \to 0$, with

$$S_\Omega(z_j) \geq 1 - \epsilon_j \cdot \mathrm{dist}(z, b\Omega), \tag{6}$$

then Ω is biholomorphic to the unit ball.

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Let $\Omega \subset \mathbb{C}^n$ ($n \geq 2$) be a $C^{k,\alpha}$-smooth strictly pseudoconvex domain, $k \geq 2, \alpha \geq 0$. What is the optimal estimate for

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Problem

Let $A = A(a,b) = \{z \in \mathbb{C} : a < |z| < b\}$ *for* $0 < a < b < \infty$*. Find a precise formula for* $S_A(z) = S_A(|z|)$*.*