Weighted Bergman spaces of domains with Levi-flat boundary: two case studies (arXiv:1703.08165)

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Introduction

X: **cx. mfd.,** $\Omega \Subset X$: **domain,** $\partial\Omega = M$: **smooth real hypersurface.**

Levi problem in a generalized sense

What kinds of geometry of M control holomorphic functions on Ω?

Classical results:
(Oka, Bremermann, Norguet) $X = \mathbb{C}^n$, M: ψ**c** $\implies$ Ω **is Stein.**
(Grauert) M: **s**ψ**c** $\implies$ Ω **is a proper modification of a Stein space.**

M: **Levi-flat, i.e., foliated by cx. hypersurfaces of** X **(Levi foliation).**

Levi problem for domains with Levi-flat boundary

What kinds of dynamical property of the Levi foliation control holomorphic functions on Ω?

Let us see an example.

Holomorphic disk bundles and flat circle bundles

Σ: closed Riemann surface of genus ≥ 2. Fix its uniformization $\Sigma \simeq \mathbb{D}/\pi_1(\Sigma)$. Take a representation $\rho\colon \pi_1(\Sigma) \to \mathrm{Aut}(\mathbb{D}) \subset \mathrm{Aut}(\mathbb{CP}^1)$, $\mathrm{Diffeo}^+(S^1)$.

Definition (suspension)

$$
\begin{aligned}
X_\rho &:= \Sigma \times_\rho \mathbb{CP}^1 && := \mathbb{D} \times \mathbb{CP}^1/(z, w) \sim (\gamma z, \rho(\gamma) w) \text{ for } \gamma \in \pi_1(\Sigma).\\
\Omega_\rho &:= \Sigma \times_\rho \mathbb{D} && := \mathbb{D} \times \mathbb{D}/\pi_1(\Sigma).\\
M_\rho &:= \Sigma \times_\rho S^1 && := \mathbb{D} \times S^1/\pi_1(\Sigma).
\end{aligned}
$$

We regard X_ρ, Ω_ρ, M_ρ as $\mathbb{CP}^1$-bundle, $\mathbb{D}$-bundle, S^1-bundle over Σ respectively by first projection. They are flat bundles in the sense that horizontal disks $\mathbb{D} \times \{t\}$ ($t \in \mathbb{CP}^1$) give a holomorphic foliation $\mathcal{F}_\rho$ by Riemann surfaces on X_ρ and preserve Ω_ρ and M_ρ. Hence, M_ρ is a Levi-flat real hypersurface.

Theorem of Grauert, Diederich–Ohsawa

Σ, ρ: as before. New representations called conjugation are given by $\pi_1(\Sigma) \to \mathrm{Aut}(\mathbb{D})$, $\gamma \mapsto \alpha \circ \rho(\gamma) \circ \alpha^{-1}$ for each $\alpha \in \mathrm{Aut}(\mathbb{D})$. Conjugations do not change complex/CR str. of X_ρ, Ω_ρ, M_ρ.

Theorem (Grauert, Diederich–Ohsawa)

1. **ρ is not to conjugate to rotations**
   $\Longrightarrow$ Ω_ρ is a proper modification of a Stein space.
2. **ρ is conjugate to rational rotations**
   $\Longrightarrow$ Ω_ρ is holomorphically convex, but not Stein.
3. **ρ is conjugate to rotations including an irrational rotation**
   $\Longrightarrow$ $\dim \mathcal{O}(\Omega_\rho) = 1$.

Examples

1. $\rho(\gamma) = \gamma$ **(Deck transformation), $[\rho] \in \mathcal{T}(\Sigma)$ (Teichmüller space).**
2. $\rho(\gamma) \equiv \mathrm{id}_{\mathbb{D}} \implies \Omega_\rho = \Sigma \times \mathbb{D}$. $\mathcal{O}(\Omega_\rho) = \mathcal{O}(\mathbb{D})$.

General Problem

Σ, ρ: as before.

Problem

Study what kinds of dynamical property of ρ control holomorphic functions on Ω_ρ and CR functions on M_ρ with growth and regularity conditions respectively.

Motivated by (A.–Brinkschulte): a curvature restriction for hypothetical smooth closed Levi-flat in $\mathbb{CP}^2$ was obtained by studying the Bergman space of its complement.

Today's goal

Study the weighted Bergman / Hardy spaces for the case $\rho_0(\gamma) = \gamma$ (Deck transformation).

Bergman space

$\Omega \Subset X$, $M = \partial\Omega$: **smooth.** dV: **volume form on** X.
$\delta : X \to \mathbb{R}$, $\Omega = \{\delta > 0\}$, $d\delta \neq 0$ **on** $\partial\Omega$. $\alpha \geq -1$.

Definition (Weighted Bergman space, Hardy space)

$$A^2_\alpha(\Omega) := \{f \in \mathcal{O}(\Omega) \mid \langle f, f\rangle_\alpha < \infty\},$$
$$\langle f, g\rangle_\alpha := \begin{cases} \int_\Omega f\overline{g}\delta^\alpha dV/\Gamma(\alpha+1) & \textbf{for } \alpha > -1, \\ \lim_{\alpha \searrow -1}\langle f, g\rangle_\alpha & \textbf{for } \alpha = -1. \end{cases}$$

The Hardy space ($\alpha = -1$) is the space of holomorphic functions which have L^2 boundary value. The boundary values are CR functions on M.

CR functions on Levi-flats

$(M, \mathcal{F})$: **compact Lev-flat CR manifold, i.e.,**
M: **compact real manifold,** $\mathcal{F}$: **real codimension one non-singular foliation by complex manifolds.**

Definition (CR function)

A CR function is a function which are holomophic along $\mathcal{F}$**.**

Note that transverse regularity are not guaranteed.

(Inaba) Any continuous CR function on M **are leafwise constant. Hence, continuous CR functions are constant if** $\mathcal{F}$ **has a dense leaf.**

(Ohsawa–Sibony, Hsiao–Marinescu; A.) The finite/infinite dimensionality of the space of CR sections of a fixed smooth CR line bundle depend on transverse regularity we require.

Liouvilleness

Σ: as before, $\rho_0 \colon \pi_1(\Sigma) \to \mathrm{Aut}(\mathbb{D}), \gamma \mapsto \gamma$ (Deck transformation).
$X := X_{\rho_0}$, $\Omega := \Omega_{\rho_0}$, $M := M_{\rho_0}$, $\Omega' := X_{\rho_0} \setminus \overline{\Omega_{\rho_0}}$, $\mathcal{F} := \mathcal{F}_{\rho_0}$.

Corollary of E. Hopf's ergodicity theorem

Ω, Ω' are Liouville, i.e., $\dim \mathcal{O} \cap L^\infty(\Omega) = \dim \mathcal{O} \cap L^\infty(\Omega') = 1$.

Proof.

- **Let $f \in \mathcal{O} \cap L^\infty(\Omega)$ or $\mathcal{O} \cap L^\infty(\Omega')$. Consider the boundary value of f, which is leafwise holomorphic.**
- **M is diffeomorphic to the unit tangent bundle of Σ, and $\mathcal{F}$ is isomorphic to the unstable foliation of geodesic flows of Σ.**
- **The ergodicity of geodesic flow w.r.t. the Lebesgue measure implies that any bounded leafwise harmonic function is a.e. constant. Hence, f is constant.** □

Refinement of E. Hopf's ergodicity theorem by L. Garnett gives $\dim A^2_{-1}(\Omega) = \dim A^2_{-1}(\Omega') = 1$. **Another proof based on a property of ${}_3F_2$ will be given later.**

Hardy space and weighted Bergman spaces of Ω

$$\mathcal{O}(\Omega) \simeq \{f \in \mathcal{O}(\mathbb{D} \times \mathbb{D}) \mid f(z, w) = f(\gamma z, \gamma w), \gamma \in \pi_1(\Sigma)\}.$$

$$\mathcal{O}(\Omega') \simeq \{f \in \mathcal{O}(\mathbb{D} \times \mathbb{D}) \mid f(z, w) = f(\gamma z, \overline{\gamma} w), \gamma \in \pi_1(\Sigma)\}.$$

Corollary of the technique of Berndtsson–Charpentier

$$\dim A^2_\alpha(\Omega) = \dim A^2_\alpha(\Omega') = \infty, \quad \forall \alpha > -1/2.$$

(Fu–Shaw, A.–Brinkschulte; A) The $1/2$ is the Diederich–Fornaess index of Ω and Ω', and the best possible as the DF index.

(A.) The $1/2$ roughly corresponds to the fact that the foliated harmonic measure of the Levi foliation belongs to the Lebesgue measure class, from which the ergodicity follows.

Main result

Main result (A., arXiv:1703.08165 + in preparation.)

$$\exists I: \bigoplus_{n=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes n}) \hookrightarrow \mathcal{O}(\Omega), \quad \exists I': \bigoplus_{n=0}^{\infty} \mathrm{Ker}(\Delta - \lambda_n I) \hookrightarrow \mathcal{O}(\Omega')$$

where Δ is the Laplace–Beltrami operator of Σ w.r.t. Poincaré metric, and

$$H^0(\Sigma, K_\Sigma^{\otimes n}) := \{\textbf{holomorphic } n\textbf{-differential on } \Sigma \psi = \psi(\tau)(d\tau)^{\otimes n}\},$$
$$\mathrm{Ker}(\Delta - \lambda_n I) := \{f : \Sigma \to \mathbb{C} \mid \Delta f = \lambda_n f\}.$$

The images of I and I' are dense in compact open topology, and, moreover, contained in $A^2_\alpha(\Omega)$ and $A^2_\alpha(\Omega')$ resp. for $\forall \alpha > -1$.

Construction of I and I'

- Ω **contains a divisor** $D = \{(z,z) \mid z \in \mathbb{D}\}/\pi_1(\Sigma) \simeq \Sigma$.
- Ω' **contains a totally real** $D' := \{(z,\bar{z}) \mid z \in \mathbb{D}\}/\pi_1(\Sigma) \approx \Sigma$**, and** Ω' **is characterized as the Grauert tube of maximal radius.**

$I\colon \bigoplus_{n=0}^{\infty} H^0(\Sigma, K_\Sigma^{\otimes n}) \hookrightarrow \mathcal{O}(\Omega)$ **is given by**

$$I(\psi)(z,w) := \int_z^w \frac{1}{B(n,n)} \left(\frac{(w-\tau)(\tau-z)}{(w-z)d\tau} \right)^{\otimes(n-1)} \psi(\tau)(d\tau)^{\otimes n}$$

for $\psi \in H^0(\Sigma, K_\Sigma^{\otimes n})$**,** $n \geq 1$**, where we write** $\psi = \psi(\tau)(d\tau)^{\otimes n}$ **on the uniformizing coordinate** $\tau \in \mathbb{D}$**. The well-definedness follows from a property of cross ratios.**

$I'\colon \bigoplus_{n=0}^{\infty} \mathrm{Ker}(\Delta - \lambda_n I) \hookrightarrow \mathcal{O}(\Omega')$ **is given by the analytic continuation of** $f \in \mathrm{Ker}(\Delta - \lambda_n I)$ **regarded as a real-analytic function on** D' **to** $\mathbb{D} \times \mathbb{D}$**. The well-definedness follows from a known fact on PDE.**

Outline of proof of integrability

I is obtained by optimal L^2-jet extension from D to Ω. We may regard $\psi \in H^0(\Sigma, K_\Sigma^{\otimes n})$ as an n-th order jet of holo func along D via

$$K_\Sigma \simeq T_\Sigma^* \simeq T_D^* \simeq N_{D/\Omega}^*, \quad D = \{(z,z) \mid z \in \mathbb{D}\}/\pi_1(\Sigma) \subset \Omega,$$

Step 1. We work on a non-holomorphic coordinate of $\mathbb{D}_z \times \mathbb{D}_w$ given by (z,t), $t := (w-z)(1-\overline{z}w)^{-1}$. Expand $f = f(z,w) \in \mathcal{O}(\Omega)^{\pi_1(\Sigma)}$ as $f = \sum_{n=0}^\infty f_n(z)t^n$, then $\{f_n\}$ enjoys

$$\frac{\partial f_n}{\partial \overline{z}} + \frac{nz}{1-|z|^2} f_n + \frac{n-1}{1-|z|^2} f_{n-1} = 0.$$

Put $\varphi_n := f_n(z)\left(\frac{\sqrt{2}dz}{1-|z|^2}\right)^{\otimes n} \in C^{(0,0)}(\Sigma, K_\Sigma^n)$. Then $\{\varphi_n\}$ satisfy

$$\overline{\partial}\varphi_0 = 0, \quad \overline{\partial}\varphi_n = -\frac{n-1}{\sqrt{2}}\varphi_{n-1} \otimes \omega \ (n \geq 1)$$

where $\omega = 2dz \otimes d\overline{z}/(1-|z|^2)^2$.

Outline of proof of integrability – continued

Step 2. Let $\psi \in H^0(\Sigma, K_\Sigma^{\otimes N})$. Put$\varphi_n := 0$ for $n < N$ and $\varphi_N := \psi$. We pick the L^2 minimal solution to

$$\overline{\partial}\varphi_n = -\frac{n-1}{\sqrt{2}}\varphi_{n-1} \otimes \omega$$

inductively and determine φ_n for $n > N$. The spectral decomposition of the complex laplacian tells us the L^2 minimal solutions are

$$\begin{aligned}\varphi_{N+m} &= \overline{\partial}^*_{N+m} G^{(1)}_{N+m} \left(-\frac{N+m-1}{\sqrt{2}}\varphi_{N+m-1} \otimes \omega \right) \\ &= -\frac{\sqrt{2}(N+m-1)}{m(2N+m-1)}\overline{\partial}^*_{N+m} \left(\varphi_{N+m-1} \otimes \omega\right)\end{aligned}$$

where $\overline{\partial}^*_n$ is the formal adjoint of $\overline{\partial} : C^{(0,0)}(\Sigma, K_\Sigma^{\otimes n}) \to C^{(0,1)}(\Sigma, K_\Sigma^{\otimes n})$ and $G^{(1)}_n$ is the Green operator on $C^{(0,1)}(\Sigma, K_\Sigma^{\otimes n})$.

Outline of proof of integrability – continued[3]

Step 3. For $\alpha > -1$, the convergence of $f = \sum_{n=0}^{\infty} f_n(z) t^n$, $\varphi_n = f_n(z) \left(\frac{\sqrt{2} dz}{1-|z|^2} \right)^{\otimes n}$ in $L^2_\alpha(\Omega)$ follows from

$$\begin{aligned}
\|f\|_\alpha^2 &= \pi \sum_{n=0}^{\infty} \|\varphi_n\|^2 \frac{\Gamma(n+1)}{\Gamma(n+\alpha+2)} \\
&= \pi \sum_{m=0}^{\infty} \|\varphi_{N+m}\|^2 \frac{\Gamma(N+m+1)}{\Gamma(N+m+\alpha+2)} \\
&= \pi \|\psi\|^2 \sum_{m=0}^{\infty} \frac{\Gamma(N+m+1)}{\Gamma(N+m+\alpha+2)} \frac{(2N-1)!}{\{(N-1)!\}^2} \frac{\{(N+m-1)!\}^2}{m!(2N+m-1)!} \\
&= \pi \|\psi\|^2 \frac{\Gamma(N+1)}{\Gamma(N+2+\alpha)} {}_3F_2 \left(\begin{matrix} N+1, N, N \\ 2N, N+2+\alpha \end{matrix}; 1 \right) < \infty
\end{aligned}$$

where we used $\delta := 1 - \left| \frac{w-z}{1-\bar{z}w} \right|^2$, $dV = \frac{4}{|1-\bar{z}w|^4} \frac{i}{2} dz \wedge d\bar{z} \wedge \frac{i}{2} dw \wedge d\overline{w}$.
Similar computation shows $f \in \mathcal{O}(\Omega)$, and the Liouvilleness of Ω.

Outline of proof of integrability – continued[4]

Step 4. **Want to show**

$$\sum_{n=0}^{\infty} f_n(z)t^n = \int_z^w \frac{1}{B(N,N)} \left(\frac{(w-\tau)(\tau-z)}{(w-z)d\tau} \right)^{\otimes(N-1)} \psi(\tau)(d\tau)^{\otimes N}.$$

Enough to show the desired equality on $\{0\} \times \mathbb{D}$.

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n(0)t^n &= \frac{(2N-1)!}{(N-1)!} \sum_{m=0}^{\infty} \frac{(N+m-1)!}{(2N+m-1)!} \frac{1}{m!} \frac{\partial^m \psi}{\partial z^m}(0) t^{N+m} \\
&= \frac{(2N-1)!}{(N-1)!} t^N \int_0^1 dt_N \ldots \int_0^{t_3} dt_2 \int_0^{t_2} t_1^{N-1} \psi(tt_1) dt_1 \\
&= \frac{(2N-1)!}{(N-1)!} t^N \int_0^1 \frac{t_1^{N-1}(1-t_1)^{N-1}}{(N-1)!} \psi(tt_1) dt_1 \\
&= \int_0^t \frac{1}{B(N,N)} \left(\frac{(t-\tau)\tau}{t} \right)^{(N-1)} \psi(\tau) d\tau. \qquad \square
\end{aligned}$$

A Forelli–Rudin construction

The reproducing kernel B_α of A^2_α is called the weighted Bergman kernel. $B_\alpha(z,w) = \sum_{j=1}^{\infty} e_j(z)\overline{e_j(w)}$ for any orthonormal basis of A^2_α $\{e_1, e_2, \dots\}$.

Corollary

The weighted Bergman kernel of Ω is

$$B_\alpha((z,w);(z',w')) = \frac{\Gamma(\alpha+2)}{\pi^2(4g-4)} +$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{c_{n,\alpha}}\frac{1}{B(n,n)^2}\int_{\tau\in\overline{zw}}\int_{\tau'\in\overline{z'w'}}\frac{B_n(\tau,\tau')(d\tau\otimes\overline{d\tau'})^{\otimes n}}{([w,\tau,z]\otimes\overline{[w',\tau',z']})^{\otimes(n-1)}}.$$

where g is the genus of Σ, $B_n(\tau,\tau')(d\tau\otimes\overline{d\tau})^{\otimes n}$ is the Bergman kernel of holomorphic n-differentials, i.e. of $H^0(\Sigma, K_\Sigma^{\otimes n})$, and

$$c_{n,\alpha} = \frac{\Gamma(n+1)}{\Gamma(n+2+\alpha)}\,{}_3F_2\left(\begin{matrix} n+1, n, n \\ 2n, n+2+\alpha \end{matrix}; 1\right), [w,\tau,z] = \frac{(w-z)d\tau}{(w-\tau)(\tau-z)}.$$