RECENT DEVELOPMENT IN NEVANLINNA THEORY AND DIOPHANTINE APPROXIMATION

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Abstract. In this set of notes, we'll give a survey with details about the recent development about the quantitive results (in the spirit of the Second Main type Theorem) for holomorphic mappings from the complex plane into algebraic varieties intersecting divisors. Our method is to use the classical H. Cartan's Cecond Main Theorem.

1. Logarithmic derivative lemma and Nevanlinna's Second Main Theorem for Meromorphic Functions

Let $f \not\equiv 0$ be a meromorphic function on $\mathbf{C}$. Let us denote by $n_f(r, a)$ the number of solutions of the equation $f(z) = a$ in the disk $|z| < r$, counting multiplicity. Here $a \in \mathbf{C}$. By the Argument Principle and the Cauchy-Riemann equations we have

$$n_f(r, a) - n_f(r, \infty) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f'}{f-a} dz = \frac{r}{2\pi}\frac{d}{dr}\int_0^{2\pi} \log|f(re^{i\theta}) - a| d\theta.$$

Let

$$N_f(r, \infty) = \int_{r_0}^r n_f(t, \infty)\frac{dt}{t}$$

and $N_f(r, a) = N_{1/(f-a)}(r, \infty)$. Then we have

$$\int_0^{2\pi} \log|f(re^{i\theta}) - a| d\theta = N_f(r, a) - N_f(r, \infty) + O(1). \tag{1}$$

Define the Nevanlinna's proximity function $m_f(r, \infty)$ by

$$m_f(r, \infty) = \int_0^{2\pi} \log^+ |f(re^{i\theta})|\frac{d\theta}{2\pi},$$

where $\log^+ x = \max\{0, \log x\}$. For any complex number a, let

$$m_f(r, a) = m_{1/(f-a)}(r, \infty).$$

The **Nevanlinna's characteristic function** of f is defined by

$$T_f(r) = m_f(r, \infty) + N_f(r, \infty).$$

$T_f(r)$ measures the growth of f. For example: $T_f(r) = O(1)$ if and only if f is constant; $T_f(r) = O(\log r)$ if and only if f is a rational function. (1) gives

The First Main Theorem: $T_f(r) = m_f(r,a) + N_f(r,a) + O(1)$..

The proof of the Second Main Theorem is based on the so-called logarithmic derivative lemma.

Theorem 1.1 (Logarithmic Derivative Lemma (LDL). *Let $f(z)$ be a meromorphic function. Then, for $\delta > 0$*

$$\int_0^{2\pi} \log^+ \left| \frac{f'}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq \left(1 + \frac{(1+\delta)^2}{2}\right) \log^+ T_f(r) + \frac{\delta}{2} \log r + O(1) \|_{E(\delta)}$$

where $\|_E$ means that the inequality holds for all r except the set E with finite Lebesgue measure.

To prove LDL, we recall

Lemma 1.1 (Calculus Lemma). *Let T be a strictly nondecreasing function of class C^1 defined on $(0,\infty)$. Let $\gamma >$ be a number such that $T(\gamma) \geq e$. Let ϕ be a strictly positive nondecreasing function such that*

$$\int_e^\infty \frac{1}{t\phi(t)} dt = c_0(\phi) < \infty.$$

Then the inequality

$$T'(r) \leq T(r)\phi(T(r))$$

holds for all $r \geq \gamma$ outside a set of Lebesgue measure $\leq c_0(\phi)$.

Proof. Let $A \subset [\gamma, \infty)$ be the set of r such that $T'(r) \geq T(r)\phi(T(r))$. Then

$$\text{meas}(A) = \int_A dr \leq \int_\gamma^\infty \frac{T'(r)}{T(r)\phi(T(r))} dr = \int_e^\infty \frac{dt}{t\phi(t)} = c_0(\phi),$$

which proves the lemma.

The typical use of the calculus lemma is as follows: Let Γ be a non-negative function on $\mathbf{C}$, define

$$T_\Gamma(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} \Gamma \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}.$$

Then we have, for every $\delta > 0$,

$$2\int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} \leq (T_\Gamma(r))^{1+\delta} (br T_\Gamma^{1+\delta}(r))^\delta \|_{E(\delta)}. \ (*)$$

Proof of DLD. For $w \in \mathbf{C}$, we define an surface element as follows:

$$\Phi = \frac{1}{(1 + \log^2 |w|)|w|^2} \frac{\sqrt{-1}}{4\pi^2} dw \wedge d\bar{w}.$$

This is a $(1,1)$ form on $\mathbf{C}$ with singularities at $w=0,\infty$. By computation

$$\int_{\mathbf{C}}\Phi=\int_{\mathbf{C}}\frac{1}{(1+\log^2 r)|r|^2}\frac{1}{2\pi^2}rdrd\theta=1.$$

By the change of the variable formula (or notice that $n_f(t,w)$ is the number of times that the point $w\in\mathbf{C}$ is covered by $f(D(t))$, where $D(t)=\{|\zeta|<t\}$) we have (consulting Theorem 2.14 of the book "Functions of one complex variable" by J.B. Conway)

$$\int_{\triangle(t)}f^*\Phi=\int_{w\in\mathbf{C}}n_f(t,w)\Phi(w).$$

Thus, by letting $\mu(r):=\int_1^r\frac{dt}{t}\int_{\triangle(t)}f^*\Phi$, we have

$$\begin{aligned}\mu(r)&=\int_1^r\frac{dt}{t}\int_{\triangle(t)}\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\frac{\sqrt{-1}}{4\pi^2}dz\wedge d\bar{z}\\&=\int_{w\in\mathbf{C}}\int_1^r\frac{dt}{t}n_f(t,w)\Phi(w)=\int_{w\in\mathbf{C}}N_f(r,w)\Phi(w)\le T_f(r)+O(1)\end{aligned}$$

where the last inequality holds is due to the the First Main Theorem. By the calculus lemma (see (*) above), we get

$$\frac{1}{\pi}\int_{|z|=r}\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\frac{d\theta}{2\pi}\le(\mu(r))^{(1+\delta)^2}r^\delta b^\delta\|_{E_\delta}$$

where b is a constant. By making use of this, the Calculus lemma and the concavity of the logarithm function, we carry the following computations:

$$\begin{aligned}\int_0^{2\pi}\log^+\left|\frac{f'}{f}(re^{i\theta})\right|\frac{d\theta}{2\pi}&=\frac{1}{4\pi}\int_{|z|=r}\log^+\left(\frac{|f'|^2}{(1+\log^2|f|)|f|^2}((1+\log^2|f|)\right)d\theta\\&\le\frac{1}{4\pi}\int_{|z|=r}\log^+\left(\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\right)d\theta\\&\quad+\frac{1}{4\pi}\int_{|z|=r}\log^+(1+(\log^+|f|+\log^+(1/|f|))^2)d\theta\\&\le\frac{1}{4\pi}\int_{|z|=r}\log\left(1+\frac{|f'|^2}{(1+\log^2|f|)|f|^2}\right)d\theta\\&\quad+\frac{1}{2\pi}\int_{|z|=r}\log^+(\log^+|f|+\log^+(1/|f|))d\theta+\frac{1}{2}\log 2\\&\le\frac{1}{2}\log\left(1+\frac{1}{2\pi}\int_{|z|=r}\frac{|f'|^2}{(1+\log^2|f|)|f|^2}d\theta\right)\\&\quad+\frac{1}{2\pi}\int_{|z|=r}\log(1+\log^+|f|+\log^+(1/|f|))d\theta+\frac{1}{2}\log 2\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\log\left(1+\frac{1}{2}\mu^{(1+\delta)^2}(r)r^{\delta}b^{\delta}\right)\\
&\quad +\log\left(1+m(r,f)+m(r,1/f)\right)+\frac{1}{2}\log 2\|_{E_\delta}\\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2}(\mu(r))^{(1+\delta)^2}r^{\delta}b^{\delta}\right)+\log^+ T_f(r)+O(1)\|_{E_\delta}\\
&\leq \left(1+\frac{(1+\delta)^2}{2}\right)\log^+ T_f(r)+\frac{\delta}{2}\log r+O(1)\|_{E(\delta)}.
\end{aligned}
$$

This proves the theorem.

Remark: LDL can also be proved by the "negative curvature method" (G-J+Calculus Lemma), i.e. instead of using

$$\int_{\triangle(t)} f^*\Phi = \int_{w\in\mathbf{C}} n_f(t,w)\Phi(w),$$

we can use

$$
\begin{aligned}
&\partial_z\partial_{\bar z}(1+(\log|f|^2)^2)^{1/2} = (1+(\log|f|^2)^2)^{-3/2}|\partial_z\log f|^2\\
&+(1+(\log|f|^2)^2)^{-1/2}(\log|f|^2)\partial_z\partial_{\bar z}\log|f|^2.
\end{aligned}
$$

The Second Main Theorem (Nevanlinna). *Let f be a non-constant meromorphic function on $\mathbf{C}$ and let $a_1,\dots,a_q\in\mathbf{C}$ be distinct points. Then, for any $\delta>0$*

$$
\begin{aligned}
&\sum_{j=1}^{q} m_f(r,a_j)+m_f(r,\infty)+N_(r,R_f)\\
\leq\ & 2T_f(r)+O(\log^+ T_f(r))+\delta\log r+O(1)\ \|_{E(\delta)}.
\end{aligned}
$$

Proof. We just outline the proof here. A complete proof can be founded at any standard Nevanlinna theory book. Let $d=\min_{i\neq j}\{|a_i-a_j|,4q\}$, and let A_j be those θ such that $|f(re^{i\theta})-a_j|<d/4q$. Then $A_j, j=1,\dots,q$ are disjoint and

$$
\begin{aligned}
\sum_{j=1}^{q} m_f(r,a_j) &\leq \sum\int_{A_j}\log\frac{1}{|f-a_j|}\frac{d\theta}{2\pi}+q\log\frac{4q}{d}\\
&= \sum\int_{A_j}\log\frac{|f'|}{|f-a_j|}\frac{d\theta}{2\pi}+\int_0^{2\pi}\frac{1}{|f'(re^{i\theta})|}\frac{d\theta}{2\pi}+O(1)\\
&\leq \sum_{j=1}^{q} m_{(f-a_j)'/(f-a_j)}(r,\infty)+m_{f'}(r,0)+O(1),
\end{aligned}
$$

where we use the important property that $(f - a_j)' = f'$. Start from here and using the logarithmic derivative lemma, we can derive the SMT above. Note that the factor 2 before $T_f(r)$ comes from the use of f'.

2. The First Main Theorem

Denote by

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right), \quad \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right),$$

and

$$\partial u = \frac{\partial u}{\partial z} dz, \ \bar{\partial} u = \frac{\partial u}{\partial \bar{z}} d\bar{z},$$

$d = \partial + \bar{\partial}$, $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. Write $z = re^{i\theta}$, then $d^c u = \frac{1}{4\pi}(\frac{r\partial u}{\partial r} d\theta - r^{-1}\frac{\partial u}{\partial \theta} dr)$. By Stoke's theorem, it is easy to derive the following result.

Theorem 2.1 (Green-Jensen formula for C^2 functions) *For any smooth α,*

(1)

$$\int_{|z|<r} dd^c\alpha = \int_{S_r} d^c\alpha.$$

(2)

$$\int_0^r \frac{dt}{t}\int_{S_t} d^c\alpha = \frac{1}{2}\int_{S_r}\alpha\sigma - \frac{1}{2}\alpha(0).$$

(3) Hence

$$\int_0^r \frac{dt}{t}\int_{|z|<t} dd^c\alpha = \frac{1}{2}\int_{S_r}\alpha\sigma - \frac{1}{2}\alpha(0).$$

We would also have to deal with α with certain type of singularities. Let α be on $\mathbf{C}$ either smooth (**Type I**), or locally $\log|g|^2$ (**Type II**) or $\log(1 + (|g|^2 h)^\lambda)$ with $0 < \lambda \leq 1$, where $h > 0$ smooth, g is holomorphic (**Type III**).

Theorem 2.2 (Green-Jensen formula) *For any admissible α (Type I to Type III) which is smooth at the origin, then*

(1)

$$\int_{B_r} dd^c\alpha = \int_{S_r} d^c\alpha - Sing_\alpha(r),$$

where

$$Sing_\alpha(r) = \lim_{\epsilon\to 0}\int_{S(sing_\alpha,\epsilon)(r)} d^c\alpha.$$

(2)

$$\int_0^r \frac{dt}{t}\int_{S_t} d^c\alpha = \frac{1}{2}\int_{S_r}\alpha\sigma - \frac{1}{2}\alpha(0).$$

(3) Thus,

$$\mathcal{I}_r(dd^c[\alpha]) = \frac{1}{2}\mathcal{A}_r(\alpha) - \frac{1}{2}\alpha(0)$$

where $dd^c[\alpha] = dd^c\alpha + Sing_\alpha(r)$ *(so* $\{$**current** $dd^c[\alpha]\} = \{$**diff. form** $dd^c\alpha\} + \{Sing_\alpha\}$)

Now we are ready to state the First Main Theorem. Let X be a projective variety and let L be an ample divisor on L. Let $f : \mathbf{C} \to X$ be a holomorphic map.

We first give some definitions:

The *Height or characteristic function*: Let $f : \mathbf{C} \to X$, and let $L \to X$ be a positive line bundle having a metric with h. $T_{f,L}(r)$ of f with respective to (L, h) is defined by

$$T_{f,L}(r) = \int_0^r \frac{dt}{t}\int_{B_t} f^*c_1(L, h).$$

It can be easily proved that $T_{f,L}(r)$ is essentially independent (up to a bounded term) of the choice of the metric and is determined by the bundle itself. It can also be proved that f must be constant if L is ample (i.e. $c_1(L, h) > 0$) and $T_{f,L}(r)$ is bounded. We can also prove that f is rational if $T_f(L, r) = O(logr)$ (assuming L is ample).

The Weil-function of D and the Proximity function of f with respect to D (assuming that $\mathcal{O}(D)$ has an Hermitian metric), we defined the Weil function of D as

$$\lambda_D(x) := -\log \|s_D(x)\|$$

s_D is a canonical meromorphic section associated with D. The proximity function is defined by

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

As an example, the Weil function for the hyperplanes $H = \{a_0x_0 + \cdots + a_nx_n = 0\}$ is given by

$$\lambda_H(x) = \log \frac{\max_{0\le i\le n}|x_i| \max_{0\le i\le n}|a_i|}{|a_0x_0 + \cdots + a_nx_n|}.$$

Lemma 2.1 *The Weil functions λ_D for Cartier divisors D on a complex projective variety X satisfy the following properties.*

(*a*) **Additivity**: *If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1+\lambda_2$ extends uniquely to a Weil function for D_1+D_2.*

(*b*) **Functoriality**: *If λ is a Weil function for a Cartier divisor D on X, and if $\phi : X' \to X$ is a morphism such that $\phi(X') \not\subset SuppD$, then $x \mapsto \lambda(\phi(x))$ is a Weil function for the Cartier divisor ϕ^*D on X'.*

(*c*) **Normalization**: *If $X = \mathbb{P}^n$, and if $D = \{z_0 = 0\} \subset X$ is the hyperplane at infinity, then the function*

$$\lambda_D([z_0 : \cdots : z_n]) := \log \frac{\max\{|z_0|, \ldots, |z_n|\}}{|x_0|}$$

is a Weil function for D.

(*d*) **Uniqueness**: *If both λ_1 and λ_2 are Weil functions for a Cartier divisor D on X, then $\lambda_1 = \lambda_2 + O(1)$.*

(*e*) **Boundedness from below**: *If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.*

(*f*) **Principal divisors**: *If D is a principal divisor (f), then $-\log|f|$ is a Weil function for D.*

The *Counting function of f with respect to $D = [s = 0]$, where $s \in H^0(M, L)$* is

$$N_f(r, D) = \int_0^r n_f(t, D)\frac{dt}{t},$$

where $n_f(t, D)$ is the number of zeros of $s \circ f = 0$ inside $|z| < t$, counting multiplicities.

Theorem 2.3 (First Main Theorem) *Let $f : \mathbf{C} \to X$ be holomorphic, $L \to X$ Hermitian line bundle, $s \in H^0(X, L)$ with $D = [s = 0]$. Assume that $s \circ f \not\equiv 0$, then*

$$T_{f,L}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

Proof. By definition, on U_α, $\|s_D\|^2 = |s_\alpha|^2 h_\alpha$, so by Poincare-Lelong formula,

$$dd^c[\log \|s_D\|^2] = -c_1(L, h) + [D].$$

The FMT is thus obtained by applying the Green-Jensen formula.

3. H. Cartan's Second Main Theorem

Recall we have established the First Main Theorem for $f : \mathbf{C} \to X$ for a general compact complex manifold (see my another notes). We now derive the the Second Main Theorem for the case that $X = \mathbf{P}^n(\mathbf{C})$ and for divisors of hyperplanes. We write $T_f(r) := T_f(L, r)$ which is called the Cartan's characteristic function, where $L = \mathcal{O}_{\mathbf{P}^n}(1)$. In the case $X = \mathbf{P}^n$. Recall that $|Z|$ defines an Hermitian norm in tautological bundle mentioned earlier. Its dual bundle, the hyperplane section bundle, denoted by $\mathcal{O}_{\mathbf{P}^n}(1)$, has transition function $g_{\alpha,\beta} = z_\alpha/z_\beta$, where $U_\alpha = \{z_\alpha \neq 0\}$. The sections of L are $s_H = \{< \mathbf{a}, Z > /z_\alpha\}$ with $[s_H = 0] = H = \{a_0 z_0 + \cdots + a_n z_n = 0\}$. The metric on L is give $h_\alpha = |z_\alpha|^2/\|Z\|^2$. Thus it first Chen form is

$$c_1(L, h) = -dd^c \log h_\alpha = dd^c \log \|Z|^2.$$

It is called the Fubini-Study metric on $\mathbf{P}^n$. Hence, by Green-Jensen formula,

$$\begin{aligned} T_f(r) = \int_{r_0}^{r} \frac{dt}{t} \int_{|\zeta|\leq t} f^* c_1(L, h) &= \int_{r_0}^{r} \frac{dt}{t} \int_{|\zeta|\leq t} dd^c \log \|\mathbf{f}\|^2 \\ &= \int_0^{2\pi} \log \|\mathbf{f}(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1), \end{aligned}$$

where $\mathbf{f} = (f_0, \dots, f_n)$ is a reduced representation of f, i.e. $f_0, \dots, f_n$ have no common zeros.

$$\lambda_H(x) = \log \frac{\|x\| \|\mathbf{a}|}{| < x, \mathbf{a} > |}.$$

Given hyperplanes $H_1, \dots, H_q$ (or $\mathbf{a}_1, \dots, \mathbf{a}_q$). We say that $H_1, \dots, H_q$ are **in general position** if for any injective map $\mu : \{0, 1, \dots, n\} \to \{1, \dots, q\}$, $\mathbf{a}_{\mu(0)}, \dots, \mathbf{a}_{\mu(n)}$ are linearly independent. For hyperplanes $H_1, \dots, H_q$ in general position we have the following product to the sum estimate.

Lemma 3.1 (Product to the sum estimate) *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$, located in general position. Denote by T the set of all injective maps $\mu : \{0, 1, \dots, n\} \to \{1, \dots, q\}$. Then*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq \int_0^{2\pi} \max_{\mu \in T} \sum_{i=0}^{n} \lambda_{H_{\mu(i)}}(f(re^{i\theta})) \frac{d\theta}{2\pi} + O(1).$$

Theorem 3.1 (Cartan's Second Main Theorem) *Let $H_1, \dots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position. Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerated holomorphic curve (i.e. its image is not contained in any proper subspaces). Then for any $\delta > 0$ the inequality*

$$\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0)$$

$$\leq \quad (n+1)T_f(r) + O(\log^+ T_f(r)) + \delta \log r + O(1)\|_{E_\delta}.$$

We outline here a proof of SMT of Cartan:

- We will use the following properties of the Wronski determinants.
  a) $W(f_0, \ldots, f_n) \not\equiv 0$ iff $f_0, \ldots, f_n$ are linearly independent.
  b) If $(g_0, \ldots, g_n) = (f_0, \ldots, f_n)B$ where B is an invertible matrix, then $W(g_0, \ldots, g_n) = \det BW(f_0, \ldots, f_n)$.
  c) $W(gg_0, \ldots, gg_n) = g^{n+1}W(f_0, \ldots, f_n)$.
  d) et $A(f_0, \ldots, f_n) := W(f_0, \ldots, f_n)/(f_0 \cdots f_n)$, Then, $A(gg_0, \ldots, gg_n) = A(f_0, \ldots, f_n)$, and form LDL, $m(r, A(f_0, \ldots, f_n) = O(\log T_f(r) + \log r)\|_E$.
- If $H_j : L_j(x) = 0, 1 \leq j \leq q$ are hyperplanes in general position (see the definition below), then, for every $z \in \mathbf{C}$,

$$\frac{\|f(z)\|^q}{|L_1(f)(z) \cdots L_q(f)(z)|} \leq C \frac{\|f(z)\|^{n+1}}{|L_{i_1}(f)(z) \cdots L_{i_{n+1}}(f)(z)|},$$

  or

$$\|f(z)\|^{q-(n+1)} \left| \frac{W(f_0, \ldots, f_n)}{L_1(f)(z) \cdots L_q(f)(z)} \right| \leq C \left| \frac{W(L_{i_0}(f), \ldots, L_{i_n}(f))}{L_{i_0}(f) \cdots L_{i_n}(f)} \right|$$

$$= CA(f_0, \ldots, f_n),$$

  here we used the property that $W(L_{i_0}(f), \ldots, L_{i_n}(f))= C_{i_0, \ldots, i_n} W(f_0, \ldots, f_n)$.
- If f is linearly non-degenerate, then $W(f_0, \ldots, f_n) \not\equiv 0$.

The above outline of proof actually gives the following more general form of SMT, which is more convenient to use.

Theorem 3.2 (The general theorem of Cartan)). *Let $f = [f_0 : \cdots : f_n] : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve whose image is not contained in any proper subspaces. Let $H_1, ..., H_q$ (or $\mathbf{a}_1, \ldots, \mathbf{a}_q$) be arbitrary hyperplanes in $\mathbf{P}^n(\mathbf{C})$. Denote by $W(f_0, \ldots, f_n)$ the Wronskian of $f_0, \ldots, f_n$. Then, for any $\delta > 0$, the inequality*

$$\int_0^{2\pi} \max_K \sum_{k \in K} \lambda_{H_k}(f(re^{i\theta})) \frac{d\theta}{2\pi} + N_W(r, 0)$$
$$\leq \quad (n+1)T_f(r) + O(\log T_f(r)) + \delta \log r + O(1)\|_{E_\delta}$$

where the maximum is taken over all subsets K of $\{1, \ldots, q\}$ such that $\mathbf{a}_j, j \in K$, are linearly independent.

4. The Second Main Theorem for General Divisors on Projective Varieties

The Basic Theorem:

The starting point is the following result which is basically a reformulation of H. Cartan's theorem (the general form). We call it the "Basic Theorem".

Theorem 4.1 (Basic Theorem) [Ru-Vojta, 2017]. *Let X be a complex projective variety and let D be a Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X, \mathscr{O}(D))$, and let $s_1, \ldots, s_q$ be nonzero elements of V. For each $i = 1, \ldots, q$, let D_j be the Cartier divisor (s_j). Let $f\colon \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \le (\dim V + \epsilon)T_{f,D}(r) \ \|.$$

Here the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j\in J}$ are linearly independent.

Proof. Let $d = \dim V$. We may assume that $d > 1$ (otherwise, all D_j are the same divisor, and the sets J have at most one element each, so the theorem follows immediately from the First Main Theorem.

Let $\Phi\colon X \dashrightarrow \mathbb{P}^{d-1}$ be the rational map associated to the linear system V. Let X' be the closure of the graph of Φ, and let $p\colon X' \to X$ and $\phi\colon X' \to \mathbb{P}^{d-1}$ be the projection morphisms. Let $\tilde{f} :: \mathbb{C} \to X'$ be the lifting of f.

Note that, even though Φ extends to the morphism $\phi\colon X' \to \mathbb{P}^{d-1}$, the linear system of $H^0(X', p^*\mathscr{O}(D))$ corresponding to V may still have base points. What is true, however, is that there is an effective Cartier divisor B on X' such that, for each nonzero $s \in V$, there is a hyperplane H in $\mathbb{P}^{d-1}$ such that $p^*(s) - B = \phi^*H$. (More precisely, $\phi^*\mathscr{O}(1) \cong \mathscr{O}(p^*D - B)$. The map

$$\alpha\colon H^0(X', \mathscr{O}(p^*D - B)) \to H^0(X, \mathscr{O}(p^*D))$$

defined by tensoring with the canonical global section s_B of $\mathscr{O}(B)$ is injective, and its image contains $p^*(V)$. The preimage $\alpha^{-1}(p^*(V))$ corresponds to a base-point-free linear system for the divisor $p^*D - B$.)

For each $j = 1, \ldots, q$, let H_j be the hyperplane in $\mathbb{P}^{d-1}$ for which $p^*(s_j) - B = \phi^*H_j$. Then,

$$\lambda_{p^*D_j} = \lambda_{\phi^*H_j} + \lambda_B + O(1)\ . \tag{2}$$

By functoriality of Weil functions, $\lambda_{p^*D_j}(\tilde{f}(z)) = \lambda_{D_j}(f(z))$. Therefore it will suffice to prove the inequality

$$\int_0^{2\pi}\left(\max_J \sum_{j\in J}\lambda_{H_j}(\phi(\tilde{f})(re^{i\theta})) + \lambda_B(\tilde{f}(re^{i\theta}))\right)\frac{d\theta}{2\pi} \leq_{exc} (\dim V+\epsilon)T_{f,D}(r). \tag{3}$$

For any subset J of $\{1,\ldots,q\}$, the sections s_j, $j\in J$, are linearly independent elements of V if and only if the hyperplanes H_j, $j\in J$, lie in general position in $\mathbb{P}^{d-1}$. Thus we may apply the above H. Cartan's Theorem to obtain that

$$\int_0^{\infty}\max_J \sum_{j\in J}\lambda_{H_j}(\phi(\tilde{f})(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{exc} (\dim V+\epsilon)T_{\phi(\tilde{f})}(r). \tag{4}$$

From (2), we get $T_{\phi(\tilde{f})}(r) = T_{f,D}(r) - T_{\tilde{f},B}(r) + O(1)$. On the other hand, since each set J as above has at most $\dim V$ elements and B is effective, we get

$$(\#J)\lambda_B(x) \leq (\dim V)\lambda_B(x) + O(1)$$

for all $x\in X'$. Hence

$$\begin{aligned}&\int_0^{2\pi}\left(\max_J \sum_{j\in J}\lambda_{H_j}(\phi(\tilde{f})(re^{i\theta})) + \lambda_B(\tilde{f}(re^{i\theta}))\right)\frac{d\theta}{2\pi}\\ \leq_{exc}\ & (\dim V+\epsilon)T_{f,D}(r) - (\dim V+\epsilon)T_{\tilde{f},B}(r) + (\dim V)m_{\tilde{f}}(r,B)\\ \leq_{exc}\ & (\dim V+\epsilon)T_{f,D}(r),\end{aligned}$$

where, in the last inequality, we used the first main theorem that $m_{\tilde{f}}(r,B) \leq T_{\tilde{f},B}(r) + O(1)$. This finishes the proof.

Nevanlinna Constant:

The above Basic Theorem motivates the following notation of the Nevanlinna constant: Let X be a smooth projective variety and D be an effective Cartier divisor on X. For any section $s\in H^0(X,\mathcal{O}(D))$, we use $ord_E s$, or $ord_E(s)$, to denote the coefficients of (s) in E where (s) is the divisor on X associated to s.

Definition. *Let X be a smoothl complex projective variety, and D be an effective Cartier divisor on X. The Nevanlinna constant of D, denoted by $Nev(D)$, is given by*

$$Nev(D) := \inf_N\left(\inf_{\{\mu_N,V_N\}}\frac{\dim V_N}{\mu_N}\right),$$

where the infimum "$\inf_N$" is taken over all positive integers N and the infimum "$\inf_{\{\mu_N,V_N\}}$" is taken over all pairs $\{\mu_N, V_N\}$ where μ_N is a positive *real number and* $V_N \subset H^0(X, \mathcal{O}(ND))$ *is a linear subspace with* $\dim V_N \geq 2$ *such that, for all* $P \in suppD$*, there exists a basis* B *of* V_N *with*

$$\sum_{s \in B} ord_E(s) \geq \mu_N ord_E(ND) \tag{5}$$

for all irreducible component E *of* D *passing through* P*. If* $\dim H^0(X, \mathcal{O}(ND)) \leq 1$ *for all positive integers* N*, we define* $Nev(D) = +\infty$*.*

Theorem 4.2 [Ru, J. of Geometric Analysis, 2016]. *Let* X *be a complex smooth projective variety and* D *be an effective Cartier divisor on* X*. Then, for every* $\epsilon > 0$*,*

$$m_f(r, D) \leq (Nev(D) + \epsilon)\, T_{f,D}(r) \ \|$$

holds for any Zariski dense holomorphic mapping $f : \mathbf{C} \to X$*.*

Outline of the proof: Denote by σ_0 the set of all prime divisors occurring in D, so we can write

$$D = \sum_{E \in \sigma_0} \mathrm{ord}_E(D) E.$$

Let

$$\Sigma := \{\sigma \subset \sigma_0 \mid \cap_{E \in \sigma} E \neq \emptyset\}.$$

For an arbitrary $x \in X$, from the claim above, pick $\sigma \in \Sigma$ (depends on x) for which

$$\lambda_D(x) \leq \lambda_{D_{\sigma,1}}(x)$$

where $D_{\sigma,1} := \sum_{E \in \sigma} \mathrm{ord}_E(D) E$. Now for each $\sigma \in \Sigma$, by definition, there is a basis B_σ of $V_N \subset H^0(X, ND)$ such that

$$\sum_{s \in B_\sigma} \mathrm{ord}_E(s) \geq \mu_N \, \mathrm{ord}_E(ND),$$

at some (and hence all) points $P \in \cap_{E \in \sigma} E$. Since Σ is finite, $\{B_\sigma \mid \sigma \in \Sigma\}$ is a finite collection of bases of V_N. Thus, we have, using the property of Weil function that, if $D_1 \geq D_2$, then $\lambda_{D_1} \geq \lambda_{D_2}$, we get that,

$$\lambda_{ND}(x) \leq \frac{1}{\mu_N} \max_{\sigma \in \Sigma} \sum_{s \in B_\sigma} \lambda_s(x).$$

The theorem can thus be derived by taking $x = f(re^{i\theta})$, by taking integration and then by applying the Basic Theorem above.

Define $\delta_f(D)$, the *Nevanlinna defect of* f *with respect to* D, by

$$\delta_f(D) := \liminf_{r \to +\infty} \frac{m_f(r, D)}{T_{f,D}(r)}.$$

Corollary 4.1[Defect Relation]. *Let D be an effective Cartier divisor on a smooth complex projective variety X. Then*

$$\delta_f(D) \leq Nev(D)$$

for any Zariski dense holomorphic map $f : \mathbf{C} \to X$.

Corollary 4.2. *Let D be an effective Cartier divisor on a complex normal projective variety X. If $Nev(D) < 1$, then every holomorphic map $f : \mathbf{C} \to X \setminus D$ is not Zariski dense, i.e., the image of f must be contained in a proper subvariety of X.*

Proof. Note that $f : \mathbf{C} \to X \setminus D$ implies that $m_f(r, D) = T_{f,D}(r) + O(1)$. So $\delta_f(D) = 1$. Assume that f is Zariski dense, then above Corollary implies that

$$1 = \delta_f(D) \leq \mathrm{Nev}(D) < 1$$

which gives a contradiction. So f is not Zariski dense. Previous results can be derived by computing the Nevanlinna constant $\mathrm{Nev}(D)$

Example 4.1. Let $X = \mathbb{P}^n$ and $D = H_1 + \cdots + H_q$ where $H_1, \cdots, H_q$ are hyperplanes in $\mathbb{P}^n$ in general position. We take $N = 1$ and consider $V_1 := H^0(\mathbb{P}^n, \mathcal{O}(D)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then $\dim V_1 = \binom{q+n}{n}$. For each $P \in \mathrm{Supp} D$, since $H_1, \cdots, H_q$ are in general position, $P \in H_{i_1} \cap \cdots \cap H_{i_l}$ with $\{i_1, \ldots, i_l\} \subset \{1, \ldots, q\}$ and $l \leq n$. Without loss of generality, we can just assume $H_{i_1} = \{z_1 = 0\}, \cdots, H_{i_l} = \{z_l = 0\}$ by taking proper coordinates for $\mathbb{P}^n$. Now we take the basis $B = \{z_0^{i_0} \cdots z_n^{i_n} \mid i_0 + \cdots + i_n = q\}$ for $V_1 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then, for each irreducible component E of D containing P, say $E = \{z_{j_0} = 0\}$ with $1 \leq j_0 \leq l$, we have $\mathrm{ord}_E\{z_j = 0\} = 0$ for $j \neq j_0$, $\mathrm{ord}_E\{z_{j_0} = 0\} = 1$ and thus $\mathrm{ord}_E D = 1$. On the other hand,

$$\sum_{s \in B} \mathrm{ord}_E\, s = \sum_{\vec{i}} i_{j_0} = \frac{1}{n+1} \sum_{\vec{i}} (i_0 + \cdots + i_n) = \frac{q}{n+1} \binom{q+n}{n} = \frac{q}{n+1} \dim V_1,$$

where, in above, the sum is taken for all $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$, and we used the fact that the number of choices of $\vec{i} = (i_0, \ldots, i_n)$ with $i_0 + \cdots + i_n = q$ is $\binom{q+n}{n}$. Thus we can take $\mu_1 = \frac{q}{n+1} \dim V_1$, and hence,

$$\mathrm{Nev}(D) \leq \frac{\dim V_1}{\mu_1} = \frac{n+1}{q}.$$

5. Results derived by computing Nevanlinna's constant

In this section, we establishing the Second Main Theorem results by computing the Nevanlinna constant. To be more convenient, we indeed don't directly compute the Nevanlinna constant, rather we apply the Basic Theorem, in a similar way in proof Theorem 4.2 above.

An Preliminary Result:

Let $D_1, \dots, D_q$ be effective divisors on X. We say that $D_1, \dots, D_q$ are in *l-subgeneral position* on X if for any subset $I \subseteq \{1, \dots, q\}$ with $\#I \leq l+1$,

$$\dim \bigcap_{i \in I} \text{Supp } D_i \leq l - \#I,$$

where $\dim \emptyset = -1$. In particular, the supports of any $l+1$ divisors in l-subgeneral position have empty intersection. If $l = \dim X$, then we say the divisors are in *general position* on X. Let $\mathscr{L}$ be a line sheaf on X, we use $h^0(\mathscr{L})$ to denote $\dim H^0(X, \mathscr{L})$, and $\mathscr{L}(-D)$ to denote the sheaf $\mathscr{L} \otimes \mathscr{O}(-D)$ for a given divisor D on X.

Definition Let $\mathscr{L}$ be a line sheaf and D be a nonzero effective Cartier divisor on a projective variety X. We define

$$\gamma(\mathscr{L}, D) := \inf_N \frac{N h^0(\mathscr{L}^N)}{\sum_{m \geq 1} h^0(\mathscr{L}^N(-mD))},$$

where N passes over all positive integers such that $h^0(\mathscr{L}^N(-D)) \neq 0$. If no such N exists, then we define $\gamma(\mathscr{L}, D) = +\infty$ (Note that $|\mathscr{L}^N|$ does not have to be base point free.)

Theorem 5.1 [Preliminary Result]. *Let X be a complex projective variety of dimension n and let $D_1, \dots, D_q$ be effective Cartier divisors, located in l-subgeneral position on X with $l+n-2>0$. Let $\mathscr{L}$ be a line sheaf on X with $h^0(\mathscr{L}^N) \geq 2$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^q m_f(r, D_j) \leq l \left(\max_{1 \leq j \leq q} \gamma(\mathscr{L}, D_j) + \epsilon \right) T_{f,\mathscr{L}}(r) \ \|.$$

Sketch of Proof. let $\delta > 0$ be a sufficiently small number. We choose N large enough such that, for each i,

$$\frac{N h^0(\mathscr{L}^N)}{\sum_{m \geq 1} h^0(\mathscr{L}^N(-mD_i))} \leq \max_{1 \leq j \leq q} \gamma(\mathscr{L}, D_j) + \delta.$$

Let $x \in X$. Since $D_i, 1 \leq i \leq q$, are in ℓ-sub-general position, we have

$$\sum_{i=1}^q \lambda_{D_i}(x) \leq \ell \lambda_{D_{i_0}}(x) + O(1),$$

for some i_0 with $1 \leq i_0 \leq q$, where i_0 depends on the point x, but $O(1)$ is independent of x. We consider the following filtration of $H^0(X, \mathscr{L}^N)$:

$$H^0(X, L^N) \supseteq H^0(X, \mathscr{L}^N(-D_{i_0})) \supseteq \cdots \supseteq H^0(X, \mathscr{L}^N(-mD_{i_0})) \\ \supseteq H^0(X, \mathscr{L}^N(-(m+1)D_{i_0})) \supseteq \cdots$$

and choose a basis $s_1, \cdots, s_l \in H^0(X, \mathscr{L}^N)$, where $l = h^0(X, \mathscr{L}^N)$ according to this filtration. Notice that for any section $s \in H^0(X, \mathscr{L}^N(-mD_{i_0}))$, $(s) \geq mD_{i_0}$. So we have

$$\begin{aligned}\sum_{j=1}^{l}(s_j) &\geq \left(\sum_{m=0}^{\infty} m[h^0(\mathscr{L}^N(-mD_{i_0})) - h^0(\mathscr{L}^N(-(m+1)D_{i_0}))]\right) D_{i_0} \\ &= \left(\sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_{i_0}))\right) D_{i_0}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{j=1}^{l}\lambda_{s_j} &\geq \left(\sum_{m=1}^{\infty} h^0(\mathscr{L}^N(-mD_{i_0}))\right)\lambda_{D_{i_0}} \\ &\geq \frac{Nh^0(\mathscr{L}^N)}{\max_{1\leq j\leq q}\gamma(\mathscr{L}, D_j)+\delta}\lambda_{D_{i_0}}.\end{aligned}$$

Note that the basis $\{s_1, \cdots, s_M\}$ depends only on i_0, so the number of such choices is finite, since $i_0 \in \{1, \ldots, q\}$, while x varies. We denote the set of base as $J_1, \ldots, J_T$. Thus we get, for every $x \in X$,

$$\sum_{i=1}^{q}\lambda_{D_i}(x) \leq \frac{\ell(\max_{1\leq j\leq q}\gamma(\mathscr{L}, D_j)+\delta)}{N\cdot h^0(\mathscr{L}^N)} \max_{1\leq t\leq T}\sum_{j\in J_t}\lambda_{(s_j)}(x) + O(1).$$

By taking $x = f(re^{i\theta})$ and then taking the integration, it then follows from the Basic Theorem and a suitable choice of δ. This finished the proof.

We now compute $\gamma(\mathscr{L}, D_j)$ in some cases: Let $D := D_1 + \cdots + D_q$ where $D_1, \ldots, D_q$ are effective Cartier divisors on X. Write

$$\gamma(D_j) = \gamma(\mathscr{O}(D), D_j).$$

To compute $\gamma(D_j)$, we consider the following two cases.

Case 1: The divisors are ample and linearly equivalent: Assume that each D_j, $1 \leq j \leq q$, is linearly equivalent to a fixed ample divisor A on X. We write $h^0(D) := h^0(\mathscr{O}(D))$. By the Riemann-Roch theorem, with $n = \dim X$, we have

$$h^0(ND) = h^0(qNA) = \frac{(qN)^n A^n}{n!} + o(N^n)$$

and

$$h^0(ND - mD_j) = h^0((qN-m)A) = \frac{(qN-m)^n A^n}{n!} + o(N^n).$$

Thus

$$\sum_{m\geq 1} h^0(ND - mD_j) = \frac{A^n}{n!}\sum_{l=0}^{qN-1} l^n + o(N^{n+1}) = \frac{A^n(qN-1)^{n+1}}{(n+1)!} + o(N^{n+1}).$$

Hence

$$\gamma(D_j) = \lim_{N\to\infty} \frac{N\frac{(qN)^n A^n}{n!} + o(N^{n+1})}{\frac{A^n(qN-1)^{n+1}}{(n+1)!} + o(N^{n+1})} = \frac{n+1}{q}.$$

Case 2: Big and nef case: The important result associated to the concept of equi-degree is the following lemma regarding $D := D_1 + \cdots + D_q$ where each D_j is only assumed to be big and nef for $1 \le j \le q$.

Lemma 5.1[Lemma 9.7 in Levin Annals paper]. *Let X be a projective variety of dimension n. If $D_j, 1 \le j \le q$, are big and nef Cartier divisors, then there exist positive real numbers r_j such that $D = \sum_{j=1}^q r_j D_j$ has equi-degree.*

So we only need to compute $\gamma(D_j)$ under an additional assumption that $D_1, \ldots, D_q$ are of equi-degree, i.e.

$$D_j.D^{n-1} = \frac{1}{q} D^n \text{ for } j = 1, \ldots, q$$

where $D := D_1 + \cdots + D_q$.

We use the following lemma from Autissier (see his Duke paper)

Lemma 5.2 *Suppose E is a big and base-point free Cartier divisor on a projective variety X of dimension n, and F is a nef Cartier divisor on X such that $F - E$ is also nef. Let $\beta > 0$ be a positive real number. Then, for any positive integers N and m with $1 \le m \le \beta N$, we have*

$$\begin{aligned} h^0(NF - mE) \quad \ge \quad & \frac{F^n}{n!} N^n - \frac{F^{n-1} . E}{(n-1)!} N^{n-1} m \\ & + \frac{(n-1)F^{n-2} . E^2}{n!} N^{n-2} \min\{m^2, N^2\} + O(N^{n-1}) \end{aligned}$$

where O depends on β.

Let $n = \dim X$, and assume that $n \ge 2$. Fix $1 \le i \le q$ and apply Lemma 5.2 by taking $\beta = \frac{D^n}{nD^{n-1}.D_i}$, we get

$$\begin{aligned} & \sum_{m=1}^{\infty} h^0(ND - mD_i) \\ & \ge \sum_{m=1}^{[\beta N]} \left(\frac{D^n}{n!} N^n - \frac{D^{n-1} . D_i}{(n-1)!} N^{n-1} m + \frac{D^{n-2}.D_i^2}{n!} N^{n-2} \min\{m^2, N^2\} \right) + O(N^n) \\ & \ge \left(\frac{D^n}{n!} \beta - \frac{D^{n-1} . D_i}{(n-1)!} \frac{\beta^2}{2} + \frac{D^{n-2}.D_i^n}{n!} g(\beta) \right) N^{n+1} + O(N^n) \\ & = \left(\frac{\beta}{2} + \frac{D^{n-2}.D_i^2}{D^n} g(\beta) \right) D^n \frac{N^{n+1}}{n!} + O(N^n) \ge \left(\frac{\beta}{2} + \hat{\alpha} \right) N h^0(ND) + O(N^n) \end{aligned}$$

where $\hat{\alpha} := \frac{\min_{1\le j\le q} D_j^n}{D^n} g(\beta)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $g(x) = \frac{x^3}{3}$ if $x \le 1$ and $g(x) = x - \frac{2}{3}$ for $x \ge 1$. Note that $\beta = \frac{D^n}{nD^{n-1}.D_i} = \frac{q}{n}$, so $g(\beta) \ge \frac{1}{3n^3}$. Hence,

$$\gamma(D_i) = \inf_N \frac{Nh^0(ND)}{\sum_{m\ge 1} h^0(ND - mD_i)} \le \frac{1}{\frac{\beta}{2} + \hat{\alpha}} = \frac{2n}{q + 2n\hat{\alpha}}.$$

Notice that

$$\hat{\alpha} = \frac{\min_{1\le j\le q} D_j^n}{D^n} g(\beta) \ge \frac{\min_{1\le j\le q} D_j^n}{3n^3 D^n}.$$

So the preliminary result gives the following theorem:

Theorem 5.2. *Let X be a complex projective variety of dimension $n \ge 2$, and let $D_1, \dots, D_q$ be effective, big, and nef Cartier divisors on X, located in l-subgeneral position. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ has equi-degree (such numbers exist due to Lemma* **??***). Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then*

$$\sum_{j=1}^q r_j m_f(r, D_j) \le \frac{l(l-1)}{(l+n-2)} \frac{2n}{q+C} \left(\sum_{j=1}^q r_j T_{f,D_j}(r) \right) \ \|$$

where

$$C = \frac{\min_{1\le j\le q}(r_j^n D_j^n)}{6n^2 4^n D^n}.$$

The Improvement in the case $X = \mathbb{P}^n$:

In this subsection, we improve the result stated above in the case when $X = \mathbb{P}^n$. The new technique is to use "multi-index filtration".

Theorem 5.3 (SMT for hypersurfaces)[Ru, Amer. J. of Math. 2004] *Let $f : \mathbf{C} \to \mathbf{P}^n$ be a Zariski-dense holomorphic map. Let $D_1, \dots, D_q$ be hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ of degree d_j, located in general position. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^q d_j^{-1} m_f(r, D_j) \le (n+1) + \epsilon) T_f(r) \ \|_E.$$

Define the defect

$$\delta_f(D) = \liminf_{r\to+\infty} \frac{m_f(r, D)}{dT_f(r)}.$$

Then we have the following defect relation

Corollary(Defect Relation) *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be an algebraically non-degenerate holomorphic map, and let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ in general position. Then we have*

$$\sum_{j=1}^{q} \delta_f(D_j) \leq n+1.$$

Proof of Ru's theorem. The proof is similar to the "Preliminary Theorem" above. Let $Q_j, 1 \leq j \leq q$, be the homogeneous polynomials in $\mathbf{C}[X_0, \ldots, X_n]$ of degree d_j defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where d is the the l.c.m of $d_j's$, we can assume that $Q_1, \ldots, Q_q$ have the same degree of d.

Given $z \in \mathbf{C}$ there exists a numbering $\{i_1, ..., i_q\}$ of the indices $1, ..., q$ such that

$$|Q_{i_1} \circ f(z)| \leq \cdots \leq |Q_{i_q} \circ f(z)|.$$

Since $Q_1, ..., Q_q$ are in general position, Hilbert Nullstellensatz implies that for any integer $k, 0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk}(x_0, ..., x_n) Q_{i_j}(x_0, ..., x_n),$$

where $b_{jk}, 1 \leq j \leq n+1, 0 \leq k \leq n$, are homogeneous forms with coefficients in $\mathbf{C}$ of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_1}(f)(z)|, ..., |Q_{i_{n+1}}(f)(z)|\},$$

where c_1 is a positive constant depending only on the coefficients of b_{jk}, thus depends only on the coefficients of Q_j. Therefore,

$$\|f(z)\|^d \leq c_1 \max\{|Q_{i_1}(f)(z)|, ..., |Q_{i_{n+1}}(f)(z)|\}.$$

Thus

$$\prod_{j=1}^{q} \frac{\|f(z)\|^d}{|Q_j(f)(z)|} \leq c_1^{q-n} \prod_{k=1}^{n} \frac{\|f(z)\|^d}{|Q_{i_k}(f)(z)|},$$

i.e.,

$$\sum_{j=1}^{q} \lambda_{D_j}(f(z)) \leq \sum_{k=1}^{n} \lambda_{D_{i_k}}(f(z)) + O(1). \tag{6}$$

Note that the indices $i_1, \ldots, i_n$ depends on z.

Let $\gamma_1 := Q_{i_1}, \cdots, \gamma_n := Q_{i_n}$. For $N \in \mathbf{N}$, let V_N be the space of homogeneous polynomials of $n+1$ variables of degree N and fix a (arbitrary) basis $\phi_1, ..., \phi_l$, where $l = \dim V_N = \binom{N+n}{n}$. Arrange the $n-$tuples $\mathbf{i} = (i_1, ..., i_n)$ of non-negative integers by lexicographic order. Define, for the

$n-$tuples $\mathbf{i} = (i_1, ..., i_n)$ o non-negative integers with $\sigma(\mathbf{i}) := \sum_j i_j \leq N/d$, the spaces $W_{\mathbf{i}} := W_{N,\mathbf{i}}$ by

$$W_{N,\mathbf{i}} = \sum_{\mathbf{e} \geq \mathbf{i}} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{N-d\sigma(\mathbf{e})}.$$

Clearly, $W_{(0,...,0)} = V_N$ and $W_{\mathbf{i}} \supset W_{(\mathbf{i}')}$ if $\mathbf{i}' \geq \mathbf{i}$, so that the $\{W_{\mathbf{i}}\}$ in fact defines a filtration of V_N.

Lemma 5.3 [Lemma 3.3 in Ru' Amer. J. Math. 2004 paper]. *There exists an integer N0 dependent only on $\gamma_1, \ldots, \gamma_n$ such that for $d\sigma(\mathbf{i}) < N - N_0$, we have*

$$\triangle_{\mathbf{i}} := \dim \left(W_{\mathbf{i}} / W_{\vec{i}'} \right) = d^n,$$

where $W_{\mathbf{i}} \supseteq W_{\mathbf{i}'}$ with $\mathbf{i}' \geq \mathbf{i}$ which is next to $W_{\mathbf{i}}$. Also the remaining n-tuples $\mathbf{i}$, $\triangle_{\mathbf{i}}$ *is bounded by* $\dim V_{N_0}$

We claim that, for every $1 \leq j \leq n$,

$$\sum_{\mathbf{i}} i_j \triangle_{\mathbf{i}} = \frac{N^{n+1}}{d(n+1)!} + O(N^n). \tag{7}$$

Indeed, note that $l = \dim V_N = \frac{N^n}{n!} + O(N^{n-1})$.

$$\begin{aligned}
\triangle &= \sum_{d\sigma(\mathbf{i}) \leq N} i_j \triangle_{\mathbf{i}} = \sum_{d\sigma(\mathbf{i}) \leq N - N_0} i_j \ \triangle_{\mathbf{i}} + O(1) = d^n \sum_{i_1 + \cdots + i_n \leq \frac{N-N_0}{d}} i_j + O(1) \\
&= d^n \sum_{i_0 + \cdots + i_n = \frac{N}{d} - n} i_j + O(1) \\
&= \frac{d^n}{n+1} \sum_{i_0 + \cdots + i_n = \frac{N-N_0}{d}} \sum_{\eta=0}^{n} i_\eta + O(1) \\
&= \frac{d^n}{n+1} \sum_{i_0 + \cdots + i_n = \frac{N-N_0}{d}} \left(\frac{N - N_0}{d} \right) + O(1) \\
&= \frac{d^n}{n+1} \binom{N/d}{n} \left(\frac{N - N_0}{d} \right) = \left(\frac{N^{n+1}}{d(n+1)!} + O(N^n) \right),
\end{aligned}$$

where, in above, we used the fact that the number of nonnegative integer m-tuples with sum $\leq T$ for a positive integer T is equal to the number of non-negative integer $(m+1)$-tuples with sum exactly T, which is $\left(\begin{array}{c} T+m \\ m \end{array} \right)$. This proves the claim.

We now choose a basis $s_1, ..., s_l$ for V_N according to this basis. Then, $\gamma_1 := Q_{i_1}, \cdots, \gamma_n := Q_{i_n}$,

$$\sum_{i=1}^{l}(s_i) \geq \left(\frac{N^{n+1}}{d(n+1)!} + O(N^n)\right)(D_{i_1} + \cdots + D_{i_n}).$$

Therefore,

$$\sum_{i=1}^{l}\lambda_{s_i} \geq \left(\frac{N^{n+1}}{d(n+1)!} + O(N^n)\right)(\lambda_{D_{i_1}} + \cdots + \lambda_{D_{i_n}}).$$

The rest of the proof is similar to the proof of Theorem 5.1 by using (6) and the basic Theorem.

Ru's Annals paper result.

The above result about hypersurfaces in $\mathbf{P}^n$ has been extended by Ru in 2009.

Theorem 5.4 [Ru, Annals of Math., 2009]. *Let X be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exist an ample divisor A on X and positive integers d_j such that $D_j \sim d_j A$ for $j = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon)T_{f,A}(r) \ \|.$$

Proof. We only need to compute the Nevanlinna constant (by applying Theorem 4.2). Since A is very ample, $\phi_A : X \to \mathbb{P}^u$, the canonical map associated to A, is an embedding. Let $Q_1, \ldots, Q_q$ be the linear forms in $(u+1)$-variables such that $D_i = \phi_A^*\{Q_i = 0\}$. Let

$$\psi : X \to \mathbb{P}^{q-1}, \quad x \mapsto [Q_1(\phi_A(x)), \ldots, Q_q(\phi_A(x))].$$

Let $Y := \psi(X) \subset \mathbb{P}^{q-1}$. By the general position assumption for $D_1, \ldots, D_q$, ψ is a finite morphism from X to Y.

On $\mathbb{P}^{q-1}$, we have for all $N \in \mathbb{N}$ a short exact sequence

$$0 \to \mathcal{I}_Y(N) \to \mathcal{O}_{\mathbb{P}^{q-1}}(N) \to \mathcal{O}_Y(N) \to 0.$$

The beginning of the corresponding long exact sequence reads

$$0 \to H^0(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) \to H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(N)) \xrightarrow{\tau} H^0(Y, \mathcal{O}_Y(N))$$
$$\to H^1(\mathbb{P}^{q-1}, \mathcal{I}_Y(N))$$

where τ denotes the restriction map. Since $H^1(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) = 0$ for N big enough, we have, for N big enough,

$$\begin{aligned} (8) \qquad H^0(Y, \mathcal{O}_Y(N)) &\cong H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(N))/\ker(\tau) \\ &\cong H^0(\mathbb{P}^{q-1}, \mathcal{O}_{\mathbb{P}^{q-1}}(N))/H^0(\mathbb{P}^{q-1}, \mathcal{I}_Y(N)) \\ &\cong \mathbb{C}[Y_0, \dots, Y_{q-1}]_N/(I_Y)_N, \end{aligned}$$

where $(I_Y)_N$ denotes the set of those homogeneous polynomials of degree N vanishing on Y. We now estimate the Nevanlinna constant by letting, for $\tilde{N} = \frac{N}{q}$, and N is a multiple of q and big enough,

$$V_{\tilde{N}} := \psi^* H^0(Y, \mathcal{O}_Y(N)) \subset H^0(X, \mathcal{O}(\frac{N}{q} D)) = H^0(X, \mathcal{O}(\tilde{N} D)).$$

Since $\psi : X \to Y$ is a finite surjective morphism, by using (8)

$$\dim(V_{\tilde{N}}) = \dim H^0(Y, \mathcal{O}_Y(N)) = \dim (\mathbb{C}[Y_0, \dots, Y_{q-1}]_N/(I_Y)_N) = H_Y(N),$$

where $H_Y(N)$ is the Hilbert function of Y.

To continue, let $P \in \mathrm{Supp} D$. The condition that $D_1, \dots, D_q$ are in general position implies that $P \in \cap_{t=1}^{l} (\phi_A^* \{Q_{i_t} = 0\})$ for some distinct $Q_{i_1}, \dots, Q_{i_l} \in \{Q_1, \dots, Q_q\}$ with $l \le n$. Without loss of generality, we can assume that $l = n$ (otherwise we just add more polynomials). Let $\vec{c} = (c_1, \dots, c_q)$ be the q-vector whose i_j-th entry $(1 \le j \le n)$ is 1, with all other entries being 0. Let $\vec{y}^{\vec{a}^{(1)}}, \dots, \vec{y}^{\vec{a}^{(H_Y(N))}}$ be monomials such that their equivalence classes in $\mathbb{C}[Y_0, \dots, Y_{q-1}]_N/(I_Y)_N$ give a basis and such that

$$S_Y(N, \vec{c}) = \sum_{i=1}^{H_Y(N)} \vec{a}^{(i)} \bullet \vec{c},$$

where $S_Y(N, \vec{c})$ is the N-th Hilbert weight and the bullet denotes the usual dot product. Recall that the N-th Hilbert weight of Y with respect to the weight $\vec{c}$ is given by

$$S_Y(N, \vec{c}) = \max \sum_{i=1}^{H_Y(N)} \vec{a}^{(i)} \bullet \vec{c},$$

where the maximum is taken over all sets of monomials $\vec{y}^{\vec{a}^{(1)}}, \dots, \vec{y}^{\vec{a}^{(H_Y(N))}}$ whose residue class modulo I_Y form a basis of $\mathbb{C}[Y_0, \dots, Y_{q-1}]_N/(I_Y)_N$. For $\nu = 1, \dots, H_Y(N)$, and N a positive multiple of q, let

$$s_\nu = (Q_1^{a_1^{(\nu)}} \dots Q_q^{a_q^{(\nu)}})|_{\phi_A(X)}.$$

These functions form a basis for $V_{\tilde{N}}$ understood as a subspace of $H^0(X, \mathcal{O}(\tilde{N} D))$.

We recall the following key lemma which is combination of Theorem 2.1 and Lemma 3.2 in Ru's Annals paper.

Lemma 5.3 *Let $Y \subset \mathbb{P}^N$ be an algebraic variety of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $\vec{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}_{\geq 0}$. Let $\{i_0, \dots, i_n\}$ be a subset of $\{0, \dots, N\}$ such that*

$$Y \cap \{y_{i_0} = 0, \dots, y_{i_n} = 0\} = \emptyset.$$

Then

$$\frac{1}{mH_Y(m)} S_Y(m, \vec{c}) \geq \frac{1}{(n+1)}(c_{i_0} + \cdots + c_{i_n}) - \frac{(2n+1)\triangle}{m} \cdot \left(\max_{i=0,\dots,N} c_i\right).$$

We now continue our proof. Let E be an irreducible component of D with $P \in E$. We assume that E is contained in $\phi_A^*\{Q_{j_0} = 0\}$. With our chosen $\vec{c}$ and $\vec{a}^{(i)}$, using Lemmas 5.3 (notice the condition that $D_1, \dots, D_q$ are in general position on X), and the symmetry property of the $\vec{a}^{(1)}, \dots, \vec{a}^{(H_Y(N))}$,

$$\begin{aligned}
&\frac{1}{\mathrm{ord}_E D} \sum_{\nu} \mathrm{ord}_E\, s_\nu = \sum_{\nu=1}^{H_Y(N)} a_{j_0}^{(\nu)} = \frac{1}{n} \sum_{\nu=1}^{H_Y(N)} \vec{a}^{(\nu)} \bullet \vec{c} \\
&= \frac{1}{n} S_Y(N, \vec{c}) \geq \frac{1}{n} \frac{1}{n+1} N H_Y(N) \Big(\sum_{j=1}^{n} c_{i_j}\Big) + O(H_Y(N)) \\
&= \frac{1}{n+1} N(H_Y(N) + o(H_Y(N))).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum \mathrm{ord}_E\, s_\nu &\geq \frac{q}{n+1}(H_Y(N) + o(H_Y(N))\, \mathrm{ord}_E\left(\frac{N}{q} D\right) \\
&= \frac{q}{n+1}(H_Y(N) + o(H_Y(N))\, \mathrm{ord}_E(\tilde{N} D).
\end{aligned}$$

Therefore, from the definition of $\mathrm{Nev}(D)$, we have

$$\begin{aligned}
\mathrm{Nev}(D) &\leq \liminf_{\tilde{N} \to +\infty} \frac{\dim V_{\tilde{N}}}{\frac{q}{n+1}(H_Y(N) + o(H_Y(N))} \\
&= \liminf_{N \to +\infty} \frac{H_Y(N)}{\frac{q}{n+1}(H_Y(N) + o(H_Y(N))} = \frac{n+1}{q}.
\end{aligned}$$

The Theorem is thus proved by applying Theorem 4.2.

The Recent Result of Ru-Vojta:

The recent result of Ru-Vojta improves Theorem 5.1 in the case $l = n$, i.e. $D_1, \dots, D_q$ are in general position.

Theorem 5.5[Ru-Vojta, 2017]. *Let X be a complex projective variety and let $D_1, \dots, D_q$ be effective Cartier divisors intersecting properly on X. Let $D = D_1 + \cdots + D_q$. Let $\mathscr{L}$ be a line sheaf on X with $h^0(\mathscr{L}^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$m_f(r, D) \leq \left(\max_{1\leq j\leq q} \gamma(\mathscr{L}, D_j) + \epsilon\right) T_{f,\mathscr{L}}(r) \ \|.$$

The proof of uses the the filtration constructed by Pascal Autissier (see his Duke paper). We first review his results.

Let $D_1, \dots, D_r$ be effective Cartier divisors on a projective variety X. Assume that they intersect properly on X, and that $\bigcap_{i=1}^r D_i$ is non-empty. Let $\mathscr{L}$ be a line sheaf over X with $l := h^0(\mathscr{L}) \geq 1$.

Definition. A subset $N \subset \mathbb{N}^r$ is said to be **saturated** if $\mathbf{a}+\mathbf{b} \in N$ for any $\mathbf{a} \in \mathbb{N}^r$ and $\mathbf{b} \in N$.

Lemma 5.4 [Lemma 3.2 in Autissier's paper]. *Let A be a local ring and $(\phi_1, \dots, \phi_r)$ be a regular sequence of A. Let M and N be two saturated subsets of $\mathbb{N}^r$.Then*

$$\mathcal{I}(M) \cap \mathcal{I}(N) = \mathcal{I}(M \cap N),$$

where, for $N \subset \mathbb{N}^r$, $\mathcal{I}(N)$ is the ideal of A generated by $\{\phi_1^{b_1} \cdots \phi_q^{b_r} \mid \mathbf{b} \in N\}$.

We use the Lemma in the following particular situation: Let

$$\triangle = \{\mathbf{t} = (t_1, \dots, t_r) \in (\mathbb{R}^+)^r \mid t_1 + \cdots + t_r = 1\}.$$

For each $\mathbf{t} \in \triangle$ and $x \in \mathbb{R}^+$, let

$$N(\mathbf{t}, x) = \{\mathbf{b} \in \mathbb{N}^r \mid t_1 b_1 + \cdots + t_r b_r \geq x\}.$$

Notice that $N(\mathbf{t}, x) \cap N(\mathbf{u}, y) \subset N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda)y)$ for all $\lambda \in [0, 1]$. So, from Lemma, we have

$$\mathcal{I}(N(\mathbf{t}, x)) \cap \mathcal{I}(N(\mathbf{u}, y)) \subset \mathcal{I}(N(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}, \lambda x + (1-\lambda)y)) \tag{9}$$

for any $\mathbf{t}, \mathbf{u} \in \triangle$; $x, y \in \mathbb{R}^+$; and $\lambda \in [0, 1]$.

Definition. Let W be a vector space of finite dimension. A **filtration** of W is a family of subspaces $\mathcal{F} = (\mathcal{F}_x)_{x\in\mathbb{R}^+}$ of subspaces of W such that $\mathcal{F}_x \supseteq \mathcal{F}_y$ whenever $x \leq y$, and such that $\mathcal{F}_x = \{0\}$ for x big enough. A basis $\mathcal{B}$ of W is said to be **adapted to** $\mathcal{F}$ if $\mathcal{B} \cap \mathcal{F}_x$ is a basis of $\mathcal{F}_x$ for every real number $x \geq 0$.

Lemma 5.5[See Levin's annals paper] *Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations of W. Then there exists a basis of W which is adapted to both $\mathcal{F}$ and $\mathcal{G}$.*

For any fixed $\mathbf{t} \in \triangle$, we construct a filtration of $H^0(X, \mathscr{L})$ as follows: for $x \in \mathbb{R}^+$, one defines the ideal $\mathcal{I}(\mathbf{t}, x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}(\mathbf{t}, x) = \sum_{\mathbf{b} \in N(\mathbf{t}, x)} \mathscr{O}_X(-\sum_{i=1}^r b_i D_i) \ , \tag{10}$$

and let

$$\mathcal{F}(\mathbf{t})_x = H^0(X, \mathcal{I}(\mathbf{t}, x) \otimes \mathscr{L}) \ . \tag{11}$$

Then $(\mathcal{F}(\mathbf{t})_x)_{x \in \mathbb{R}^+}$ is a filtration of $H^0(X, \mathscr{L})$.

For $s \in H^0(X, \mathscr{L}) - \{0\}$, let $\mu_{\mathbf{t}}(s) = \sup\{y \in \mathbb{R}^+ \mid s \in \mathcal{F}(\mathbf{t})_y\}$. Also let

$$F(\mathbf{t}) = \frac{1}{h^0(\mathscr{L})} \int_0^{+\infty} (\dim \mathcal{F}(\mathbf{t})_x) \, dx \ . \tag{12}$$

Remark 5.1. Let $\mathcal{B} = \{s_1, \ldots, s_l\}$ be a basis of $H^0(X, \mathscr{L})$ with $l = h^0(\mathscr{L})$. Then we have

$$F(\mathbf{t}) \geq \frac{1}{l} \int_0^{\infty} \#(\mathcal{F}(\mathbf{t})_x \cap \mathcal{B}) dx = \frac{1}{l} \sum_{k=1}^{l} \mu_{\mathbf{t}}(s_k),$$

where equality holds if $\mathcal{B}$ is adapted to the filtration $(\mathcal{F}(\mathbf{t})_x)_{x \in \mathbb{R}^+}$.

The key result we will use about this filtration is the following Proposition.

Proposition 5.1 *With the notations and assumptions above, let $F : \triangle \to \mathbb{R}^+$ be the map defined in (12). Then F is concave. In particular, for $\mathbf{t} \in \triangle$,*

$$F(\mathbf{t}) \geq \min_i \left(\frac{1}{h^0(\mathscr{L})} \sum_{m \geq 1} h^0(\mathscr{L}(-mD_i)) \right) \ . \tag{13}$$

Proof. For any $\mathbf{t}, \mathbf{u} \in \triangle$ and $\lambda \in [0, 1]$, we need to prove that

$$F(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u}) \geq \lambda F(\mathbf{t}) + (1 - \lambda) F(\mathbf{u}). \tag{14}$$

By Lemma 5.5, there exists a basis $\mathcal{B} = \{s_1, \ldots, s_l\}$ of $H^0(X, \mathscr{L})$ with $l = h^0(\mathscr{L})$, which is adapted both to $(\mathcal{F}(\mathbf{t})_x)_{x \in \mathbb{R}^+}$ and to $(\mathcal{F}(\mathbf{u})_y)_{y \in \mathbb{R}^+}$. For $x, y \in \mathbb{R}^+$, by Lemm 5.4 (or the remark after the Lemma), since $D_1, \ldots, D_r$ intersect properly on X

$$\mathcal{F}(\mathbf{t})_x \cap \mathcal{F}(\mathbf{u})_y \subset \mathcal{F}(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u})_{\lambda x + (1 - \lambda) y}.$$

For $s \in H^0(X, \mathscr{L}) - \{0\}$, we have, from the definition of $\mu_{\mathbf{t}}(s)$ and $\mu_{\mathbf{u}}(s)$, $s \in \mathcal{F}(\lambda \mathbf{t} + (1 - \lambda)\mathbf{u})_{\lambda x + (1 - \lambda) y}$ for $x < \mu_{\mathbf{t}}(s)$ and $y < \mu_{\mathbf{u}}(s)$, and thus

$$\mu_{\lambda \mathbf{t} + (1 - \lambda)\mathbf{u}}(s) \geq \lambda \mu_{\mathbf{t}}(s) + (1 - \lambda) \mu_{\mathbf{u}}(s).$$

Taking $s = s_j$ and summing it over $j = 1, \dots, l$, we get, by Remark 5.1,

$$F(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}) \geq \lambda\frac{1}{l}\sum_{j=1}^{l}\mu_{\mathbf{t}}(s_j) + (1-\lambda)\frac{1}{l}\sum_{j=1}^{l}\mu_{\mathbf{u}}(s_j).$$

On the other hand, since $\mathcal{B} = \{s_1, \dots, s_l\}$ is a basis adapted to both $\mathcal{F}(\mathbf{t})$ and $\mathcal{F}(\mathbf{u})$, from Remark 5.1, $F(\mathbf{t}) = \frac{1}{l}\sum_{j=1}^{l}\mu_{\mathbf{t}}(s_j)$ and $F(\mathbf{u}) = \frac{1}{l}\sum_{j=1}^{l}\mu_{\mathbf{u}}(s_j)$. Thus

$$F(\lambda\mathbf{t} + (1-\lambda)\mathbf{u}) \geq \lambda F(\mathbf{t}) + (1-\lambda)F(\mathbf{u}),$$

which proves that F is a convex function.

To prove (13), let $\mathbf{e}_1 = (1, 0, \dots, 0)$, $\cdots$, $\mathbf{e}_r = (0, 0, \dots, 1)$ be the natural basis of $\mathbb{R}^r$, and write, for $\mathbf{t} \in \triangle$, $\mathbf{t} = t_1\mathbf{e_1} + \cdots + t_r\mathbf{e}_r$. Then, notice that $t_1 + \cdots + t_r = 1$, from the convexity of F, we get

$$F(\mathbf{t}) = F(t_1\mathbf{e_1} + \cdots + t_r\mathbf{e}_r) \geq (t_1 + \cdots + t_r)\min_i F(\mathbf{e}_i) = \min_i F(\mathbf{e}_i)$$

and, obviously, $F(\mathbf{e}_i) = \frac{1}{h^0(\mathscr{L})}\sum_{m\geq 1} h^0(\mathscr{L}(-mD_i))$ for $i = 1, \dots, r$. This finishes the proof.

Proof of Theorem 5.5. Let $\epsilon > 0$, and pick a positive integer N such that

$$\max_{1\leq j\leq q}\frac{Nh^0(\mathscr{L}^N)}{\sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_j))} < \max_{1\leq j\leq q}\gamma(\mathscr{L}, D_j) + \frac{\epsilon}{4}\ . \tag{15}$$

Let

$$\Sigma = \left\{\sigma \subseteq \{1, \dots, q\} \mid \bigcap_{j\in\sigma}\operatorname{Supp} D_j \neq \emptyset\right\}\ .$$

For any $z \in \mathbf{C}$, since $D_i, 1 \leq i \leq q$, are in ℓ-sub-general position, there is $\sigma \in \Sigma$ such that

$$\sum_{i=1}^{q}\lambda_{D_i}(x) \leq \sum_{j\in\sigma}\lambda_{D_j}(x) + O(1) \tag{16}$$

where σ depends on the point x, but $O(1)$ is independent of x. Consider the following filtration of $H^0(X, \mathscr{L}^N)$ with respect to σ: Let

$$\triangle_\sigma = \left\{\mathbf{a} = (a_i) \in \mathbb{N}^{\#\sigma} \mid \sum_{i\in\sigma} a_i = b\right\}.$$

For $\mathbf{a} \in \triangle_\sigma$ (hence $\frac{1}{b}\mathbf{a} \in \triangle$), as above, one defines (see (10), (11), and (12)) the ideal $\mathcal{I}(x)$ of $\mathscr{O}_X$ by

$$\mathcal{I}(x) = \sum_{\mathbf{b}}\mathscr{O}_X\left(-\sum_{i\in\sigma} b_i D_i\right)$$

where the sum is taken for all $\mathbf{b} \in \mathbb{N}^{\#\sigma}$ with $\sum_{i\in\sigma} a_i b_i \geq bx$. Let

$$\mathcal{F}(\sigma;\mathbf{a})_x = H^0(X, \mathscr{L}^N \otimes \mathcal{I}(x)) ,$$

which we regard as a subspace of $H^0(X, \mathscr{L}^N)$, and let

$$F(\sigma;\mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)} \int_0^{+\infty} (\dim \mathcal{F}(\sigma;\mathbf{a})_x)\, dx .$$

Applying Proposition 5.1 with the line sheaf being taken as $\mathscr{L}^N$, we have

$$F(\sigma;\mathbf{a}) \geq \min_{1\leq i\leq q} \left(\frac{1}{h^0(\mathscr{L}^N)} \sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i)) \right).$$

Let $\mathcal{B}_{\sigma;\mathbf{a}}$ be a basis of $H^0(X, \mathscr{L}^N)$ adapted to the above filtration $\{\mathcal{F}(\sigma;\mathbf{a})_x\}_{x\in\mathbb{R}^+}$. By Remark 5.1, $F(\sigma,\mathbf{a}) = \frac{1}{h^0(\mathscr{L}^N)} \sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}} \mu(s)$, where $\mu(s)$ is the largest rational number for which $s \in \mathcal{F}(\sigma;\mathbf{a})_\mu$. Hence

$$\sum_{s\in\mathcal{B}_{\sigma;\mathbf{a}}} \mu(s) \geq \min_{1\leq i\leq q} \sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i)) . \tag{17}$$

It is important to note that the set $\bigcup_{\sigma;\mathbf{a}} \mathcal{B}_{\sigma;\mathbf{a}}$ is a finite set.

By a compactness argument, there exist a finite covering $\{U_j\}_{j\in J_{\sigma,\mathbf{a},s}}$ of X by Zariski-open sets and a finite set $K_{\sigma,\mathbf{a},s} \subseteq \mathbb{N}^{\#\sigma}$ such that

$$s = \sum_{\mathbf{b}\in K_{\sigma,\mathbf{a},s}} f_{s,j;\mathbf{b}} \prod_{i\in\sigma} 1_{D_i}^{b_i} \tag{18}$$

on U_j for all $j \in J_{\sigma,\mathbf{a},s}$, where 1_{D_i} is the canonical section of $\mathscr{O}(D_i)$ for each i and $f_{s,j;\mathbf{b}} \in \Gamma(U_j, \mathscr{L}^N(-\sum_{i\in\sigma} b_i D_i))$ and all $\mathbf{b} \in K$ satisfy $\sum_{i\in\sigma} a_i b_i \geq b\mu(s)$. Hence

$$\lambda_s(f(z)) \geq \min_{\mathbf{b}\in K_{\sigma,\mathbf{a},s}} \sum_{i\in\sigma} b_i \lambda_{D_i}(f(z)) + O(1).$$

Let $c \geq 1$ be an integer such that $h^0(\mathscr{L}^N(-cD_j)) = 0$ for $j = 1, \ldots, q$ and fix an integer b with $b \geq \frac{cn}{N\epsilon_0}$, where $\epsilon_0 > 0$ is chosen such that

$$\epsilon_0 < \frac{\epsilon}{(\max_{1\leq j\leq q} \gamma(\mathscr{L}, D_j) + 1 + \epsilon)(4\max_{1\leq j\leq q} \gamma(\mathscr{L}, D_j) + 1 + \epsilon)}.$$

Therefore, by the choice of b, we may assume that all $\mathbf{b} \in K_{\sigma,\mathbf{a},s}$ satisfies

$$\sum_{i\in\sigma} b_i \leq nc \leq bN\epsilon_0 . \tag{19}$$

Choose $\mathbf{a} = (a_i) \in \triangle_\sigma$ such that

$$\left| \frac{\lambda_{D_i}(f(z))}{\sum_{j\in\sigma} \lambda_{D_j}(f(z))} - \frac{a_i}{b} \right| \leq \frac{1}{b} \text{ for all } i \in \sigma,$$

i.e.

$$\begin{aligned}
\lambda_s(f(z)) &\geq \min_{\mathbf{b}\in K_{\sigma,\mathbf{a},s}} \sum_{i\in\sigma} b_i \lambda_{D_i}(f(z)) + O_v(1) \\
&\geq \left(\sum_{j\in\sigma} \lambda_{D_j}(f(z))\right) \min_{\mathbf{b}\in K\sigma,\mathbf{a},s} \sum_{i\in\sigma} b_i \frac{a_{P,v;i}-1}{b} + O_v(1) \qquad (20)\\
&\geq (\mu(s) - N\epsilon_0)\left(\sum_{j\in\sigma} \lambda_{D_j}(f(z))\right) + O_v(1),
\end{aligned}$$

Therefore, by (20), (17) and (16), we have

(21)

$$\begin{aligned}
\sum_{s\in\mathcal{B}_\sigma} \lambda_{s,v}(f(z)) &\geq \left(\sum_{s\in\mathcal{B}_\sigma} (\mu(s) - N\epsilon_0)\right)\left(\sum_{i\in\sigma} \lambda_{D_i}(f(z))\right) + O_v(1) \\
&\geq \left(\min_{1\leq i\leq q} \sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i)) - Nl\epsilon_0\right)\left(\sum_{i\in\sigma} \lambda_{D_i}(f(z))\right) + O_v(1) \\
&\geq \left(\min_{1\leq i\leq q} \sum_{m\geq 1} h^0(\mathscr{L}^N(-mD_i)) - Nl\epsilon_0\right) \sum_{i=1}^{q} \lambda_{D_i}(f(z)) + O_v(1)\ ,
\end{aligned}$$

where $l = h^0(\mathscr{L}^N)$. The rest of teh argument is similar to above by applying the Basic Theorem. This finishes the proof.

Note that, by the computation we dis above, if each D_j, $1 \leq j \leq q$, is linearly equivalent to a fixed ample divisor A on X, then

$$\gamma(D_j) = \lim_{N\to\infty} \frac{N\frac{(qN)^n A^n}{n!} + o(N^{n+1})}{\frac{A^n(qN-1)^{n+1}}{(n+1)!} + o(N^{n+1})} = \frac{n+1}{q}.$$

So the Theorem of Ru-Vojta recovers Theorem 5.4 (as well as giving an alternative proof of Theorem 5.4).