Smale's Mean Value Conjecture

and Related Problems

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1. Smale's Mean Value Conjecture

Let P be a non-linear polynomial; then b is a **critical point** of P if $P'(b) = 0$, and v is a **critical value** of P if $v = P(b)$ for some critical point b of P.

In 1981, Stephen Smale proved the following

Theorem 1. *Let P be a non-linear polynomial and $a \in \mathbb{C}$ such that $P'(a) \neq 0$. Then there exists a critical point b of P such that*

$$\left|\frac{P(a) - P(b)}{a - b}\right| \leq 4|P'(a)| \tag{1.1}$$

Or equivalently, we have

$$\min_{b, P'(b)=0} \left|\frac{P(a) - P(b)}{a - b}\right| \leq 4|P'(a)| \tag{1.2}$$

Smale then asked whether one can replace the factor 4 in the upper bound in (1.1) by 1, or even possibly by $\frac{d-1}{d}$, where $d = \deg P$.

He also pointed out that the number $\frac{d-1}{d}$ would, if true, be the best possible bound here as it is attained (for any nonzero λ) when $P(z) = z^d - \lambda z$ and $a = 0$ in (1.1).

Note that if b_i are the critical points of $P(z) = z^d - \lambda z$ and $a = 0$, then

$$\left|\frac{P(b_1)}{b_1 P'(0)}\right| = \cdots = \left|\frac{P(b_{d-1})}{b_{d-1} P'(0)}\right| = \frac{d-1}{d}.$$

Q: Is it also true for **all** extremal polynomials ?

$$\min_{b, P'(b)=0} \left| \frac{P(a) - P(b)}{a - b} \right| \le 4|P'(a)| \tag{1.2}$$

Let M be the least possible values of the factor in the upper bound in (1.2) for all non-linear polynomials and M_d be the corresponding value for the polynomial of degree d.

Then Smale's theorem and example show that

$\frac{d-1}{d} \le M_d \le 4$ and $1 \le M \le 4$.

Smale's Mean Value conjecture:

$M = 1$ or even $M_d = \frac{d-1}{d}$, where $d = \deg P$.

• S. Smale, The fundamental theorem of algebra and complexity theory, *Bull. Amer. Math. Soc.* 4 (1981), 1-36.

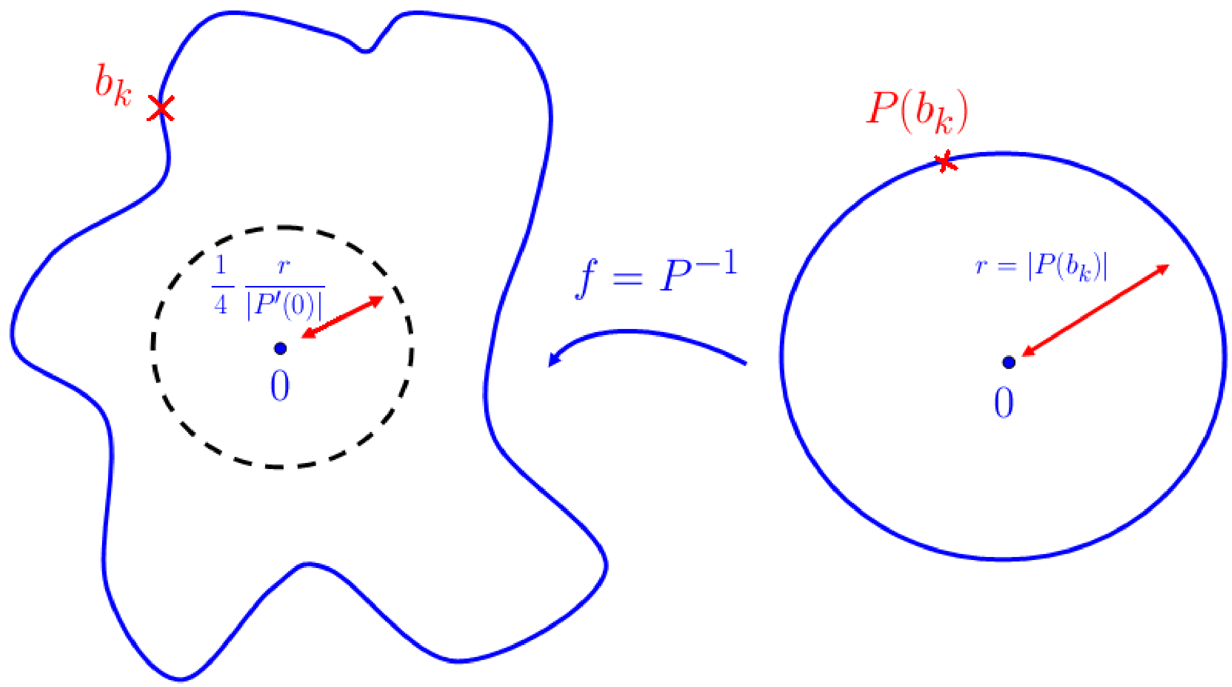
• S. Smale, Mathematical Problems for the Next Century, *Mathematics: frontiers and perspectives*, eds. Arnold, V., Atiyah, M., Lax, P. and Mazur, B., Amer. Math. Soc., 2000.

Smale's mean value conjecture is equivalent to the following

Normalised conjecture : Let P be a monic polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) = 1$. Let $b_1, \ldots, b_{d-1}$ be its critical points. Then

$$\min_i \left| \frac{P(b_i)}{b_i P'(0)} \right| \leq \frac{d-1}{d} \qquad (*)$$

Smale's proof: $P(0) = 0$, $P'(0) \neq 0$, apply Koebe $1/4$ Thm.



$$\frac{1}{4}\frac{|P(b_k)|}{|P'(0)|} = \frac{1}{4}\frac{r}{|P'(0)|} \leq |b_k|$$

Smale's Mean Value conjecture is true for the following polynomials.

A) Polynomials of degree $d = 2, 3, 4, 5$.

B) Polynomials with real zeros only.

C) Polynomials with zero constant term and all the zeros have the same modulus.

D) Polynomials with zero constant term and all the critical points are fixed points.

- D. Tischler (1989).

E) Polynomials with zero constant term and all the **critical points** have the same modulus.

F) Polynomials with zero constant term and all the **critical values** have the same modulus.

- V.V. Andrievskii and S. Ruscheweyh (1998).

G) Polynomials with critical points lying on two rays.

- A. Hinkkanen and I. Kayumov (2010).

- Q.I. Rahman and G. Schmeisser, *Analytic theory of Polynomials*, OUP, 2002.
- T. Sheil-Small, *Complex polynomials*. CUP, 2002.

Estimates of M_d

Tischler (1989), Crane (2006), Sendov (2006)

For $2 \leq d \leq 5$,

$$M_d = \frac{d-1}{d}.$$

Beardon, Minda and N. (2002)

$$M_d \leq 4^{1-\frac{1}{d-1}} = 4 - \frac{4\log 4}{d} + O\left(\frac{1}{d^2}\right).$$

Conte, Fujikawa and Lakic (2007)

$$M_d \le 4\frac{d-1}{d+1} = 4 - \frac{8}{d} + O\left(\frac{1}{d^2}\right) .$$

Fujikawa and Sugawa (2006)

$$M_d \le 4\left(\frac{1+(d-2)4^{-1/(d-1)}}{d+1}\right) = 4 - \frac{8+4\log 4}{d} + O\left(\frac{1}{d^2}\right).$$

Crane (2007)

For $d \geq 8$,

$$M_d \leq 4 - \frac{2}{\sqrt{d}}.$$

By applying some results on univalent functions with omitted values, we have

Theorem 2. (N., 2003) *Let P be a polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) \neq 0$. Let $b_1, \ldots, b_{d-1}$ be its critical points such that $|b_1| \leq |b_2| \leq \cdots \leq |b_{d-1}|$. Suppose that $b_2 = -b_1$, then*

$$\min_i \left| \frac{P(b_i)}{b_i} \right| \leq 2|P'(0)|.$$

G. Schmieder (2002-2003), A proof of Smale's mean value conjecture (math.CV/0206174).

J. Borcea (2003), Maximal and inextensible polynomials and the geometry of the spectra of normal operators (math.CV/0309233).

Motivation.

Smale discovered the Mean Value theorem as a by product of his investigations of the efficiency of zero finding algorithms.

Newton's map of P: $N_P(z) = z - \frac{P(z)}{P'(z)}$.

Choose an initial point z_0 suitably and let

$$z_{n+1} = N_P(z_n) = z_n - \frac{P(z_n)}{P'(z_n)},$$

then the sequence $\{z_n\}$ will converge to a zero of P.

If we consider the Taylor's series of P at z_n, then we have

$$P(z_{n+1}) = P(z_n) + \sum_{i=1}^{d} (-1)^i \frac{P^{(i)}(z_n)}{i!} \left(\frac{P(z_n)}{P'(z_n)} \right)^i .$$

It follows that $\frac{P(z_{n+1})}{P(z_n)} = 1 + \sum_{i=1}^{d} (-1)^i \frac{P^{(i)}(z_n) P(z_n)^{i-1}}{i! P'(z_n)^i}$ and hence the efficiency of Newton's method mainly depends on the growth of

$$\frac{P^{(i)}(z_n) P(z_n)^{i-1}}{i! P'(z_n)^i},$$

$i = 2, 3, ..., d; n = 0, 1, ...$

By using Löwner's theorem, Smale proved the following result.

Theorem 3. (Smale, 1981) *Let a be any non-critical point of P. Then there exists a critical point b of P such that for each $k \geq 2$,*

$$\left|\frac{P^{(k)}(a)}{k!P'(a)}\right|^{\frac{1}{k-1}} |P(a) - P(b)| \leq 4|P'(a)| \qquad (**)$$

Let K be the least possible values of the factor in the upper bound in $(**)$ and $K_{d,i}$ be the corresponding value for the polynomial of degree d and $k = i$.

Smale suggested six open problems (Problem 1A-1F) related to the inequality (**).

Most of these problems are about the precise values of K and $K_{d,i}$.

Smale also gave an example to show that $1 \leq K \leq 4$ and conjectured that $K = 1$.

Problem 1A: Reduce K from 4.

Problem 1B, 1C and 1D are about $K_{d,2}$.

Porblem 1E is the mean value conjecture.

The constant K is quite important for estimating the efficiency of Newton's Method.

Theorem 4. (Smale,1981) *Let* $R_0 = \min_{b, P'(b)=0}\{|P(b)|\} > 0$. *If* $|P(w)| < \frac{R_0}{3K+1}$, *then the iterations of Newton's method starting at* w *will converge to some zero of* P. *In addition, if* $P(w) \neq 0$, *one has*

$$\frac{|P(w')|}{|P(w)|} < \frac{1}{2}$$

where $w' = w - \frac{P(w)}{P'(w)}$.

When $i = 2$, Smale showed that for some critical point b,

$$\left|\frac{P^{(2)}(a)}{2P'(a)}\right| |P(a) - P(b)| \; = \; \left|\frac{1}{2}\sum_{j=1}^{d-1} \frac{1}{a-b}\right| |P(a) - P(b)| \leq 2|P'(a)|$$

Problem 1B asked whether 2 can further be reduced to $\frac{d-1}{2d}$, i.e.

$$\left|\sum_{j=1}^{d-1} \frac{1}{a-b}\right| |P(a) - P(b)| \leq \frac{d-1}{d}|P'(a)|.$$

- $K_{d,2} = \frac{d-1}{2d}$ when $d = 2$, i.e. $K_{2,2} = \frac{1}{4}$.

For Problem 1B, Y.Y. Choi, P.L. Cheung and N. showed that

$$K_{3,2} = \frac{4}{6\sqrt{3}} = 0.3845... > \frac{2}{6}$$

$$K_{4,2} \geq 0.473... > \frac{3}{8}, \quad K_{d,2} = ?$$

For Problem 1A, we also showed that

$$K \leq 4^{\frac{d-2}{d-1}}.$$

$$\left|\frac{P^{(i)}(a)}{i!P'(a)}\right|^{\frac{1}{i-1}}|P(a)-P(b)| \leq K_{d,i}|P'(a)|.$$

For $i = d$, V.N. Dubinin (2006), applies the method of dissymmetrization to prove the **sharp** inequality.

$$\left|\frac{P^{(d)}(a)}{d!P'(a)}\right|^{\frac{1}{d-1}}|P(a)-P(b)| \leq \frac{d-1}{d^{\frac{d}{d-1}}}|P'(a)|.$$

Hence, $K_{d,d} = \frac{d-1}{d^{\frac{d}{d-1}}}$.

2. Introduction to theory of amoeba

Let $f = f(z_1, ..., z_n)$ be a non-constant polynomial.

Let $Z_f = \{(z_1, ..., z_n) \in \mathbb{C}_*^n \,|\, f(z_1, .., z_n) = 0\}$ be the hypersurface defined by f.

The amoeba $\mathcal{A}_f$ is defined to be the image of Z_f under the map $\mathrm{Log} : \mathbb{C}_*^n \to \mathbb{R}^n$ defined by

$$\mathrm{Log}(z_1, ..., z_n) = (\log |z_1|, ..., \log |z_n|).$$

- Introduced by Gelfand, Kapranov and Zelevinsky in 1994.

I.N. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Math. Theory Appl., Birkhauser, Boston, 1994.

M. Forsberg, M. Passare, and A. Tsikh, Laurent determinants and arrangements of hyperplane amoebas, *Adv. Math.* **151** (2000), 45–70.

Components of the complement

Theorem (GKZ, 1994). $\mathcal{A}_f$ is closed and any connected component of $\mathcal{A}_f^c = \mathbb{R}^n \backslash \mathcal{A}_f$ is convex.

Ronkin function for the hypersurface, $N_f : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$N_f(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\mathsf{Log}^{-1}(\mathbf{x})} \log |f(\mathbf{z})| \frac{\mathsf{d}z_1}{z_1} \wedge \ldots \wedge \frac{\mathsf{d}z_n}{z_n}.$$

Theorem (Ronkin,2001). N_f is convex. It is affine on each connected component of $\mathcal{A}_f^c$ and strictly convex on $\mathcal{A}_f$.

Proposition (FPT,2000). The derivative of N_f with respect to x_j is the real part of

$$\nu_j(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\mathsf{Log}^{-1}(\mathbf{x})} \frac{\partial f}{\partial z_j} \frac{z_j}{f(\mathbf{z})} \frac{\mathsf{d}z_1}{z_1} \wedge \ldots \wedge \frac{\mathsf{d}z_n}{z_n}$$

- $\nu_j(\mathbf{x})$ is an integer in a connected component of $\mathcal{A}_f^c$.

Let $f = \sum_{\omega \in I} b_\omega \mathbf{z}^\omega$ and Δ be the Newton polygon of f (that is, the convex hull of the elements ω of I for which $b_\omega \neq 0$.)

Theorem (FPT,2000). The map

$$\begin{aligned} \text{ord} : \mathcal{A}_f^c \;&\to\; \Delta \cap \mathbb{Z}^n \\ \mathbf{x} \;&\mapsto\; (\nu_1(\mathbf{x}), \ldots, \nu_n(\mathbf{x})) \end{aligned}$$

sends two different connected components to two different points.

Proposition (FPT, 2000). Let $\mathbf{x} \in \mathcal{A}_f^c$ and $\mathbf{u} \in \mathrm{Log}^{-1}(\mathbf{x})$. For each $1 \leq i \leq n$, define

$$f^{i,\mathbf{u}}(z) = f(u_1, ..., u_{i-1}, z, u_{i+1}, ..., u_n).$$

Then,

$\nu_i(\mathbf{x})$ is equal to the number of roots of $f^{i,\mathbf{u}}(z)$ inside $\{|z| < e^{x_i}\}$.

3. A problem on extremal polynomials.

Recall that we can always assume that the polynomials are **monic**. Note that any monic polynomial with zero constant term is determined uniquely by its critical points.

Let $B = (b_1, \ldots, b_{d-1}) \in \mathbb{C}^{d-1}$ and $P_B(z)$ be a degree d monic polynomial whose critical points are $b_1, \ldots, b_{d-1}$.

If $P_B(0) = 0$, then $P_B(z) = d \int_0^z (w - b_1) \cdots (w - b_{d-1}) dw$.

Assume that 0 is not be a critial point of $P_B(z)$.Then, $P_B'(0) \neq 0$ or $\prod b_i \neq 0$.

Let $\lambda \neq 0$. Consider

$$P_{\lambda B}(z) = d \int_0^z (w - \lambda b_1) \cdots (w - \lambda b_{d-1}) dw.$$

Then,

$$\frac{P_{\lambda B}(\lambda b_i)}{\lambda b_i P'_{\lambda B}(0)} = \frac{P_B(b_i)}{b_i P'_B(0)}.$$

Therefore, we may further assume that B is in the set

$$E = \left\{ (z_1, ..., z_{d-1}) \in \mathbb{C}^{d-1} | \prod z_i = \frac{(-1)^{d-1}}{d} \right\}$$

so that $P'_{\lambda B}(0) = 1$.

Define $S_i : E \to \mathbb{C}$ by

$$S_i(B) = S_i(b_1, \ldots, b_{d-1}) = \frac{P_B(b_i)}{b_i P_B'(0)} = \frac{P_B(b_i)}{b_i}.$$

To solve Smale's conjecture, we need to show that

$$\sup_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} = \frac{d-1}{d}$$

- Not clear if a maximum point exists.

Theorem 5. (E. Crane, 2006). *There exists some B^* such that*

$$\max_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

We know that

$$\max_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} < 4.$$

Hence, $|S_1(B^*)| = \cdots = |S_{d-1}(B^*)| < 4$. It can then be proven that $|b_i^*| < 4^{\frac{d+1}{d-1}}$ for all $1 \le i \le d-1$.

This gives an **explicit compact set** of polynomials in which **at least one** extremal polynomial must be found.

Crane mentioned in the paper that he is able to use verifiable interval arithmetic to confirm that

$$M_5 = \frac{4}{5}.$$

Theorem 6. (E. Crane, 2006) *If $M_{d+1} > M_d$, then for* **all** *degree d extremal polynomial P_{B^*},*

$$\max_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

Conjecture 1: For **all** degree d extremal polynomial P_{B^*}, we have

$$\max_{B \in E}\{\min_{1 \le i \le d-1} |S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

- True when $2 \le d \le 5$.

Consider the map $S : E \to \mathbb{C}^{d-1}$ defined by $S(B) = (S_1(B), ..., S_{d-1}(B))$.

As pointed out in Crane's paper, it follows from the properness of the maps S and $(\log|\cdot|, ..., \log|\cdot|)$ that we have the following:

If Conjecture 1 is true, then for each $d \geq 2$, the set of all degree d extremal polynomials P_{B^*} is **compact**.

An amoeba approach

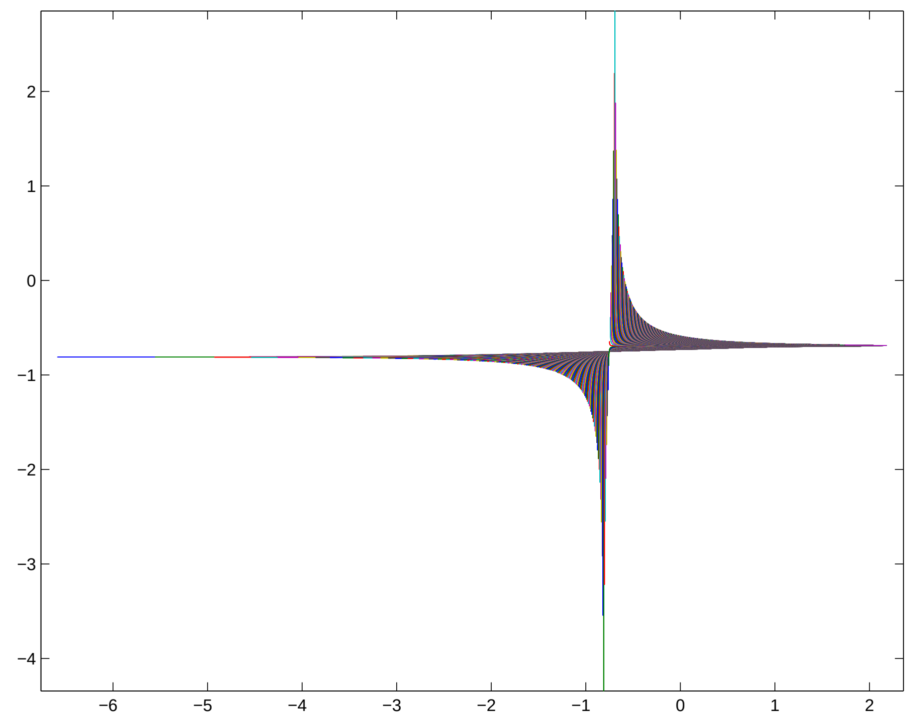
Note that we have $d-1$ homogeneous $S_i(b_1, ..., b_{d-1})$ polynomial in $d-1$ variables, so there is a unique non-constant irreducible symmetric complex polynomial $f = f_d$ such that

$$f(S_1(B), \ldots, S_{d-1}(B)) = 0$$

whenever $B \in \mathbb{C}_*^{d-1}$.

Let $\mathcal{A}_f$ be the amoeba of f. It follows from Smale's theorem that for all $t > 4$, $\mathbf{t} = (\log t, ..., \log t)$ lies in $\mathcal{A}_f^c$.

For $d = 3$, $f(z_1, z_2) = 18z_1z_2 - 9z_1 - 9z_2 + 4$



Theorem 7. *f has a leading term of the form $z_1^k \cdots z_{d-1}^k$ for some $k \in \mathbb{N}$. Let U be the unbounded component of $\mathcal{A}_f^c$ containing $(\log 4, ..., \log 4) + \mathbb{R}_+^{d-1}$ and $\mathbf{d} = (\log \frac{d-1}{d}, ..., \log \frac{d-1}{d})$. Then the following are equivalent:*

1) Smale's mean value conjecture is true for degree d;

2) U contains $\mathbf{d} + \mathbb{R}_+^{d-1}$;

3) U contains the ray $\{t(1, ..., 1) : t > \log \frac{d-1}{d}\}$;

4) $\mathbf{d}$ is a boundary point of U;

5) $N_f(\mathbf{d}) = k(d-1) \log \frac{d-1}{d}$.

Theorem 5 (E. Crane, 2006). There exists some B^* such that

$$\max_{B \in E} \{ \min_{1 \le i \le d-1} |S_i(B)| \} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

Max-Min and Min-Max problem on hypersurfaces in $\mathbb{C}^n$

For a non-constant polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ and the hypersurface $Z_f \subset \mathbb{C}_*^n$ defined by f, let

$$C(f) = \sup_{\mathbf{z} \in Z_f} \left(\min_{1 \leq i \leq n} |z_i| \right) .$$

Problem: Characterize those polynomials f for which $C(f)$ is finite, and for such a polynomial to determine whether the bound is attained by some point $\mathbf{x} \in \mathbb{C}^k$.

A monomial term of the polynomial f is the **dominant monomial** of f if it is of maximal degree in each variable separately.

Theorem 8. (Crane, 2006) *$f \in \mathbb{C}[z_1, \ldots, z_n]$ has a dominant monomial if and only if $C(f) < \infty$.*

Theorem 9. (Crane,2006) *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be non-constant. If $C(f) < \infty$, then there exists some $(z_1, \ldots, z_n) \in Z_f$ such that*

$$|z_1| = \cdots = |z_n| = C(f).$$

Proof of Theorem 9. Let $c = \log C(f)$ and $C = (c, ..., c) \in \mathbb{R}^n$. For a fixed $\delta > 0$, let $C_\delta = (c + \delta, ..., c + \delta) \in \mathbb{R}^n$.

By the definition of $C(f)$, it follows that $C_\delta + \mathbb{R}^n_{\geq 0}$ belongs to some component $\mathcal{C}$ of $\mathcal{A}_f^c$.

Assume that there is no points $(z_1, \ldots, z_n) \in Z_f$ such that $|z_1| = \cdots = |z_n| = C(f)$, then we must have the ray $t(1, ..., 1), t \geq c$ belongs to $\mathcal{A}_f^c$ and hence the ray is actually inside the component $\mathcal{C}$.

Since $\mathcal{C}$ is open, we can find some point $A = (a, ..., a) \in \mathcal{C}$ with $a < c$.

It follows from the definition of $C(f)$ that for a sufficiently small $\epsilon > 0$, there exists some point $X = (x_1, ..., x_n) \in \mathcal{A}_f$ such that

$$a < c - \epsilon < \min_{1 \leq i \leq n} x_i \leq c < c + \delta.$$

WLOG assume that $x_1 = \min_{1 \leq i \leq n} x_i$.

Consider the straight line $A + t(X - A)$ and let $Y = (y_1, ..., y_n)$ be the point on this line corresponding to the parameter $t = \frac{c+\delta-a}{x_1-a} > 1$. Then $y_i = a + (c + \delta - a)\frac{x_i-a}{x_1-a} \geq a + (c + \delta - a) \cdot 1 = c + \delta$ for all i.

Therefore, $Y \in C_\delta + \mathbb{R}^n_{\geq 0} \subset \mathcal{C} \subset \mathcal{A}_f^c$.

We know that $\mathcal{C}$ is convex and it follows that the whole line segment joining A and Y belongs to $\mathcal{C}$ and in particular $X \in \mathcal{C}$ which is a contradiction.

Theorem 5 (E. Crane, 2006). There exists some B^* such that

$$\max_{B\in E}\{\min_{1\le i\le d-1} |S_i(B)|\} = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|.$$

Proof of Theorem 5. Let $f = f_d$ be the implicit polynomial obtained from the rational functions $S_i(B)$. By Smale's mean value theorem, we have $C(f) \le 4$.

It then follows from Theorem 9 that there exists an extremal polynomial P_{B^*} such that $M_d = |S_1(B^*)| = \cdots = |S_{d-1}(B^*)|$.

Related to the above max-min problem, we consider the dual min-max problem.

For a non-constant polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ and the hypersurface $Z_f \subset \mathbb{C}_*^n$ defined by f, let

$$D(f) = \inf_{\mathbf{z} \in Z_f} \left(\max_{1 \le i \le n} |z_i| \right) .$$

$$D(f) = \inf_{\mathbf{z} \in Z_f} \left(\max_{1 \le i \le n} |z_i| \right)$$

Theorem 10. *$f \in \mathbb{C}[z_1, \ldots, z_n]$ has a non-zero constant term if and only if $D(f) > 0$.*

Theorem 11. *If $D(f) > 0$, then there exists at least one $(z_1, \ldots, z_n) \in Z_f$ such that*

$$|z_1| = \cdots = |z_n| = D(f).$$

$$f_d(S_1(B), \ldots, S_{d-1}(B)) = 0.$$

Note that f_d has a non-zero constant term. Apply the previous result to $f = f_d$.

Theorem 12. *There exists some $N_d > 0$ such that if P be a monic polynomial of degree d ≥ 2 with $P(0) = 0$ and $P'(0) = 1$ and $b_1, \ldots, b_{d-1}$ are its critical points, then*

$$\max_i \left| \frac{P(b_i)}{b_i P'(0)} \right| \geq N_d \, .$$

Moreover, at least one of the extremal polynomials for N_d satisfies the condition

$$\left|\frac{P(b_1)}{b_1 P'(0)}\right| = \cdots = \left|\frac{P(b_{d-1})}{b_{d-1} P'(0)}\right|. \qquad (*)$$

Dual Mean Value Conjecture:

Let P be a monic polynomial of degree $d \geq 2$ such that $P(0) = 0$ and $P'(0) = 1$. Let $b_1, \ldots, b_{d-1}$ be its critical points. Then

$$\max_i \left| \frac{P(b_i)}{b_i P'(0)} \right| \geq \frac{1}{d}.$$

It is conjectured that the extremal polynomial should be $P(z) = (z-a)^d - (-a)^d$, where a is some non-zero complex number.

Note that Dubinin and Sugawa (2009) have also discovered this dual mean value conjecture around the same time independently and they are able to show that $N_d \geq 1/(d4^d)$ which has been improved by Ng and Zhang (2016) to $N_d \geq 1/(4^d)$ recently.

Applications to $K_{d,i}$

Note that the existence of the extremal polynomials for any $K_{d,i}$ has never been proven and it is not clear if they exist at all because the parameter space for the normalized polynomials is not compact.

Using the amoeba theory, one can prove that for each $K_{d,i}$, at least one extremal polynomial exists .

Applications to Pareto optimal points

Recall that

$$S_i(B) = S_i(b_1, \ldots, b_{d-1}) = \frac{P_B(b_i)}{b_i P_B'(0)}.$$

In Problem 1D, Smale suggested to look for the *Pareto optimal points* of those attain the following optimization problem:

$$\max_{B \in E} \{ \min_{1 \le i \le d-1} |S_i(B)| \}$$

Definition: $B^* = (b_1^*, \ldots, b_{d-1}^*) \in \mathbb{C}_*^{d-1}$ is a *Pareto optimal point* if there is no $B \in \mathbb{C}_*^{d-1}$ such that $S_j(B) \geq S_j(B^*)$ for all $1 \leq j \leq d-1$ with strict inequality for some j.

For the past thirty years, no one knows if a Pareto optimal point exists.

Using the amoeba theory, one can show that such a Pareto optimal point does exist if the set of extremal polynomials is compact.

Problems for amoeba

Q1: How to determine if $\mathcal{A}_g^c$ of a given polynomial g has a bounded component if one knows a parametrization of g ?

Q2: Given a parametrization of g, how to determine if a point is in $\mathcal{A}_g^c$?

The answers to these questions would have important consequences to the study of Smale's problems and the mean value conjecture.