Weil-Petersson metric and hyperbolicity problems of some families of polarized manifolds

Conference on Complex Geometry, Dynamic Systems and Foliation Theory
Institute for Mathematical Sciceces
National University of Singapore
May 15-19, 2017

Sai-Kee Yeung
Purdue University

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- (ii). Log-Kodaira dimension of moduli

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- (iv) General type, or Log-general type properties, $\dim\Gamma(M, aK_M) \geqslant ca^n$ i.e. $\kappa(K_M) = n$, or $\dim\Gamma(M, a(K_M + D)) \geqslant ca^n$, i.e. $\kappa(K_M + D) = n$.

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- Similarly, we may consider $\mathcal{M}_{g,n}$, moduli of Riemann surfaces of genus g with n punctures.
- $\mathcal{M}_g$, $\mathcal{M}_{g,n}$ for $g \geqslant 2$ share the following properties: negatively curved, hyperbolic, and are of log-general type.

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- Kobayashi hyp $\Leftarrow$ Brody hyp if M compact (Brody Reparametrization)
  In general '$\not\Leftarrow$' if M non-compact

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$$\frac{f^*g}{g_{\Delta_R}} \leqslant \frac{1}{c}.$$

  Poincaré metric $g_{\Delta_R} = \frac{R^2|dz|^2}{(R^2-|z|^2)^2}$.
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  At $z = 0$, $g_{\Delta_R}(0) = \frac{|dz|^2}{R^2}$.
- (ii) Apply Lemma to $f : \Delta_R \to M$, at 0, with $df(\frac{\partial}{\partial z}) = v$,
  $\Longrightarrow R$ is bounded above
  $\Longrightarrow |v|_{g_K} > 0$.

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- eg 3. $\mathcal{M}_g$ is hyperbolic, $g \geqslant 2$.

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- (II) $\Longrightarrow$ (I), note $P^1_{\mathbb{C}} - \{0, 1, \infty\}$ and $T - \{0\}$ are hyperbolic.

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  - In fact, $-c$ depends only on the Chern number c_1^n of a fiber, similar to $\mathcal{M}_g$ case.

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- Computation of curvature of Weil-Petersson metric for higher dimensional manifolds begins with a paper of Siu in 1986.

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Let $\pi : X \to S$ be an effectively parametrized holomorphic family of compact polarized Kähler manifolds of zero first Chern class over a complex manifold S. Then S admits a $\mathcal{C}^\infty$ $Aut(\pi)$-inv Finsler metric, with holomorphic sectional curvature $\leqslant -c < 0$, where c is a constant.
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- For this article, we call M 'log-canonically polarized'.

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- ω given at t by

$$\partial\overline{\partial}\left(\frac{|dV_{\overline{M}_t}|}{\|s_t\|^2(\log\|s_t\|^2)^2}\right).$$

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- Theorem (To-Yeung (c))

  Let $\pi : X \to S$ *be an effectively parametrized holomorphic family of log-canonically polarized manifolds with bounded variation over a complex manifold* S*. Then* S *admits a* $\mathcal{C}^\infty$ *Aut*(π)*-inv Finsler metric, with holomorphic sectional curvature* $\leqslant -c < 0$*, where* c *is a constant. Hence* S *is Kobayashi hyperbolic.*

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  - For (b), all depends on existence of a Viehweg-Zuo subsheaf.
  - (Viehweg-Zuo) There exists a big subsheaf $\mathcal{F}$ of $\otimes^m \Omega(S, D)$ for some $m \in \mathbb{Z}^+$ (for canonically polarized family).

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- Theorem (To-Yeung)

Let $\pi : \chi \to S$ be an effectively parametrized family of manifolds which are one of the following types
(a) canonically polarized,
(b) log-polarized Kähler-Ricci flat,
(c) log-canonically polarized.
Assume that $S = \overline{S} - D$, where D is a simple normal crossing divisor. Then
(i). There exists explicitly a Viehweg-Zuo subsheaf of $\otimes^m \Omega(S, D)$ for some m.
(ii). S is of log-general type.

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$$\omega_{\mathcal{M}} = \frac{2\pi}{k} c_1(K^{-1}_{X|S}, g).$$

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  All Lie derivatives later are taken with respect to such vector fields.
- Hence Kodaira-Spencer Map $\rho_t : T_t S \to H^1(M_t, T_{M_t})$ is represented by $\Phi(u(t))$, a $\Box_t = \partial\partial^* + \partial^*\partial$ harmonic bundle-valued form on M_t.

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R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = {} & k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_i, \Phi_j \rangle) \cdot \langle \Phi_k, \Phi_\ell \rangle \frac{\omega^n}{n!} \\
& + k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_k, \Phi_j \rangle) \cdot \langle \Phi_i, \Phi_\ell \rangle \frac{\omega^n}{n!} \\
& + k \int_{M_t} \langle (\Box - k)^{-1} \mathcal{L}_{v_i} \Phi_k, \mathcal{L}_{v_j} \Phi_\ell \rangle \frac{\omega^n}{n!} \\
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$$\begin{aligned} \partial_i \|\Psi\|_2^2 &= \frac{\partial}{\partial t^i} \int_{M_t} \langle \Psi, \Psi \rangle \frac{\omega^n}{n!} \\ &= \int_{M_t} \langle \mathcal{L}_{v_i} \Psi, \Psi \rangle \frac{\omega^n}{n!} + \int_{M_t} \langle \Psi, \mathcal{L}_{\overline{v_i}} \Psi \rangle \frac{\omega^n}{n!} \\ &= \int_{M_t} \langle \mathcal{L}_{v_i} \Psi, \Psi \rangle \frac{\omega^n}{n!}. \end{aligned}$$

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- Key point: To handle each terms by integration by part guided by geometry.
- Obvious strategy: Control the last term by the others. (People tried for years.)

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& \qquad -k((\Box - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\
& \qquad -k((\Box - k)^{-1}(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) \\
& \qquad -\Big|(\mathcal{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\Big|^2 \\
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- To obtain the identity:
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  Completing of squares (Bochner type arguments).

III. Idea of proof of (a)

- Here $\overline{\Phi_i} \cdot \Psi_J \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell-1} TM_t)$ has components given by

$$(\overline{\Phi_i} \cdot \Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta}_1 \cdots \overline{\beta}_{\ell-1}} = \overline{(\Phi_i)^{\sigma}_{\overline{\gamma}}} \cdot (\Psi_J)^{\gamma \alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\sigma} \overline{\beta}_1 \cdots \overline{\beta}_{\ell-1}}.$$

- To obtain the identity:
  Need various integration by parts
  Regrouping of terms guided by geometry
  Completing of squares (Bochner type arguments).
- In restrospect, a similar expression was obtained independently by Schumacher (12) in a slightly different form.

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- The above implies

$$\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 \geq \frac{\|\Psi_J\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|\Psi_J\|_2^2}.$$

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- Let $\Psi_J = \|\Psi_J\|_2^2$. Then

$$\begin{array}{lll} \text{level 1} & \partial_v \overline{\partial}_v \log h^{(1)} & \geqslant \frac{h^{(1)}}{h^{(0)}} - \frac{h^{(2)}}{h^{(1)}} \\ \text{level 2} & \partial_v \overline{\partial}_v \log h^{(2)} & \geqslant \frac{h^{(2)}}{h^{(1)}} - \frac{h^{(3)}}{h^{(2)}} \\ & \cdot & \\ & \cdot & \\ \text{level n} & \partial_v \overline{\partial}_v \log h^{(n)} & \geqslant \frac{h^{(n)}}{h^{(n-1)}} - \frac{h^{(n+1)}}{h^{(n)}} \end{array}$$

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- Proposition

  Let $\sigma = \max\{\ell : \Psi_J \neq 0\}$, $N = n!$, $C_1 = \min\{1, \frac{k^n n!}{(2\pi)^n K_{M_t}^n}\}$,

  $C_\sigma = \frac{\sigma 1}{3^{\sigma-1}}, a_\ell = (\frac{3}{C_1})^{\frac{N(N^{\ell-1}-1)}{N-1}}$. *Then for*

$$
\begin{aligned}
h(v,\overline{v}) &:= (\sum_{\ell=1}^{\sigma} a_\ell \|\Psi_J\|_2^{2N/\ell})^{1/2N}, \\
\partial_v \overline{\partial}_v \log h(v,\overline{v}) &\geqslant \frac{C_\sigma}{\sigma^{1/N} a_\sigma^{1+1/N}} h(v,\overline{v}).
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- **Remark** For $n=1$, get back the results for Riemann surfaces.

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- **Remark** For $n = 1$, get back the results for Riemann surfaces.
- **Remark** Note that the sum stops at σ, which is important for Part III(ii).

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  Here $\phi_t : M_0 \to M_t$ is induced from a smooth trivialization $\phi : M_0 \times I \to \mathcal{X}$.
- Analogous to the work of Siu, Nannicini (86) obtained

$$\begin{aligned} R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) &= -\frac{1}{4V}(h_{i\bar{j}}h_{l\bar{k}} + h_{i\bar{k}}h_{l\bar{j}}) \\ &\quad - \int_{M_t} \langle(\mathcal{L}_{v_i}\Phi_k, \mathcal{L}_{v_j}\Phi_\ell\rangle \frac{\omega^n}{n!} \\ &\quad + \int_{M_t} \langle H(\Phi_i \owedge \Phi_k), H(\Phi_j \owedge \Phi_\ell)\rangle \frac{\omega^n}{n!}, \end{aligned} \tag{1}$$

here V is the volume of M_o.

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we prove

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& + ((H(\mathcal{L}_{v_i} \Psi_J), \mathcal{L}_{v_i} \Psi_J) - \left| (\mathcal{L}_{v_i} \Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2}) \right|^2 \\
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- Use bootstraping argument to construct a Finsler metric of negative holomorphic sectional curvature.

IV. About the proof of (b)

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- The same scheme works for orbifolds. Need to make sure that Hodge Decomposition, Green's kernels make sense for orbifolds.

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  (3) Need to use some sort of Maximum Principle for complete non-compact manifolds.
  (4) The above for tensors obtained after Lie derivatives with respect to the canonical (horizontal) lifts.

III(ii). About the proof of (generalized) Viehweg Conjecture

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- Proposition

  There exists a Viehweg-Zuo sheaf in cases (a), (b), (c)

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  - **Idea of Proof**
  - Consider first a Zariski open set U of $\mathcal{M}$ on which it is effectively parametrized.
  - Take a basis $\frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^m}$ of $T_t S$, and let Φ_i be the harmonic representative of $\rho_t(\frac{\partial}{\partial t^i})$ on M_t as before.

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  - Consider the map $\rho_t^{(\ell)} : S^\ell(T_t S) \to \mathcal{A}^{0,\ell}(\wedge^\ell TM_t)$ given by

$$\rho_t^{(\ell)}(\frac{\partial}{\partial t^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial t^{j_\ell}}) = \Psi_J := H(\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}).$$

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- $g_{WP,\sigma}$ is non-degenerate on $\mathcal{V}$ from defintiion.

- Computation shown earlier for Ψ_J on U_ℓ gives,
- $$\begin{aligned}
&\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 \\
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& \qquad\qquad -k((\Box - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\
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- For $\ell = \sigma$, the last term $H(\Phi_i \owedge \Psi_J) = 0$.

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  where ν_i is the canonical lift of Φ_i.
- We get a Griffith positive subsheaf $\mathcal{V}$ of $S^\ell(\Omega_S)$.

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- For this we used Theorem 1a, -ve hol sectional curv, to estimate the augmented Finsler metric by the Poincaré metric g_P in a neighborhood of D, using Ahlfors Schwartz Lemma.
- This in terms bounds Weil-Petersson metric $g_{WP,1}$ by g_P, from which we can show that L^2 sections of $\mathcal{V}|_S$ extends as log sections to $\overline{S}$ to conclude Proposition 1.

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- Appropriate modifications of the arguments can be applied to (b) and (c).