Building Blocks of Polarized Endomorphisms of Normal Projective Varieties

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Preliminary

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Let X be a (normal) projective variety.
Let $f : X \to X$ be a surjective endomorphism.
Denote by $\mathrm{N}^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron-Severi group $\mathrm{NS}(X) = Pic(X)/Pic^0(X)$.
f induces an automorphism $f^* : \mathrm{N}^1(X) \to \mathrm{N}^1(X)$.

Definition (Polarized Endomorphsim)

Let $f : X \to X$ be a surjective endomorphism of a (normal) projective variety X. We say that f is polarized if there is an ample Cartier divisor H such that $f^*H \sim qH$ for some integer $q > 1$.

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Proposition

The following are equivalent.

- *(1) f is polarized.*
- *(2) $f^*H \equiv qH$ in $\mathrm{N}^1(X)$ for some ample $\mathbb{R}$-divisor H.*

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Question: what about high dimensional case?

Main Results

Theorem (S. Meng and D.-Q. Zhang)

Let X be $\mathbb{Q}$-factorial lc and $f : X \to X$ be a polarized endomorphism. Then any MMP (with finitely many steps) starting from X can be run f-equivariantly (after iteration).

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Sketch of the idea: Let $f_i = f|_{X_i}$ and $\pi : X_i \dashrightarrow X_{i+1}$ the ith step. We need to show (1) f_i may descend to an endomorphism $f_{i+1} : X_{i+1} \to X_{i+1}$; (2) f_{i+1} is again polarized.

Main Results

For (1), the key observation is the following:

Lemma

Let (X, f) be a polarized pair. Suppose $A \subseteq X$ is a closed subvariety with $f^{-i}f^{i}(A) = A$ for all $i \geq 0$. Then $f^{\pm 1}(A) = A$ (after iteration).

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Lemma

Let X be a $\mathbb{Q}$-factorial lc normal projective variety and $f : X \to X$ a surjective endomorphism. Let $\pi : X \to Y$ be a contraction of a K_X-negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$. Suppose that $E \subseteq X$ is a subvariety such that $\dim(\pi(E)) < \dim(E)$ and $f^{-1}(E) = E$. Then replacing f by a positive power, $f(R_C) = R_C$; hence, π is f-equivariant.

Main Results

For (2), first note that by pull back of π_i, $\mathrm{N}^1(X_{i+1})$ can be regarded as an $((f^*)^{\pm 1}$-invariant) hyperplane of $\mathrm{N}^1(X_i)$ (divisoral and Fano contractions) or just $\mathrm{N}^1(X_i)$ (flip).

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Remark

1. f_i^* is a scalar iff so is f_{i+1}^*.
2. The ample cones $Amp(X_{i+1}) \cap Amp(X_i) = \varnothing$.

Main Results

Proposition (S. Meng and D.-Q. Zhang)

Let $f : V \to V$ be a (linear) automorphism of a positive dimensional real vector space V such that $f^{\pm 1}(C) = C$ for a closed cone $C \subseteq V$ which spans V and contains no line. Let q be a positive number. Then (1) and (2) below are equivalent.

(1) $f(x) = qx$ *for some* $x \in C^{\circ}$.

(2) *There exists a constant* $N > 0$, *such that* $\frac{||f^i||}{q^i} < N$ *for any* $i \in \mathbb{Z}$.

If (1) or (2) above is true, then f is a diagonalizable linear map with all eigenvalues of modulus q.

Main Results

Application: $C = Nef(X) := \overline{Amp(X)}$ or $C = PEC(X) := \overline{Big(X)}$

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Proposition (S. Meng and D.-Q. Zhang)

Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \to X$ and $g : Y \to Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. If f is polarized. Then g is polarized.

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As an application, we have:

Lemma (S. Meng and D.-Q. Zhang)

Let (X, f) be a polarized pair with X being a $\mathbb{Q}$-factorial klt normal projective variety. Assume that K_X is pseudo-effective. Then X is Q-abelian.

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Remark

Let X be a Q-abelian variety and $f : X \to X$ a surjective endomorphism. Assume the existence of a non-empty closed subset $Z \subsetneq X$ and $s > 0$, such that $f^{-s}(Z) = Z$. Then f is not polarized.

Main Results (Summarize)

Theorem (S. Meng and D.-Q. Zhang)

Let (X, f) be a polarized pair such that X has at worst $\mathbb{Q}$-factorial klt singularities. Then, replacing f by a positive power, there exist a Q-abelian variety Y, a morphism $X \to Y$, and an f-equivariant relative MMP over Y

$$X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y$$

(i.e. $f = f_1$ descends to polarized f_i on each X_i), such that we have:

(1) *If K_X is pseudo-effective, then $X = Y$ and it is Q-abelian.*

(2) *If K_X is not pseudo-effective, then for each i, $X_i \to Y$ is an equi-dimensional morphism with every fibre (irreducible) rationally connected. The $X_{r-1} \to X_r = Y$ is a Fano contraction.*

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Question

Let (X, f) be a polarized pair. Assume that X is a rationally connected variety with at worst $\mathbb{Q}$-factorial terminal singularities. Is a positive power of f^* a scalar?

True when $\dim(X) \leq 3$.

Thanks!

Q & A