Talk on May 15, 2017

Growth of the number of periodic points for meromorphic maps

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difficulty

Strategy

II. Background. We lay down the background of the talk in the second part

First of all, I would like to thank the organisors for the wonderful invitation. In this talk I will speak about some recent joint works with T.-C. DINH & T.-T. TRUONG & D.-V. VU. Here is the plan of my talk: which consists of 3 parts. In the first part (I. Introduction) I will state the main questions, ~~prepare some background~~ and state the main results. The ~~second~~ part (III Main Tools) is devoted to the two theories ~~develop~~ which have been developed by T.-C. Dinh & N. Sibony. of intersection of positive closed currents. These theories will play a key role in talk. In the last part (IV The Key points of the proof) I will explain some key points in the proof ~~give a sketchy proof~~ of the main results.

Now let me begin with the first part. Here is the settings of our problems.

- Let X be a compact Kähler manifold of dim k.
- Let $f: X \to X$ be a dominant holomorphic/meromorphic map or correspondence (Correspondence = multivalued map).
- Let $f^n := f\circ\cdots\circ f$ (n times) be the iterate of order n of f
- Let Q_n be the set of isolated periodic points of period n. (i.e. isolated solutions of $f^n(x) = x$), counting with multiplicity.

We address the following problems.

Problem 1 Compute or estimate $\# Q_n$

~~More~~ However, we often consider the following easier variant of this problem

Problem 1' Find a good upper bound for $\# Q_n$.

We will also address very briefly the following problem although we do not have time to discuss it in length in this talk.

Problem 2: Study the distribution of Q_n as $n \to \infty$

Here is our strategy for these two problems

① Get a good upper bound for $\# Q_n$ (Problem 1').

② Construct a good family of periodic points using tools from dynamics or complex analysis

In this way, we will have a good lower bound for $\# Q_n$ together with an equidistribution property. It is worthy noting that the problems are not completely solved even for holomorphic automorphisms in dim 2, e.g. when X is a K3 surfaces.

Now we are able to state the first main result.

Theorem A (Dinh-Sibony ('2014), Dinh-N-Truong ('2015) '2016)

Let $f: X \longrightarrow X$ be a hol/mero map/correspondence on a compact Kähler manifold X. Then.

- f is an Artin-Mazur map: $\# Q_n$ grows at most exponentially fast;
- The exponential growth is bounded by the algebraic entropy ~~[illegible]~~.
- In many cases (e.g. Hénon maps), mero maps with dominant topological degree, Q_n is asymptotically equidistributed with respect to a canonical invariant proba measure, as $n \to \infty$.

We should mention the result of Kaloshin and al. ('2000, '2007) for smooth non-holo maps, $\# Q_n$ can grow arbitrarily fast. The first assertion of our theorem illustrates the essential difference between cholomorphic maps and non-hol ones.

Here is our second main result.

Theorem B (Dinh-N-Vu '2017).

Let $f: X \longrightarrow X$ be a hol map on a compact Kähler manifold X. Assume that the action of f^* on Hodge cohomology is simple. Then

$$\# Q_n \leq d^n + o(d^n) \quad \text{as } n \to \infty,$$

where d is the main dynamical degree of f.

The notions appearing in the Theorem will be explained in Part II.

Now we come to Part II of the talk:

Part II: Background + Strategy + Difficulty

- Let $f: X \to X$ be a ~~corr~~ mero correspondence of graph $\Gamma \subset X\times X$. So Γ is an effective k-cycle $\Gamma = \sum \Gamma_j$, Γ_j irreducible analytic set of dim k in $X\times X$. (We only consider here finite sums and the Γ_j's are not necessarily distinct) and π_1: the restriction of π_1 and π_2 to each Γ_j is surjective.

This can be seen as a multivalued ~~...~~ meromorphic map

- For $x \in X$, $f(x) = \pi_2(\pi_1^{-1}(x)\cap\Gamma)$ and $f^{-1}(x) = \pi_1(\pi_2^{-1}(x)\cap\Gamma)$
- If ϕ is a smooth (p,q) form on X, we can define the pull-back and the push-forward of ϕ by f by

$$f^*(\phi) := (\pi_1)_*\left(\pi_2^*(\phi)\wedge[\Gamma]\right)$$

$$f_*(\phi) := (\pi_2)_*\left(\pi_1^*(\phi)\wedge[\Gamma]\right).$$

They are L^1 (p,q)-form on X, not continuous in general.

- We cannot iterate this operation. However, f^*, f_* commute with the operator $\partial, \bar\partial$.
- Consequently, f^* induces a linear map on Hodge cohomology $f^*: H^{p,q}(X) \to H^{p,q}(X)$ and we can iterate this.

$\boxed{\text{Def}}$: For $0 \le p \le k$, the limit

- $$d_p := \lim_{n\to\infty} \left\| (f^n)^* : H^{p,p}(X) \to H^{p,p}(X) \right\|^{\frac{1}{n}}.$$

is called "the dynamical degree" of order p of f.

- $h_a := \max \log d_p$ is the algebraic entropy of f

It is relevant to recall here a classical result due to Dinh & Sibony ('2005).

Dinh-Sibony ('2005) ~~Th 0 ≤ p ≤~~

- For $0\le p\le k$, d_p exists, is finite, is bi-meromorphic invariant.
- The topological entropy of f satisfies $h_t(f) \le h_a(f)$. (Gromov '1977)

Def: The exponential growth of ~~f is~~ the number of isolated periodic points of f is

$$\lim_{n\to\infty} \frac{1}{n}\log \# Q_n$$

By Th/m A, $\overline{\lim}\, \frac{1}{n}\log\# Q_n \le d_a.$

~~Now f is m~~

Khovanskii-Teissier-Gromov: ($s \longmapsto \log d_s$ is concave on $0 \le s \le k$ ($\Leftrightarrow d_s^2 \ge d_{s-1} d_{s+1}$))

So there are $0 \le p \le p' \le k$ such that

$$d_0 < \cdots < d_p = \cdots = d_{p'} > \cdots > d_k.$$

Def:

- f is with dominant topological degree if $p = p' = k$, i.e, $d_k > d_s$ $\forall$ $0 \le s \le k-1$
- the action of f^* on Hodge cohomology is simple if this action admits a unique eigenvalue of modulus d_p which is moreover a simple eigenvalue. In this cas $d := d_p$ is the main dynamical degree.

Ex. Hol maps on $\mathbb{P}^k$,

Difficulty + Strategy

- Periodic points of period n correspond to the intersection of the graph Γ_n of f^n with the diagonal $\Delta = \{(x,x) : x \in X\}$. It may have positive dimension.
- Recall that Q_n is the set of isolated periodic points of period n, counting with multiplicity.
- By Lefschetz fixed points theorem: If $\dim(\Gamma_n \cap \Delta) = 0$ the

$$\# Q_n = \{\Gamma_n\} \smile \{\Delta\}$$

It is then easy to get an estimate on $\# Q_n$ using the dynamical degrees. As you could see, the

Main difficulty for Prob 1: the contribution of positive dimension components of the set of periodic points. In other words, we need to control some intersection with dimension excess.

Strategy of the proof of Theorem B

- Consider the graph Γ_n of f^n as a positive closed currents on $X \times X$.

Step 1: Prove that the limit $\Gamma_\infty = \lim_{n\to\infty} \frac{[\Gamma_n]}{d^n}$ exists in the sense of current. More concretely, we prove by the super-potential theory that the two main dynamical currents

$$T_+ = \lim_{n\to\infty} \frac{(f^n)^*(\omega^p)}{d^n} \quad \text{and} \quad T_- = \lim_{n\to\infty} \frac{(f^n)_*(\omega^{k-p})}{d^n}$$

exist. Hölder continuous; bounded super-potential, and by the fact that the action of f^* on Hodge cohomology

- We show that $\Gamma_\infty = T_+ \wedge T_-$ is well-defined "simply" in the sense of super-potential theory.
- We show that $\underbrace{T_+ \otimes T_-}_{\Gamma_\infty}$ $\longrightarrow$ horizontal dimension $T^+ \otimes T^-$ of tangent currents, $\dim K^0(T^+ \otimes T^-) = 0$.

Step 2: We deal with the difficulty of dimension excess in our strategy as follows.

- We prove that the dimension excess of $\Gamma_\infty \cap \Delta$ is as expected 0.
- We need an upper-semi-continuity for the intersection "

This, combined with the previous assertion and the limit $\frac{[\Gamma_n]}{d^n} \to \Gamma_\infty$ implies that the dimension excess for $\frac{\Gamma_n}{d^n} \cap \Delta$ can be asymptotically 0 as $n \to \infty$. Consequently, the density of dimension 0.

~~This upper semi-continuity~~

of $\frac{[\Gamma_n]}{d^n}$ with Δ is asymptotically smaller than the density of dimension 0 of $[\Gamma_\infty]$ with Δ. In other words,

$$\overline{\lim_{n\to\infty}} \frac{\# Q_n}{d^n} \leq [\Gamma_\infty]$$

This implies $\# Q_n = d^n + o(d^n)$. $\square$

Now we speak of the two main tools: super-potentials and densities of positive closed currents.

III Main Tools:

III.1 Super-potentials (Dinh-Sibony 2009-2010)

The starting point is that the pluripotential theory is well-developed for positive closed currents of bidegree $(1,1)$ thanks to the notion of p.s.h. functions. More precisely, if T is a positive closed $(1,1)$ current, then we can write locally $T = dd^c u$ in the sense of currents, where u is p.s.h function and $d^c = \frac{i}{2\pi}(\bar\partial - \partial)$. Working with a pointwise defined function is more comfortable than working directly with a current. For example, it allows more operations like multiplying a positive current S by u. One just has to assume that u is integrable with respect to the trace measure of S. But when T is of higher bidegree, the potential is just an L^1-form, one cannot consider their wedge product with S if S is singular.

Super-potentials are functions which play the role of quasi-potentials for positive closed currents of arbitrary bidegree. For $0 \leq p \leq k$, let $\mathscr{C}_p$ be the convex cone of positive closed currents on X.
$\mathscr{D}_p$ be the real vector space generated by $\mathscr{C}_p$.
$\mathscr{D}^0_p$ = subspace of $\mathscr{D}_p$ of currents belonging to the class 0 in $H^{p,p}(X,\mathbb{R})$. $\tilde{\mathscr{D}}^0_p$ = smooth forms of $\mathscr{D}^0_p$.

Consider the $*$-norm on $\mathscr{D}_p$:

- for $S \in \mathscr{C}_p$: $\|S\|_* := |\langle S, \omega^{k-p}\rangle|$ which is the mass of S
- for $S \in \mathscr{C}_p$: $\|S\|_* = \inf_{\substack{S = S^+ - S^- \\ S^\pm \in \mathscr{C}_p}} (\|S^+\| + \|S^-\|)$.

Consider now a current $R \in \mathscr{D}^0_{k-p+1}$ and a $(k-p,k-p)$-current U_R which is a potential of R i.e. $\boxed{dd^c U_R = R}$

Let $\alpha = (\alpha_1, \dots, \alpha_h)$ with $h = \dim H^{p,p}(X,\mathbb{R})$ be a fixed family of real smooth closed (p,p) forms ~~on X~~ such that the family of classes.

$\{\alpha\} = \{\{\alpha_1\}, \dots \{\alpha_h\}\}$ is a basis of $H^{p,p}(X, \mathbb{R})$. By adding to U_R a suitable closed smooth form, we can assume that $\langle U_R, \alpha_i \rangle = 0$ for $i = 1, \dots, h$. In this case we say that U_R is α-normalized.

Def: Let $T \in \mathcal{D}_p$. The α-normalized superpotential $\mathcal{U}_T$ of T is the function $\mathcal{U}_T : \widetilde{\mathcal{D}}^0_{k-p+1} \longrightarrow \mathbb{R}$ given by

$$\mathcal{U}_T(R) := \langle T, U_R \rangle;$$

where U_R is an α-normalized smooth potential of R.

We say that T has bounded/continuous/Hölder superpotential if $\mathcal{U}_T$ can be extended to a function on $\mathcal{D}_{k-p+1}$ which is bounded/continuous/Hölder with respect to the $*$-topology.

$\mathcal{U}_T(R)$ does not depend on the choice of an α-normalized U_R.

Assume that T has a continuous super-potential. Take any $S \in \mathcal{D}_q$.

Write $\{T\} = \sum_{i=1}^{h} a_i \{\alpha_i\}$.

Define $T \wedge S$ to be the real $(p+q, p+q)$ current satisfying

$$\langle T \wedge S, \Phi \rangle := \mathcal{U}_T\Big(\underbrace{dd^c \Phi \wedge S}_{\mathcal{D}^0_{k-p+1}}\Big) + \sum_{j=1}^{h} a_j \langle \alpha_j, \Phi \wedge S \rangle$$

Φ is a real smooth $(k-p-q, k-p-q)$-form.

$\langle T \wedge S, \Phi \rangle - \sum a_j \langle \alpha_j, \Phi \wedge S \rangle$
$= \langle \underbrace{T - \sum a_j \alpha_j}_{\in \mathcal{D}^0_p}, \Phi \wedge S \rangle = \langle dd^c U_T, \Phi \wedge S \rangle = \langle U_T, dd^c(\Phi \wedge S) \rangle$
$= \langle U_T, dd^c \Phi \wedge S \rangle \underset{\text{Stokes}}{=} \langle \mathcal{U}_T(dd^c \Phi \wedge S)$

Now we come to

III.2 Density of positive closed currents along a manifold

Consider a manifold $V \subset X$.

Consider $\pi : E \xrightarrow{} V$ the normal vector bundle of V in X.

Denote by $\overline{E}$ the natural compactification of E, i.e. $\overline{E} = \mathbb{P}(E \oplus \mathbb{C})$, where $\mathbb{C}$ is the trivial line bundle over X.

For $\lambda \in \mathbb{C}^*$ denote by A_λ the multiplication by λ on the fibers of E.
$A_\lambda(u) = \lambda u$ for $u \in E_a$ and $a \in V$.

Let $T \in \mathcal{C}_p$. Then the limit $(A_\lambda)_* T$ does not exist in general. However,

Theorem (Dinh + Sibony 2012)
All limit values of $(A_\lambda)_* T$ are in the same coho class in $H^{*,*}(\bar{E}, \mathbb{R})$.
We call this coho class the density of T along V.

Remark When V is a single point a, $\bar{E} = \mathbb{P}^k$ and
$\{$the density of T along $V\} \equiv \lambda \{\omega_{FS}\}$, $\lambda \in \mathbb{R}^+$.
λ is the Lelong number of T at a.
We have an analytic counterpart of the Fulton intersection theory.
~~We also~~ The density of positive closed currents ~~have~~ enjoys a upper-semi continuous property. as in the case of V a single point.

Finally we explain some points in the proof of Theorem B.
There are two new results of independent interest.
(~~Superpot~~ wedgeable in the sense of superpotential $\Longrightarrow$ wedgeable in the sense of tangent currents)

Prop 1: ~~Let~~ $T \in \mathcal{C}_p$, $S \in \mathcal{C}_q$ and T has a continuous super-potential. Then $T \wedge S = T \curlywedge S$.
~~This proposition allows us~~

Prop 2: Domination principle for positive closed currents.
Let $T \in \mathcal{C}_p$ s.c. the super-potential of T is bounded/continuous/Hölder continuous. Let $S \in \mathcal{C}_p$ s.c. $S \leq T$.
Then the super-potential of S is also bounded/continuous/Hölder continuous.
This prop is needed in order to show that
$$d^{-n}[\Gamma_n] \xrightarrow[n\to\infty]{} T^+ \otimes T^-$$

$\dim V = l \cdot \dim X = k$.

Subject :

Date:

R tangent current

$\{R\} = k^V(T)$ tangent class of T.

$H^*(\bar{E}, \mathbb{C})$ free ~~module~~ $H^*(V, \mathbb{C})$-module generated by $1, h, \dots, h^{k-l}$

$-h$ tautological $(1,1)$ class on $\bar{E}$.

$$\frac{\text{class}(R)}{k_V(T)} = \sum_j \pi^*(k_j^V(T)) \, h^{j-l+p}$$

$$k_j^V(T) \in H_{j,j}(V, \mathbb{C})$$

Semicontinuity $\max j \cdot k_j^V(T) \neq 0 \longrightarrow$ horizontal dimension

$$\boxed{\begin{gathered} T_n \longrightarrow T \\ \text{horizontal dim}(T) = s \cdot \lim k_s^V(T_n) \leq k_s^V(T) \end{gathered}}$$

$\mathcal{C}_q \; \mathcal{C}_q$

$T \wedge S = \lim\limits_{\lambda} A_\lambda(T \otimes S) \xrightarrow[\lambda \to 0]{} R$ unique.

$(T \wedge S)$ unique on $\underset{\underset{X}{\|}}{\Delta}$ $\mathcal{C}_{p+q}$: $\boxed{\pi^*(T \wedge S) = R}$