Rigidity Problems in Several Complex Variables and Cauchy-Riemann Geometry

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We make a discussion on various developments related to a rigidity phenomenon discovered by Poincaré 110 years ago.

Rigidity of holomorphic maps in $\mathbb{C}^n$

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• f a holomorphic function in a neighborhood U of the origin and maps $U \cap \mathbb{R} \subset \mathbb{R} \Rightarrow f(z) = \overline{f(\overline{z})}$. It gives some restriction but not much: $f = \sum_{j=0}^{\infty} a_j z^j$ with $a_j \in \mathbb{R}$.

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• For any $p \in \mathbb{C}$, $f(z) = \frac{1}{z-p}$ is holomorphic in $\mathbb{C} \setminus \{p\}$ but blows up at p. Hence for any domain $D \subset \mathbb{C}$ and $p \in \partial D$, there is $f \in \mathcal{O}_D$ but not holomorphically extendible across p.

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Poincaré (1907): Let $F = (f_1, f_2)$ be a non-constant holomorphic map from U to $\mathbb{C}^2$ with $U \cap \partial\mathbb{B}^2 \neq \emptyset$. Suppose that $F(U \cap \partial\mathbb{B}^2) \subset \partial\mathbb{B}^2$. Then F extends to a linear fractional one to one and onto holomorphic self-map $\mathbb{B}^2$. Or, F extends to an automorphism of $\mathbb{B}^2$.

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Hartogs (1906), Levi (1911): There is a smoothly bounded domain D in $\mathbb{C}^2$ diffeomorphic to the 2-ball, for which there is $p \in \partial D$ such that any holomorphic function f in D extends across p.

Rigidity of holomorphic maps $\mathbb{C}^n$

Poincare's theorem holds in higher dimension is mainly due to some geometric reason, which later led to the development of the invariant theory in CR Geometry by Segre, E. Cartan, Chern-Moser, Tanaka, etc.

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Hartogs-Levi's phenomenon is on an extension phenomenon, which led to the work of Oka, Grauert, and the work on $\overline{\partial}$-equations of Hormander, Kohn, etc.

Rigidity of holomorphic maps in $\mathbb{C}^n$

Let M be a smooth real hypersurface in $\mathbb{C}^n$ with defining equation $\rho(z,\overline{z})$ in U. Namely, $M = \{z \in U : \rho(z,\overline{z}) = 0\}$ and $d\rho|_M \neq 0$.

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Define for each $p\in M$, $T_p^{(1,0)}M := CT_pM\cap T_p^{(1,0)}\mathbb{C}^n$. Then $dim_{\mathbb{C}}T_p^{(1,0)}M\equiv n-1>0$ for $n>1$. $(M,T^{(1,0)}M:=\sqcup_{p\in M}T_p^{(1,0)}M)$ is called a CR manifold. For instance, write $L_j=\frac{\partial\rho}{\partial z_n}\frac{\partial}{\partial z_j}-\frac{\partial\rho}{\partial z_j}\frac{\partial}{\partial z_n}$ for $j=1,\cdots,n-1$. Then $T_p^{(1,0)}M=span_{\mathbb{C}}\{L_1|_p,\cdots,L_{n-1}|_p\}$.

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A crucial geometric fact to make the Poincare's rigidity theorem work: If F is a biholomorphic map sending M_1 to M_2, then for any $p\in M_1$, it holds:

$$F_*T_p^{(1,0)}M_1=T_{F(p)}^{(1,0)}M_2$$

Rigidity of holomorphic maps in $\mathbb{C}^n$

Using the invariant theory developed by Cartan-Chern-Moser, Burns-Shnider-Wells proved the following Riemann non-mapping theorem:

Burns-Shnider-Wells(1978): A generic smooth small deformation of the unit ball in $\mathbb{B}^n$ is no longer biholomorphic to $\mathbb{B}^n$

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CR version of the Chow theorem (Huang, 1994): Let F be a holomorphic map sending a piece of strongly pseudo-convex algebraic hypersurface in $\mathbb{C}^n$ into a strongly pseudo-convex algebraic hypersurface in $\mathbb{C}^N$ with $n > 1$. Then F is a Nash algebraic holomorphic map.

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- When $N = n$, the result was obtained by **Webster** in 1977.
- **Forstneric** (1989): When the two algebraic hypersurfaces are spheres, then the algebraic map is single-valued, namely, rational.
- Many other later developments

Rigidity of holomorphic maps in $\mathbb{C}^n$

Alexander: Let F be a proper holomorphic self-map of $\mathbb{B}^n$ with $n \geq 2$. Then $f \in Aut(\mathbb{B}^n)$. Namely, there is $\sigma \in Aut(\mathbb{C}^n)$ such that $\sigma \circ f = z$.

Work of Webster, Faran, Cima-Sufffridge, Forstneric, etc.

Rigidity of holomorphic maps in $\mathbb{C}^n$

Huang: Let F be a proper holomorphic map from $\mathbb{B}^n$ into $\mathbb{B}^N$, which is C^3-smooth up to the boundary. Write $P(n,0)=n,\ P(n,1)=n+n-1,\ P(n,2)=P(n,1)+n-2$, and $P(n,\kappa)=P(n,\kappa-1)+(n-\kappa)$ for $1\le\kappa\le n-1$. Then if $P(n,\kappa-1)\le N<P(n,\kappa)$, then F is $n-\kappa+1$ -linear. Namely, for any $p\in\mathbb{B}^n$, there is an affine complex subspace S_p of complex dimension $(n-\kappa+1)$ such that the restriction of F to S_p is linear fractional or total geodesic. In particular, when $N<\frac{n(n+1)}{2}$, F is at least 2-linear.

Rigidity of holomorphic maps in $\mathbb{C}^n$

Example: Consider $F : \mathbb{B}^6 \to \mathbb{B}^N$. Then F is 6-linear if $6 \leq N < 11$; 5-linear if $11 \leq N < 16$; 4-linear if $16 \leq N \leq 18$; 3-linear if $18 \leq N < 20$ and 2-linear if $20 \leq N < 21$. When $N \geq 21$, F has no linear direction in general.

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Similarity between the minimal target dimension N for which the rigidity breaks down and the minimal target dimension in the classical Cartan-Janet theorem for which there is no more obstruction to locally isometrically embed an analytic Riemannian manifold of dimension n into $\mathbb{R}^N$

Rigidity of holomorphic maps in $\mathbb{C}^n$

F is called a minimum proper map if F is not equivalent to a map of the form $(F', 0)$ where F' is a proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^{N'}$ with $N' < N$. Here two proper holomorphic maps F, G from $\mathbf{B}^n$ into $\mathbf{B}^N$ are said to be equivalent if there are $\sigma \in Aut(\mathbf{B}^n)$ and $\tau \in Aut(\mathbf{B}^N)$ such that $G = \tau \circ F \circ \sigma$.

Rigidity of holomorphic maps in $\mathbb{C}^n$

Following the way the Whitney map $F(z,w) = (z, zw, w^2)$ was constructed, Huang-Ji-Yin constructed a minimum proper monomial map from $\mathbf{B}^n$ into $\mathbf{B}^m$ $(m \geq n)$ when there does not exist a positive integer k such that $kn < m \leq (k+1)n - k(k+1)/2 - 1$. This motivated us to formulate the following general gap conjecture:

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Gap Rigidity Conjecture (Huang-Ji-Yin): *There is no minimum proper holomorphic map from $\mathbf{B}^n$ into $\mathbf{B}^N$, that is C^3-smooth up to the boundary, if and only if it holds that $kn < N \leq (k+1)n - 1 - k(k+1)/2$ for a certain positive integer k, or $N \in \mathcal{I}_k$ for a certain positive integer k, where $\mathcal{I}_k := \{m : kn < m \leq (k+1)n - 1 - k(k+1)/2\}$.*

Rigidity of holomorphic maps in $\mathbb{C}^n$

Write $\mathcal{I}_1 = [n+1, 2n-2], \mathcal{I}_2 = [2n+1, 3n-4], \mathcal{I}_3 = [3n+1, 4n-7]$, etc. Also, let $K(n) = [\frac{-1+\sqrt{1+8n}}{2}]$ if $\frac{-1+\sqrt{1+8n}}{2}$ is not an integer; and let $K(n) = \frac{-1+\sqrt{1+8n}}{2} - 1$, otherwise. Then $\mathcal{I}_k \neq \emptyset$ for $k \leq K(n)$. Note that $\mathcal{I}_k \cap \mathcal{I}_{k'} = \emptyset$ for $k \neq k'$.
The work by Huang-Ji-Yin shows that the above conjecture holds for $k \leq 3$. A solution to this conjecture would give a complete picture of the gap phenomenon for rational maps between balls.

Rigidity of holomorphic maps in $\mathbb{C}^n$

Example: Consider $F : \mathbb{B}^{12} \to \mathbb{B}^N$. Then the first gap $\mathcal{I}_1 = [13, 22]$; the second gap $\mathcal{I}_2 = [25, 32]$; the third gap $\mathcal{I}_3 = [37, 41]$. (Proved by Huang-Ji-Yin). The conjectured fourth gap the second gap $\mathcal{I}_4 = [49, 49]$.

Consider $F : \mathbb{B}^n \to \mathbb{B}^N$ with n fixed. The gap conjecture predicts there are $K(n) = [\frac{-1+\sqrt{1+8n}}{2}]$ many gap intervals if $\frac{-1+\sqrt{1+8n}}{2}$ is not an integer and $\frac{-1+\sqrt{1+8n}}{2} - 1$ many intervals otherwise. For instance, $K(12) = [4.42] = 4$. $K(100) = [13.65] = 13$.
Connection with Hilbert 17th problem: Recent work of **Ebenfelt**.

Rigidity of holomorphic maps in $\mathbb{C}^n$

A well-known super-rigidity problem in differential geometry and ergodic theory: Let G_1 and G_2 be two non-compact semi-simple Lie groups with $dim(G_2) \geq dim(G_1)$. Suppose $\Gamma_j \subset G_j$ are lattices and there is an injective homomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$. Can one then extend ϕ to a global homomorphism from G_1 into G_2?

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Work of many mathematicians Mostow, Margulis, Gromov-Piastetski-Shaprio, Johnson-Milnor, Corlette, Mok-Siu-Yeung, etc.

The remaining unsolved case is when $G_1 = Aut(\mathbf{B}^n)$ and $G_2 = Aut(\mathbf{B}^N)$ with $1 < n < N$ with Γ_1 co-compact in G_1.

Rigidity of holomorphic maps in $\mathbb{C}^n$

After applying the harmonic mapping theory and Siu's Bochner trick, the above question is reduced to the following:

Conjecture: Let $\mathbb{B}^n \subset \mathbb{C}^n$ and $\mathbb{B}^N \subset \mathbb{C}^N$ be the unit balls, and let F be a proper holomorphic embedding from $\mathbb{B}^n$ into $\mathbb{B}^N$ $(1 < n < N)$. Suppose that there is a co-compact discrete subgroup $\Gamma \subset Aut(\mathbb{B}^n)$ such that for each $\sigma \in \Gamma$, there is $\tau_\sigma \in Aut(\mathbb{B}^N)$ such that $\tau_\sigma \circ f = f \circ \sigma$. Is f then a linear embedding?

Rigidity of holomorphic maps in $\mathbb{C}^n$

Answered in the affirmative by Cao-Mok in case $(N \leq 2n-1)$.

Cao-Mok, Jun-Muk Hwang: needs only to show that for any point in the ball, there is at least one complex direction along which the map f is linear fractional. The semi-rigidity result can be used to produce the partial linearity for mappings with C^3 boundary regularity by using the boundary CR invariants in case $N < n(n+1)/2$. How to adapt the argument in the CR geometry to this setting?

CR link and rigidity of normal isolated singularity

V: a complex space with an isolated non-regular point $p \in V$. Assume $\dim_{\mathbf{C}} V \geq 2$.

Understand the complex structure of the germ of V at p through the Cauchy-Riemann (CR) links (proposed by Kuranishi).

CR link and rigidity of normal singularity

Since our concern here is mainly local, we assume that V is embedded in a complex Euclidean space $\mathbf{C}^m$ with $p = 0$. Define $\rho(z, \overline{z}) = |z|^2$.

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Definition: Let $M_\epsilon = \{z \in V : \ \rho(z, \overline{z}) = \epsilon^2\}$. The M_ϵ is called the ϵ-Milnor link of $(V, 0)$.

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Definition: Let $M_\epsilon = \{z \in V : \rho(z,\overline{z}) = \epsilon^2\}$. The M_ϵ is called the ϵ-Milnor link of $(V,0)$.

More generally, for ρ with positive complex Hessian and with 0 being an isolated zero of ρ, we define the ϵ ρ-link $M_{\epsilon,\rho} := \{z \in V : \rho(z,\overline{z}) = \epsilon^2\}$.

CR link and rigidity of normal singularity

For $0 < \epsilon << 1$, M_ϵ is a compact smooth submanifold embedded in V of real co-dimension 1.

Always assume $\dim_{\mathbf{C}} V, \ \dim_{\mathbf{C}} V' \geq 2$.

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A crucial fact that makes it possible to study $(V, 0)$ through its links is that $M_{\rho,\epsilon}$, at its smooth point, carries a partial complex structure naturally induced from V, called the inherited Cauchy-Riemann structure.

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For any $q \in M_\epsilon$, define

$$T_q^{(1,0)} M_{\epsilon,\rho} = T_q^{(1,0)} V \cap CT_q M_{\epsilon,\rho}.$$

Then the complex dimension of $T_q^{(1,0)} M_{\epsilon,\rho}$ is $\dim_{\mathbf{C}} V - 1$ for any q. Moreover, $T_q^{(1,0)} M_{\epsilon,\rho}$ depends smoothly on $q \in M_{\rho,\epsilon}$ and thus naturally defines a complex smooth vector bundle $T^{(1,0)} M_{\epsilon,\rho}$ over $M_{\epsilon,\rho}$ with $T_q^{(1,0)} M_{\epsilon,\rho}$ as its fiber space over $q \in M_{\epsilon,\rho}$.

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$M_{\epsilon,\rho}$ is CR equivalent to $M'_{\epsilon',\rho'}$ if there is a smooth diffeomorphism F from $M_{\epsilon,\rho}$ to $M'_{\epsilon',\rho'}$ that respects the CR structures defined above. Namely,

$$F_*\left(T^{(1,0)}M_{\rho,\epsilon}\right) = T^{(1,0)}M'_{\rho',\epsilon'}.$$

CR link and rigidity of normal singularity

Andreotti – Grauert's Hartogs type extension theorem: Suppose 0 is the only isolated singularity of the normal complex space V and V', respectively. Assume that F is a CR equivalence map from $M_{\epsilon,\rho}$ to $M'_{\epsilon',\rho'}$. Write $V_{\epsilon,\rho} = \{z \in V : \rho(z) < \epsilon\}$ and $V'_{\epsilon',\rho'} = \{z \in V' : \rho'(z) < \epsilon'\}$. Then F extends as a biholomorphic map from V_ϵ to $V'_{\epsilon'}$. In particular, $(V,0)$ is holomorphically equivalent to $(V',0)$.

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Unfortunately, the above reduction is not reversible. By the Chern-Moser theory, for $\epsilon \neq \epsilon'$, $M_{\epsilon,\rho}$ and $M_{\epsilon',\rho}$ have, in general, very different CR structures. Finding the invariants from the links which are directly related to the complex structure of the singularities is always an important problem in this subject of Several Complex Variables.

CR link and rigidity of normal singularity

The unit sphere in $\mathbb{C}^n$ with $n \geq 2$ may be regarded as the simplest strongly pseudoconvex CR manifold. Compact spherical CR manifolds may be regarded as the simplest strongly pseudoconvex CR manifolds which may bound normal singularities. (A CR manifold is said to have a spherical CR structure if it is locally CR equivalent to a piece of the sphere of the same dimension.)

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The unit sphere in $\mathbb{C}^n$ with $n \geq 2$ may be regarded as the simplest strongly pseudoconvex CR manifold. Compact spherical CR manifolds may be regarded as the simplest strongly pseudoconvex CR manifolds which may bound normal singularities. (A CR manifold is said to have a spherical CR structure if it is locally CR equivalent to a piece of the sphere of the same dimension.)

A typical example of spherical CR manifolds is a spherical space form:

$$M_\Gamma := \partial \mathbb{B}^n / \Gamma,$$

with Γ is a finite subgroup of $Aut(\mathbb{B}^n)$ with 0 as its only fixed point.

CR link and rigidity of normal singularity

Cartan invariant polynomial theory: One can construct an algebraic realization Ψ of $\mathbb{C}^n/\Gamma$ into $\mathbb{C}^N$ for a certain $N \geq n$ such that $\Psi(\partial\mathbf{B}^n/\Gamma)$ bounds normal isolated complex singularities with $\Psi(\partial\mathbb{B}^n/\Gamma)$ as one of its algebraic link.

CR link and rigidity of normal singularity

Explicit Example: Let $V := \{(x,y,z) \subset \mathbb{C}^3 : \quad y^2 = 2xz\}$. Then its standard link M with $\epsilon = 1$ is algebraic and spherical, for the holomorphic map $(t,\tau) \to (t^2, \sqrt{2}t\tau, \tau^2)$ is a holomorphic covering map from $\partial\mathbb{B}^3$ into M. Hence, it is a Cartan realization of $\partial\mathbb{B}^2/\mathbb{Z}_2$. Conversely, we have the following:

Huang: Suppose that ρ is algebraic and ϵ is a regular vaule of ρ. Suppose that $M_{\rho,\epsilon} := \{z \in V : \rho(z,\overline{z}) = \epsilon^2\}$ carries a compact spherical CR structure. Then there is a finite unitary subgroup $\Gamma \subset Aut(\mathbf{B}^n)$ with 0 as its only fixed point and a biholomorphic map from $\mathbf{B}^n/\Gamma$ to $V_\epsilon := \{z \in V : \quad \rho(z) < \epsilon\}$. In particular, $(V,0)$ is holomorphically equivalent to the quotient singularity $(\mathbf{B}^n/\Gamma, 0)$.

As an application of Theorem, we have the following:

Corollary: Let V be a complex analytic space embedded in $\mathbf{C}^m$ with only an isolated singularity at 0. Suppose that for a certain algebraic ρ as before, the corresponding ϵ-link $M_{\rho,\epsilon}$ carries a (compact) spherical CR structure. Then $M_{\rho,\epsilon}$ is CR equivalent to a CR spherical space form $\partial\mathbf{B}^n/\Gamma$ with $\Gamma \subset Aut(\mathbf{B}^n)$ a certain finite group with the only fixed point at 0. In particular, the fundamental group of $M_{\rho,\epsilon}$ is isomorphic to Γ.

Proof: For any point $w \in M_{\rho,\epsilon}$, by the assumption, there is a CR equivalence map Φ_w from a piece of the sphere $\mathbf{S}$ to a piece of the link $M_{\rho,\epsilon}$ near w. Since ρ is algebraic strong pseudoconvex, by the algebraicity theorem of the speaker proved in 1994, Φ_w is also an algebraic map. In particular, we see that $M_{\rho,\epsilon}$ can be locally defined by (real Nash) algebraic functions. Now, we fix one of Φ_w, denoted by Φ. Then Φ extends to an algebraic map (possibly multiple-valued) from $\mathbf{C}^n$ into $\mathbf{C}^m$. Let E be the set of poles and branching points of Φ. Then $\Phi(\mathbf{S} \setminus E) \subset M_{\rho,\epsilon}$.
Let γ be a Jordan curve in $\mathbf{S}$ with $\gamma([0, c_0)) \subset M_{\rho,\epsilon} - E$ and $\gamma(c_0) \in E$. Suppose that for a certain sequence $\{t_j\}$ with $t_j < c_0$ and $t_j \to c_0$, $\Phi(t_j) \to q$.

Let A_q be an affine complex subspace of dimension n such that there is an affine linear map π_q fixing q, projecting $\mathbf{C}^m$ to A_q, with $\pi_q|_V$ biholomorphic near q. Write $\Phi^q = \pi_q \circ \Phi$ and $N^q = \pi_q(M_{\rho,\epsilon})$ near q. Then Φ^q, N^q must also be (Nash) algebraic. Then with a careful monodromy argument, one can prove that $\Phi^q(\gamma(t))$ has limit q as $t \to c_0^-$. Now, one can further show that Φ^q extends holomorphically across $\gamma(c)$ and thus along any path inside $\mathbf{S}$. Since $\mathbf{S}$ is simply connected, we conclude that the extension of Φ, still denoted by Φ, gives a single-valued holomorphic covering map from $\mathbf{S}$ to $M_{\rho,\epsilon}$. Let $\Gamma := \{\sigma_j\}_{j=1}^k$ be the deck transformations associated with this covering.

We see easily that $\Gamma = \{\sigma_j\}$ extends to a finite subgroup of $Aut(\mathbf{B}^n)$ such that Φ induces a biholomorphic map from $\mathbf{B}^n/\Gamma$ to V_ϵ. Since Γ has no fixed point over $\mathbf{S}$, we can assume without loss of generality that 0 is the only fixed point of Γ. Now, the rest of the proof follows from Theorem 2.2 of Huang-Ji in (Math. Res. Letter, 1998). ∎

CR link and rigidity of normal singularity

A general construction of analytic spherical links with fundamental group of infinite order, by using the Grauert tube technique. This thus shows that the algebraicity assumption in the above results are crucial for the statements to hold:

CR link and rigidity of normal singularity

Let M be a complex manifold with a Hermitian metric h. The Grauert tube domain over its holomorphic tangent bundle, induced from h, is defined to be the domain $\Omega := \{v \in T^{(1,0)}M : \; h(v,v) < 1\}$. Assume that M is a ball quotient $\mathbf{B}^n/\Gamma$, where $\Gamma \subset Aut(\mathbf{B}^n)$ is a (fixed-point free) lattice. Assume that h is a hyperbolic metric with a negative constant holomorphic sectional curvature. (Such an (M,h) is called a hyperbolic space form.) The following result is classical in the literature.

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Let M be a complex manifold with a Hermitian metric h. The Grauert tube domain over its holomorphic tangent bundle, induced from h, is defined to be the domain $\Omega := \{v \in T^{(1,0)}M : \ h(v,v) < 1\}$. Assume that M is a ball quotient $\mathbf{B}^n/\Gamma$, where $\Gamma \subset Aut(\mathbf{B}^n)$ is a (fixed-point free) lattice. Assume that h is a hyperbolic metric with a negative constant holomorphic sectional curvature. (Such an (M,h) is called a hyperbolic space form.) The following result is classical in the literature.

Proposition: Assume that (M,h) is a compact hyperbolic space form of dimension $n \geq 1$. Then the Grauert tube domain of its holomorphic tangent bundle is a domain in $T^{(1,0)}M$ with real analytic spherical boundary.

Now, applying the CR embedding theorem of Boutet de Monvel and Kohn we can find a real analytic CR embedding F from $\partial\Omega$ into a certain $\mathbf{C}^N$. The extension theorem of Kohn-Rossi shows that F extends to a holomorphic map from Ω into $\mathbf{C}^N$. Apparently F must be a local holomorphic embedding from $\Omega \setminus M$ and maps M into a point which we can assume to be 0. Making use of the Kohn-Rossi theorem, the Harvey-Lawson theorem and applying a normalization to resolve the normal-crossing singularities if necessary, we can assume, without loss of generality, that $F(\partial\Omega)$ bounds a normal Stein space with a unique isolated normal singularity at 0. F is biholomorphic from $\Omega \setminus M$ to its image and $F^{-1}(\{0\}) = M$. Based on this example, we pose the following:

CR link and rigidity of normal singularity

Problem: Suppose V is a normal complex space with an isolated singularity at p. Suppose that for some $0 < \epsilon << 1$ and a real analytic strongly plurisubharmonic function ρ with $\rho(z) > \rho(0) = 0$ for $z \neq 0$, $M_{\rho,\epsilon}$ is a real analytic spherical CR manifold. What can we say about the complex structure of V at 0?

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One approach: study the Kähler-Eistein metric on V that is complete along M and possesses certain type of singularities at the normal singular points of V. Problem is reduced to the study of the CR notion of the equality case of the Miyaoka-Yau inequality.

7. Part III: Clozel-Ullmo Rigidity Problem

8. Connections of the Clozel-Ullmo problem with rigidity problems in CR Geometry

CR link and rigidity of normal singularity

For general Milnor-links, we have the following rigidity result proved in Ebenfelt-Huang-Zaitsev:

Theorem: Suppose $V, V' \subset \mathbf{C}^N$ be irreducible complex spaces with a singularity at 0, respectively. Suppose that $\dim_{\mathbf{C}} V = n \geq 4$ and $2N < 3n - 1$. For ϵ, ϵ' write M_ϵ and $M'_{\epsilon'}$ for the standard ϵ-link and ϵ'-link of V and V', respectively. Assume that M_ϵ and $M_{\epsilon'}$ are smooth near p and p', respectively; and assume that there is a local CR equivalence map from a piece of M_ϵ near p to a piece of $M'_{\epsilon'}$ near p'. Then there is a unitary map $\mathcal{U}$ such that $\epsilon' \mathcal{U}(V) = \epsilon V'$. Namely, after a scaling, $(V, 0)$ and $(V', 0)$ are unitary equivalent.

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Mumford: Suppose that V only has an isolated normal singularity at 0 and $\dim V = 2$. Then any local smooth link $M_{\rho,\epsilon}$ can not be simply connected.

CR link and rigidity of normal singularity

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Mumford: Suppose that V only has an isolated normal singularity at 0 and dim$V = 2$. Then any local smooth link $M_{\rho,\epsilon}$ can not be simply connected.

For $n \geq 3$, Mumford's result fails and , one has the following famous Brieskorn spheres:

CR link and rigidity of normal singularity

Let $\vec{a} =< a_1, \cdots, a_n >$ with $n \geq 4$. Here $a_j' s$ are positive integers bigger than 1. Define

$$V(\vec{a}) = \{(z_1, \cdots, z_n) : \sum_{j=1}^{n} z^{a_j} = 0\}.$$

Then $0 \in V(\vec{a})$ is an isolated singularity. Write $M_\epsilon(\vec{a}) = \{z \in V(\vec{a}) : \quad |z| = \epsilon\}$. By the work of Brieskorn and Milnor, for many choices of $\vec{a}$, $M_\epsilon(\vec{a})$ are the topological spheres, but may or may not be diffeomorphic to the standard sphere. In fact, all Milnor exotic spheres can be realized as the links of the above form.

CR link and rigidity of normal singularity

Siu's program: Determine the complex structure of $(V, 0)$ by using the topological structure of its link together with a little more complex geometric information:

Problem: Let $(V_0, 0)$ be an isolated normal singularity with a complete Kahler metric up to 0, which has a strongly negative sectional curvature. Let $(V, 0)$ be another complex space with a normal isolated singularity at 0. Suppose that a standard small link of V at 0 has the same topological structure of that of $(V_0, 0)$ near 0. Is then $(V, 0)$ holomorphically equivalent to $(V_0, 0)$?

The Clozel-Ulmmo rigidity problem

Let (S, ω_{st}) be one of the three complex space forms equipped with the standard canonical metrics ω_{st} :

- $\mathbb{C}^n$ with the Euclidean metric;
- $\mathbb{B}^n$ with the Poincaré metric;
- $\mathbb{CP}^n$ with the Fubini-Study metric.

The Clozel-Ulmmo rigidity problem

(S, ω_{st}): complex space form;
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- Uniqueness up to holomorphic self-isometries of S.

The Clozel-Ulmmo rigidity problem

- Motivation from arithmetic algebraic geometry

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Ω : an irreducible bounded symmetric domain.

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Question 1 (Clozel-Ullmo): Is the algebraic correspondence between S modular if it preserves the Bergman metric ω?

Question 2 (Clozel-Ullmo): How about the algebraic correspondence preserving invariant volume form ω^n?

The Clozel-Ulmmo rigidity problem

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(Ω, ω) : An irreducible bounded symmetric domain.
Let $F_j : U \subset \Omega \to \Omega$ be a holomorphic map for each $1 \leq j \leq m$.
Each F_j is of full rank at some point.

Suppose they satisfy the following volume-preserving assumption:

$$\sum_{j=1}^{m} F_j^*(\omega^n) = \lambda \omega^n.$$

Q: Does every F_j extend to a self-isometry?

The Clozel-Ulmmo rigidity problem

The compact dual version of this problem:

(M, ω) : An irreducible Hermitian symmetric space of compact type

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The Clozel-Ulmmo rigidity problem

More about Hermitian symmetric spaces of compact type:

The Clozel-Ulmmo rigidity problem

- Type I: Grassmannian G(p,q)
  Complex $p-$subspaces in $\mathbb{C}^{p+q}$
- Type II: Orthogonal Grassmannian
- Type III: Symplectic Grassmannian
- Type IV: Complex Hyperquadrics
- Two exceptional classes

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Example: $\mathbb{CP}^n = G(1,n)$ is of type I.

The Clozel-Ulmmo rigidity problem

Every Hermitian symmetric space of compact type can be canonically embedded as a projective variety into some $\mathbb{CP}^N$.

Plücker embedding:

- Plücker embedding of $G(2,3)$ into $\mathbb{CP}^9$:

$$\begin{bmatrix} z_0 & z_1 & z_2 & z_3 & z_4 \\ w_0 & w_1 & w_2 & w_3 & w_4 \end{bmatrix} \to$$

$$\left[\begin{vmatrix} z_0 & z_1 \\ w_0 & w_1 \end{vmatrix}, \begin{vmatrix} z_0 & z_2 \\ w_0 & w_2 \end{vmatrix}, \dots, \begin{vmatrix} z_3 & z_4 \\ w_3 & w_4 \end{vmatrix} \right]$$

We equip them with the induced Kähler-Einstein metric from $\mathbb{CP}^N$.

The Clozel-Ulmmo rigidity Problem

Question 2 of Clozel-Ullmo was answered by Mok-Ng in the non-compact setting.

Theorem (Mok-Ng) Let Ω be an irreducible Hermitian symmetric space of non-compact type of complex dimension $n \geq 2$ equipped with the canonical Kähler-Einstein metric ω. Let $F = (F_1, \cdots, F_m)$ be a holomorphic map from an open connected subset $U \subset \Omega \to \Omega \times \cdots \times \Omega$ such that $\sum_{j=1}^{m} F_j^*(\omega^n) = \lambda \omega^n$ for some positive constant λ. Suppose that each F_j is of full rank at some point. Then each F_j extends to a holomorphic self-isometry of (Ω, ω).

Clozel-Ullmo rigidity Problem

The Clozel-Ulmmo problem for the compact dual version is answered by Fang-Huang-Xiao

Theorem (Fang,Xiao and Huang) Let M be an irreducible Hermitian symmetric space of compact type equipped with the canonical Kähler-Einstein metric ω. Let $F = (F_1, \cdots, F_m)$ be a holomorphic map from an open connected subset $U \subset M \to M \times \cdots \times M$ such that $\sum_{j=1}^{m} F_j^*(\omega^n) = \lambda\omega^n$ for some positive constant λ. Suppose that each F_j is of full rank at some point. Then each F_j extends to a holomorphic self-isometry of (M, ω).

7 Part III: Clozel-Ullmo Rigidity Problem

8 Connections of the Clozel-Ullmo problem with rigidity problems in CR Geometry

Connections between the two types of rigidity

Basic ideas of the proof of Mok-Ng's theorem:

Connections between the two types of rigidity

Basic ideas of the proof of Mok-Ng's theorem:

- Step 1: Grauert tube trick

  (L, g) : anti-canonical line bundle of Ω
  $(\mathcal{L}, h)$: a seminegative vector bundle on $\Omega \times \cdots \times \Omega$ obtained by taking direct sum of $\pi_i^* L$.
  Unit sphere bundles S_L and $S_{\mathcal{L}}$ of $(L, \lambda g)$ and $(\mathcal{L}, h)$: real algebraic strongly pseudoconvex at generic points.

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  Unit sphere bundles S_L and $S_{\mathcal{L}}$ of $(L, \lambda g)$ and $(\mathcal{L}, h)$: real algebraic strongly pseudoconvex at generic points.
  $(F_1, \cdots, F_m)$ induces a holomorphic map $\widetilde{F}$ sending a piece of S_L to $S_{\mathcal{L}}$:

  $$\widetilde{F}(z, u) := (F_1, \cdots, F_m, \det(JF_1)u, \cdots, \det(JF_m)u).$$

  By the algebraicity result mentioned before, F is complex algebraic.

Connections between the two types of rigidity

- Step 2: Reduce to a rigidity problem in CR geometry

  Extend $(F_1, \cdots, F_m)$ holomorphically along a generic path to a small neighborhood of some point $p \in \partial\Omega$.

  By the volume-preserving assumption, at least one F_j, say F_1, maps $U \cap \partial\Omega$ to $\partial\Omega$.

  By the Alexander type result of Mok-Ng, F_1 extends to an automorphism.

Connections between the two types of rigidity

- Step 3: Prove by induction

  Eliminate F_1 as it is a self-isometry by Step 2. Then one can prove each F_j extends to an automorphism by induction.

Connections between the two types of rigidity

Let's discuss the proof for the compact case.

Segre varieties for projective varieties:

Let $V \subset \mathbb{CP}^N$ be a projective variety.

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Let $V \subset \mathbb{CP}^N$ be a projective variety.

Put $V^* = \{[\overline{z}] : [z] \in V\}$.

Fix $p = [\xi_0, \cdots, \xi_N] \in \mathbb{CP}^N$.

$H_p := \{[z_0, \cdots, z_N] : z_0\xi_0 + \cdots + z_N\xi_N = 0\}$ the hyperplane determined by p in $\mathbb{CP}^N$.

Connections between the two types of rigidity

- Let $[\xi] \in V^*$. The Segre variety Q_ξ of V associated to $[\xi]$ is defined as

$$Q_\xi := V \cap H_\xi.$$

Segre family of V :

$$\mathcal{M} := \{([z], [\xi]) \in V \times V^* : [z] \in Q_\xi\}.$$

Connections between the two types of rigidity

Let V be a Hermitian symmetric space of compact type

Recall the canonical embeddings: $V \to \mathbb{CP}^N$.

Comments:

- We indeed use the "minimal embedding".
- $V = V^*$.

Connections between the two types of rigidity

Examples:

- $V = \mathbb{CP}^n, \quad p = [\xi_0, \cdots, \xi_n] \in \mathbb{CP}^n$

  $Q_p = \{[z_0, \cdots, z_n] : \xi_0 z_0 + \cdots + \xi_n z_n = 0\}.$

  In nonhomogeneous coordinates, Q_p is defined by

  $$\xi_0 + \xi_1\eta_1 + \cdots + \xi_n\eta_n = 0.$$

  where $\eta_j = \frac{z_j}{z_0}$.

Connections between the two types of rigidity

Properties of the Segre varieties:

- $z \in Q_\xi \quad \Leftrightarrow \quad \xi \in Q_z.$
- The Segre family $\mathcal{M}$ is invariant under holomorphic isometric transformations.

Connections between the two types of rigidity

Basic ideas of the proof:

- Step 1:

Volume-preserving assumption $\Rightarrow$

At least one F_j maps Segre varieties to Segre varieties.

Call it F_1.

Connections between the two types of rigidity

Basic ideas of the proof:

- Step 2: Local property of F_1 in nonhomogeneous coordinates

Step 2 (a): $(F_1(z), \overline{F_1}(\xi))$ maps $\mathcal{M}$ to $\mathcal{M}$:

$$\rho(F_1(z), \overline{F_1}(\xi)) = 0 \text{ on } \rho(z, \xi) = 0.$$

Techniques from CR geometry
$\Rightarrow$

a system of **non-linear** equations of $F_1(z)$ on Q_p.

Connections between the two types of rigidity

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$\Rightarrow$ F_1 is complex algebraic on Segre varieties Q_p.

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Comment: The total degree of $F_1|_{Q_p}$ is controlled in terms of $\dim(M)$.
Step 2 (b): F_1 is complex algebraic.

Comment: The total degree of F_1 is controlled in terms of $\dim(M)$.

Connections between the two types of rigidity

Step 2 (c): F_1 is a rational map.

Monodromy argument in complex analysis

Single-valueness of F_1.

Comment: The degree of F_1 is controlled in terms of $\dim(M)$.

Connections between the two types of rigidity

- Step 3: Global property of F_1

Step 3 (a): F_1 is a birational map

An Iteration argument, Bézout's theorem.

Step 3(b): F_1 extends to a self-isometry

Step 4 : Eliminate F_1 and prove by induction that each F_j is an isometry