Quantum reading capacity: General definition and bounds

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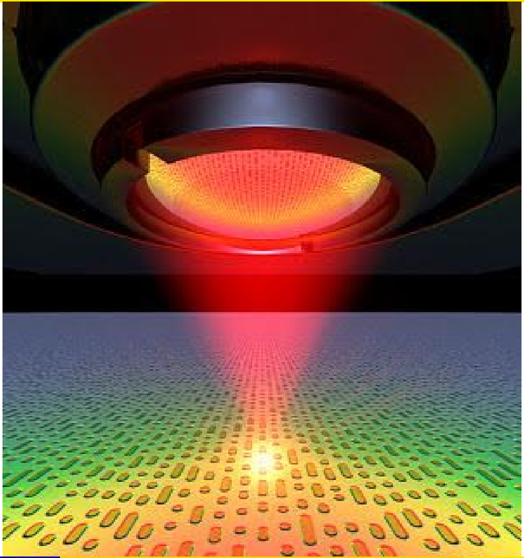
Based on arXiv:1703.03706
(with Mark M. Wilde)

Beyond I.I.D. in Information Theory 2017, IMS-NUS, Singapore

Memory device (physical model)



What is quantum reading? I

Quantum reading

Task of reading out classical information stored in a memory device such as CD or DVD [Pir11].

Principle

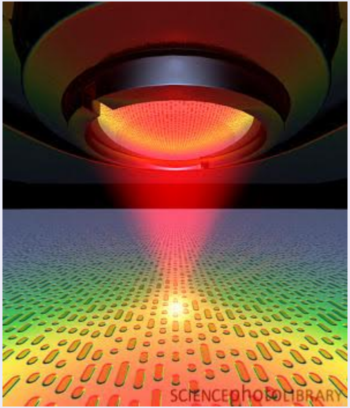
Decoding of message $\equiv$ *channel discrimination*.

Quantum reading capacity

The maximum rate at which information can be read out from a given memory encoded with a memory cell.

What is quantum reading? II

Working principle of quantum reading



Schematic diagram of quantum reading device.

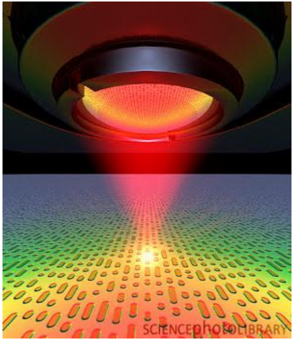
Quantum reading: Physical set up I

Memory cell

- Memory cell is a collection $\mathcal{S}_{\mathcal{X}} = \{\mathcal{N}^{x}_{A\to B}\}_{x\in\mathcal{X}}$ of quantum channels.
- Both the encoder Eddard and the reader Arya agree upon a memory cell before executing the reading protocol.
- A memory cell is used to encode information (classical message).

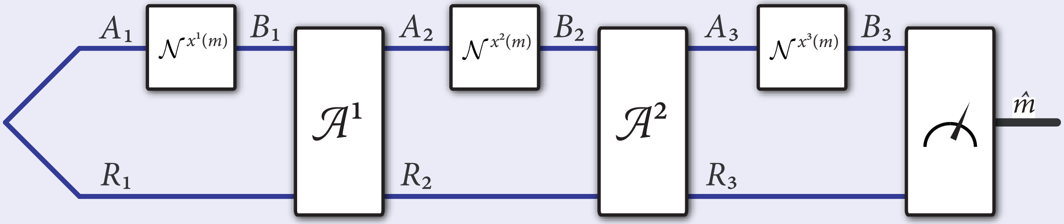
Quantum reading: Physical set up II

- Encoder: Eddard Stark encodes message in the form of a *sequence of quantum channels* chosen from a given memory cell and sends it to Arya Stark.
- Reader: The quantum reading task comprises the estimation of a message encoded in the form of a sequence of quantum channels.



Idea (Abstract setting)

Quantum reading protocol



The figure depicts a quantum reading protocol that calls a memory cell three times to decode the message m as $\hat{m}$.

Idea

Reader

Arya Stark: The transmitter and receiver are in same physical location.

Channel discrimination

In general, an adaptive strategy can give a significant advantage over a non-adaptive strategy in the context of quantum channel discrimination.

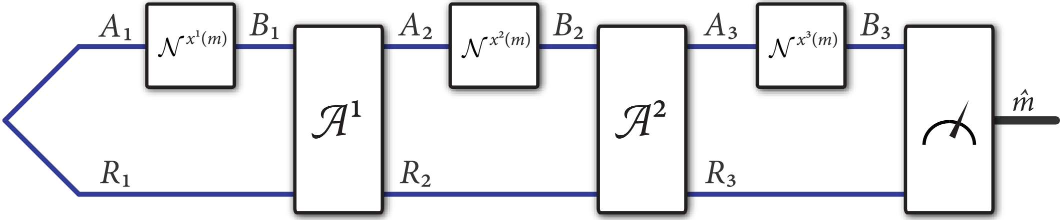
Quantum reading protocol I

Definition

An (n, R, ε) quantum reading protocol for a memory cell $\mathcal{S}_{\mathcal{X}}$ is defined by an encoding map $\mathcal{E} : \mathcal{M} \to \mathcal{X}^n$ and an adaptive strategy $\mathcal{J}_{\mathcal{S}_{\mathcal{X}}}$ with measurement $\{\Lambda^{\hat{m}}_{R_n B_n}\}_{\hat{m}\in\mathcal{M}}$. The protocol is such that the average success probability is at least $1-\varepsilon$, where $\varepsilon \in (0,1)$:

$$1-\varepsilon \leq 1 - p_{\text{err}} := \frac{1}{|\mathcal{M}|}\sum_m \operatorname{Tr}\left\{\Lambda^{(m)}_{R_nB_n}\left(\mathcal{N}^{x_n(m)}_{A_n\to B_n}\circ\mathcal{A}^{n-1}_{R_{n-1}B_{n-1}\to R_nA_n}\circ\cdots\circ\mathcal{A}^{1}_{R_1B_1\to R_2A_2}\circ\mathcal{N}^{x_1(m)}_{A_1\to B_1}\right)(\rho_{R_1A_1})\right\}. \quad (1)$$

Quantum reading protocol II



The figure depicts a quantum reading protocol that calls a memory cell three times to decode the message m as $\hat{m}$.

Quantum reading protocol III

Rate

The rate R of a given (n, R, ε) quantum reading protocol is equal to the number of bits read per channel use:

$$R := \frac{1}{n} \log_2 |\mathcal{M}|.$$

Achievable rate

A rate R is called achievable if $\forall \varepsilon \in (0, 1)$, $\delta > 0$, and sufficiently large n, there exists an $(n, R - \delta, \varepsilon)$ code.

Quantum reading capacity

The quantum reading capacity $\mathcal{C}(\mathcal{S}_{\mathcal{X}})$ of a memory cell $\mathcal{S}_{\mathcal{X}}$ is defined as the supremum of all achievable rates R.

Environment-parametrized (EP) memory cell

Definition

A set $\mathcal{E}_{\mathcal{X}} = \{\mathcal{E}^x_{A\to B}\}_{x\in\mathcal{X}}$ of quantum channels is an environment-parametrized memory cell if there exists a set $\{\theta^x_E\}_{x\in\mathcal{X}}$ of environment states and a fixed interaction channel $\mathcal{F}_{AE\to B}$ such that for all input states ρ_A and $\forall x \in \mathcal{X}$

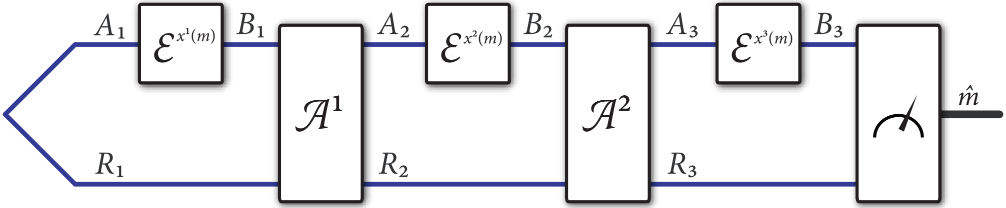
$$\mathcal{E}^x_{A\to B}(\rho_A) = \mathcal{F}_{AE\to B}(\rho_A \otimes \theta^x_E).$$

Jointly teleportation-simulable memory cell

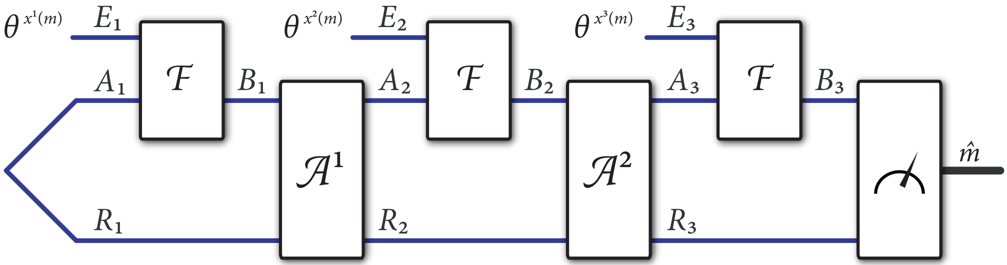
A set $\mathcal{T}_{\mathcal{X}} = \{\mathcal{N}^x_{A\to B}\}_{x\in\mathcal{X}}$ of quantum channels is a jointly teleportation-simulable memory cell if there exists a set $\{\omega^x_{RB}\}_{x\in\mathcal{X}}$ of resource states and an LOCC channel $\mathcal{L}_{ARB\to B}$ such that, for all input states ρ_A and $\forall x \in \mathcal{X}$

$$\mathcal{N}^x_{A\to B}(\rho_A) = \mathcal{L}_{ARB\to B}(\rho_A \otimes \omega^x_{RB}).$$

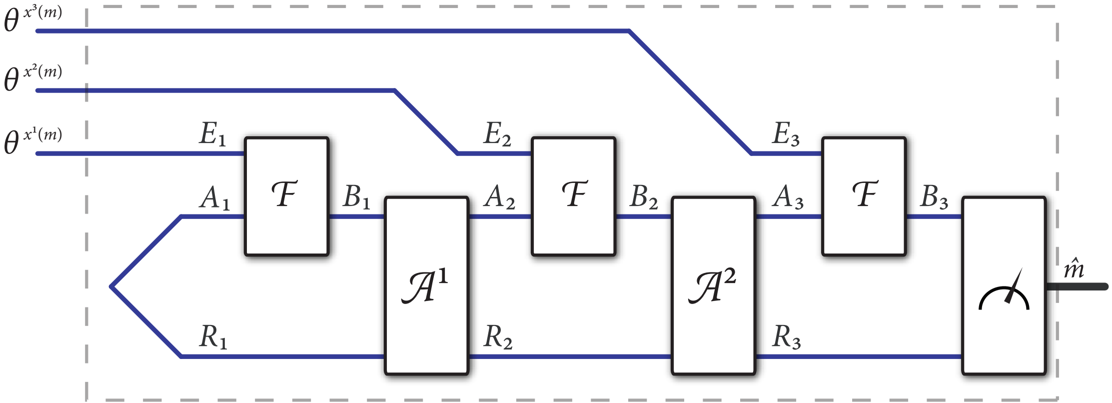
EP memory cell: Adaptive-to-non-adaptive-reduction I



EP memory cell: Adaptive-to-non-adaptive-reduction II



EP memory cell: Adaptive-to-non-adaptive-reduction III



EP memory cell: Adaptive-to-non-adaptive-reduction IV

Adaptive-to-non-adaptive reduction

$\mathcal{E}_{\mathcal{X}} = \{\mathcal{E}^{x}_{A\to B}\}_{x\in\mathcal{X}}$ is an environment-parametrized memory cell. Then any quantum reading protocol, which uses an adaptive strategy $\mathcal{J}_{\mathcal{E}_{\mathcal{X}}}$, can be simulated as a non-adaptive quantum reading protocol, in the following sense:

$$\mathrm{Tr}\left\{\Lambda^{\hat{m}}_{E_n B_n}\left(\mathcal{E}^{x_n(m)}_{A_n\to B_n}\circ\mathcal{A}^{n-1}_{E_{n-1}B_{n-1}\to E_n A_n}\circ\cdots\circ\mathcal{A}^{1}_{E_1 B_1\to E_2 A_2}\circ\mathcal{E}^{x_1(m)}_{A_1\to B_1}\right)(\rho_{E_1 A_1})\right\} = \mathrm{Tr}\left\{\Gamma^{\hat{m}}_{E^n}\left(\bigotimes_{i=1}^{n}\theta^{x_i(m)}_{E}\right)\right\},$$

for some POVM $\{\Gamma^{\hat{m}}_{E^n}\}_{\hat{m}\in\mathcal{M}}$ that depends on $\mathcal{J}_{\mathcal{E}_{\mathcal{X}}}$.

EP memory cell: Adaptive-to-non-adaptive-reduction V

Converse bound and second order asymptotics

For an (n, R, ε) quantum reading protocol for an environment-parametrized memory cell $\mathcal{E}_{\mathcal{X}} = \{\mathcal{E}^x\}_{x\in\mathcal{X}}$, the following inequality holds

$$R \leq \max_{p_X} I(X;E)_\theta + \sqrt{\frac{V_\varepsilon(\mathcal{E}_{\mathcal{X}})}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right),$$

where $\theta_{XE} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x|_X \otimes \theta_E^x$, and

$$V_\varepsilon(\mathcal{E}_{\mathcal{X}}) = \left\{ \begin{array}{ll} \min_{p_X\in P(\mathcal{E})} V(\theta_{XE}\|\theta_X\otimes\theta_E), & \varepsilon\in(0,1/2] \\ \max_{p_X\in P(\mathcal{E})} V(\theta_{XE}\|\theta_X\otimes\theta_E), & \varepsilon\in(1/2,1) \end{array} \right\},$$

where $P(\mathcal{E})$ denotes a set $\{p_X\}$ of probability distributions that achieve the maximum in $\max_{p_X} I(X;E)_\theta$.

Quantum reading capacity: Adaptive-to-non-adaptive reduction

Jointly teleportation-simulable memory cell: Quantum reading capacity

The quantum reading capacity of any jointly teleportation-simulable memory cell $\mathcal{T}_{\mathcal{X}} = \{\mathcal{N}^x_{A\to B}\}_{x\in\mathcal{X}}$ associated with a set $\{\omega^x_{RB}\}$ of resource states is bounded from above as

$$R \leq \max_{p_X} I(X;RB)_\omega + \sqrt{\frac{V_\varepsilon(\mathcal{T}_{\mathcal{X}})}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right),$$

where

$$\omega_{XRB} = \sum_{x\in\mathcal{X}} p_X(x)|x\rangle\langle x|_X \otimes \omega^x_{RB}.$$

Achievability of the reading capacity

The quantum reading capacity is achieved for a jointly teleportation-simulable memory cell $\mathcal{T}_{\mathcal{X}}$ when, for all $x\in\mathcal{X}$, ω^x_{RB} is equal to the Choi state of the channel $\mathcal{N}^x_{A\to B}$.

Example: Quantum reading protocol of $\mathcal{N}_G^{\text{cov}}$

Definition

Let $\mathcal{N}$ be a covariant channel with respect to a group G. We define the memory cell $\mathcal{N}_G^{\text{cov}}$ as

$$\mathcal{N}_G^{\text{cov}} = \{\mathcal{N}_{A\to B} \circ \mathcal{U}_A^g\}_{g\in G},$$

where $\mathcal{U}_A^g := U_A(g)(\cdot)U_A^\dagger(g)$. This is a jointly covariant memory cell.

Quantum reading capacity

The quantum reading capacity $\mathcal{C}(\mathcal{N}_G^{\text{cov}})$ of the jointly covariant memory cell $\mathcal{N}_G^{\text{cov}} = \{\mathcal{N}_{A\to B} \circ \mathcal{U}_A^g\}_{g\in G}$, is equal to the entanglement-assisted classical capacity of $\mathcal{N}$:

$$\mathcal{C}(\mathcal{N}_G^{\text{cov}}) = I(R;B)_{\mathcal{N}(\Phi)},$$

where $\mathcal{N}(\Phi) := \mathcal{N}_{A\to B}(\Phi_{RA})$ and $\Phi_{RA} \in \mathcal{D}(\mathcal{H}_{RA})$ is a maximally entangled state.

Example of $\mathcal{N}_G^{\text{cov}}$ I

Qudit erasure memory cell

The qudit erasure memory cell $\mathcal{Q}_{\mathcal{X}}^{q} = \{\mathcal{Q}_{A\to B}^{q,x}\}_{x\in\mathcal{X}}$, $|\mathcal{X}| = d^2$, consists of the following qudit channels:

$$\mathcal{Q}^{q,x}(\cdot) = \mathcal{Q}^{q}(\sigma^{x}(\cdot)(\sigma^{x})^{\dagger}), \tag{2}$$

where $\mathcal{Q}^q$ is a qudit erasure channel:

$$\mathcal{Q}^{q}(\rho_A) = (1-q)\rho + q|e\rangle\langle e| \tag{3}$$

such that $\dim(\mathcal{H}_A) = d$, $|e\rangle\langle e|$ is some state orthogonal to the support of input state ρ, and $\forall x \in \mathcal{X} : \sigma^x \in \mathbf{H}$ are the Heisenberg–Weyl operators.

Example of $\mathcal{N}_G^{\text{cov}}$ II

Qudit erasure memory cell

$\mathcal{Q}_{\mathcal{X}}^q$ is jointly covariant with respect to the Heisenberg–Weyl group $\mathbf{H}$ because the qudit erasure channel $\mathcal{Q}^q$ is covariant with respect to $\mathbf{H}$.

The quantum reading capacity $\mathcal{C}(\mathcal{Q}_{\mathcal{X}}^q)$ of the qudit erasure memory cell $\mathcal{Q}_{\mathcal{X}}^q$ (Definition 20) is equal to the entanglement-assisted classical capacity of the erasure channel $\mathcal{Q}^q$:

$$\mathcal{C}(\mathcal{Q}_{\mathcal{X}}^q) = 2(1-q)\log_2 d.$$

Weak converse bound for quantum reading protocol

Quantum reading capacity of an arbitrary memory cell

The quantum reading capacity of a quantum memory cell $\mathcal{S}_{\mathcal{X}} = \{\mathcal{N}^x\}_{\mathcal{X}}$ is bounded from above as

$$\mathcal{C}(\mathcal{S}_{\mathcal{X}}) \leq \sup_{\rho_{XRA}} \left[I(X;B|R)_\omega - I(X;A|R)_\rho \right],$$

where

$$\omega_{XRB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}^x_{A\to B}(\rho^x_{RA}),$$

$$\rho_{XRA} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho^x_{RA},$$

and the dimension of the Hilbert space $\mathcal{H}_R$ can be unbounded.

Zero-error quantum reading protocol

Zero-error quantum reading protocol

A zero-error quantum reading protocol of a memory cell $\mathcal{S}_{\mathcal{X}}$ is a particular (n, R, ε) quantum reading protocol for which $\varepsilon = 0$.

Zero-error quantum reading capacity

The zero-error quantum reading capacity $\mathcal{Z}(\mathcal{S}_{\mathcal{X}})$ of a memory cell $\mathcal{S}_{\mathcal{X}}$ is defined as the largest rate R such that there exists a zero-error reading protocol.

Zero-error quantum reading capacity I

Example

Let us consider a memory cell $\mathcal{B}_{\mathcal{X}} = \{\mathcal{M}^x_{A\to B}\}_{x\in\mathcal{X}}$, $\mathcal{X} = \{1,2\}$, consisting of the following quantum channels that map two qubits to a single qubit, acting as [HHLW10]

$$\mathcal{M}^x(\cdot) = \sum_{j=1}^{5} A^x_j(\cdot)\left(A^x_j\right)^\dagger, \; x \in \mathcal{X},$$

$$A^1_1 = |0\rangle\langle 00|,\; A^1_2 = |0\rangle\langle 01|,\quad A^1_3 = |0\rangle\langle 10|,\quad A^1_4 = \frac{1}{\sqrt{2}}|0\rangle\langle 11|,\quad A^1_5 = \frac{1}{\sqrt{2}}|1\rangle\langle 11|,$$

$$A^2_1 = |+\rangle\langle 00|,\; A^2_2 = |+\rangle\langle 01|,\; A^2_3 = |1\rangle\langle 1+|,\quad A^2_4 = \frac{1}{\sqrt{2}}|0\rangle\langle 1-|,\; A^2_5 = \frac{1}{\sqrt{2}}|1\rangle\langle 1-|,$$

and the standard bases for the channel inputs and outputs are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and $\{|0\rangle, |1\rangle\}$, respectively.

Zero-error quantum reading capacity II

Necessity of adaptive strategy for perfect channel discrimination

Channels in memory cell $\mathcal{B}_{\mathcal{X}}$ can be perfectly discriminated with finite number of channel uses only if adaptive strategy is employed.

Disadvantage of using non-adaptive strategy

The zero-error quantum reading capacity of the memory cell $\mathcal{B}_{\mathcal{X}}$ is bounded from below by $\frac{1}{2}$ whereas the zero-error non-adaptive quantum reading capacity is equal to zero.

Quantum reading

We discussed:

- Quantum reading task and working principle
- Quantum reading protocol
- Second order asymptotic bounds on rate of quantum reading for EP memory cell
- Weak converse bound for quantum reading
- Zero-error quantum reading protocol
- Thank you!

References

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