

Rényi divergences as
weighted non-commutative
vector valued L_p spaces

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Goal: find correct representation
of Rényi divergence that allows
for a generalization to
von Neumann algebras

Rényi divergence

$$D_\alpha(p||\sigma) = \frac{1}{\alpha-1} \log Q_\alpha(p||\sigma)$$

$$Q_\alpha(p||Q) = \sum_x p(x)^\alpha Q(x)^{1-\alpha}$$

quantum generalization:

Petz: $\bar{Q}_\alpha(p||\sigma) = \text{tr } \sigma^{\frac{1}{2}} p^\alpha \sigma^{\frac{1}{2}}$
operational, error exp. in quant. HT
desired properties for $\alpha \in [0, 2]$

Sandwiched: $\tilde{Q}_\alpha(p||\sigma) = \text{tr} \left(\sigma^{\frac{1}{2}} \sigma^{-\frac{1-\alpha}{2}} p^\alpha \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{1}{2}} \right)$
operational, strong converse exp. in HT
discr. prop. for $\alpha \in [\frac{1}{2}, \infty]$

Raden-Lyubskyn derivative: $\Delta(P||Q) = \frac{dP}{dQ} : \mathcal{X} \rightarrow \mathcal{R}^+$

discrete: $\frac{dP}{dQ}(x) = \frac{P(x)}{Q(x)}$ generally: $P(E) = \int_{x \in E} dP(x) = \int_{x \in E} dQ(x) \frac{dP}{dQ}(x)$

$$Q_\alpha(P||Q) = E_P[\Delta(P||Q)^\alpha]$$

$$\Delta(p||\sigma)[X] = \sigma X \sigma^\dagger$$

$$\text{simple Petz: } \langle \sigma^\dagger | \Delta(p||\sigma)^\alpha | \sigma \rangle$$

$$\text{sandwiched: } [\tilde{Q}(p||\sigma)]^\alpha = \left| \sigma^{\frac{1}{2}} \sigma^{-\frac{1-\alpha}{2}} p^\alpha \sigma^{\frac{1-\alpha}{2}} \sigma^{\frac{1}{2}} \right|_\alpha = \left| \sigma^{\frac{1}{2}} \right|_\alpha \left| \sigma^{-\frac{1-\alpha}{2}} p^\alpha \sigma^{\frac{1-\alpha}{2}} \right|_\alpha \left| \sigma^{\frac{1}{2}} \right|_\alpha$$

Lemma. For $\alpha \in [1, 2]$ and $\rho_{AB} \in \mathcal{D}(A \otimes B)$, we have that
 $\tilde{H}_\alpha^1(A|B)_\rho \leq \alpha \tilde{H}_{\frac{1}{2-\alpha}}^1(A|B)_\rho + (\alpha-1) \log |A|$ and
 $\tilde{H}_\alpha^1(A|B)_\rho \leq \frac{1}{2-\alpha} \left(\tilde{H}_{\frac{1}{2-\alpha}}^1(A|B)_\rho + (\alpha-1) \log |A| \right)$
Min-like entropies: $\alpha \in (1, \infty)$

Petz: $\bar{Q}_\alpha(p||q) = \text{tr } p^\alpha q^{1-\alpha}$

operational, error exp. in quant. HT
desirable properties for $\alpha \in [0, 2]$

Sandwiched: $\tilde{Q}_\alpha(p||q) = \text{tr} \left(p^{\frac{1}{2}} q^{\frac{1-\alpha}{\alpha}} p^{\frac{1}{2}} \right)^\alpha$

operational, strong converse exp. in HT
discrete prop. for $\alpha \in [\frac{1}{2}, \infty]$

$$Q_\alpha(P||Q) = E_P \left[\Delta(P/Q)^{\alpha-1} \right]$$

$$\Delta(p||q)[X] = p X q^{-1}$$

relative Petz: $\langle p^{\frac{1}{2}} | \Delta(p||q)^{\alpha-1} | p^{\frac{1}{2}} \rangle$

sandwiched: $[\tilde{Q}_\alpha(p||q)]^{\frac{1}{\alpha}} = \| p^{\frac{1}{2}} q^{\frac{1-\alpha}{\alpha}} p^{\frac{1}{2}} \|_\alpha = \| p \|_{\alpha, q}$

$$= \| p^{\frac{1}{2}} q^{\frac{1-\alpha}{\alpha}} \|_{2\alpha}^2 = \sup_{\omega \geq 0, \text{tr}(\omega)=1} \text{tr } \omega^{\frac{\alpha-1}{\alpha}} p^{\frac{1}{2}} q^{\frac{1-\alpha}{\alpha}} p^{\frac{1}{2}} = \langle p^{\frac{1}{2}} | \Delta(p||q)^{\frac{\alpha-1}{\alpha}} | p^{\frac{1}{2}} \rangle$$

Raden-Nykodym derivative: $\Delta(P/Q) = \frac{dP}{dQ} : \mathcal{X} \rightarrow \mathbb{R}^+$

discrete: $\frac{dP}{dQ}(x) = \frac{P(x)}{Q(x)}$

generally: $P(E) = \int_{x \in E} dP(x) = \int_{x \in E} dQ(x) \cdot \frac{dP}{dQ}(x)$