A linkage principle for Soergel bimodules

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Representation Theory of Symmetric Groups and Related Algebras
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18 December 2017

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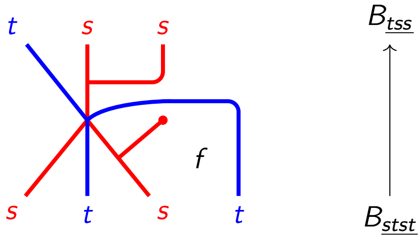

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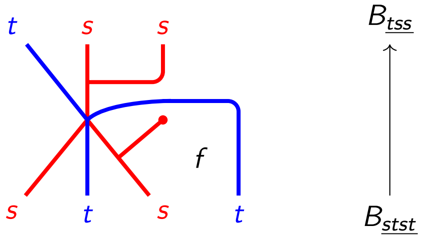

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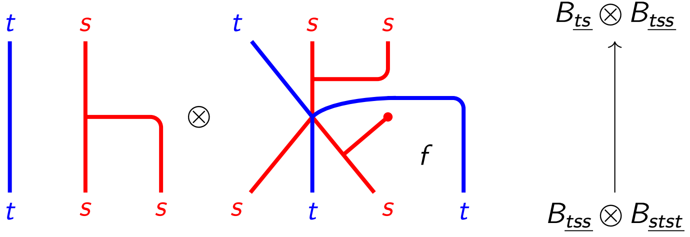

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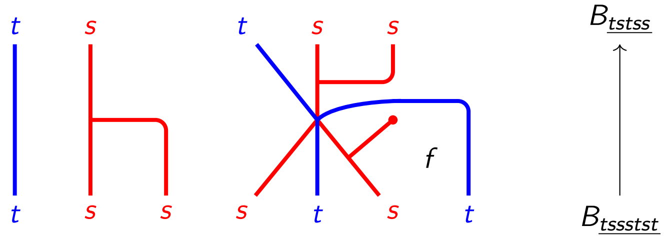

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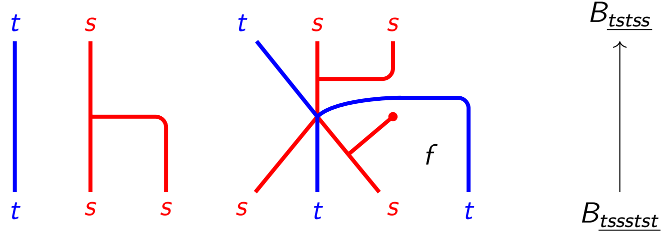

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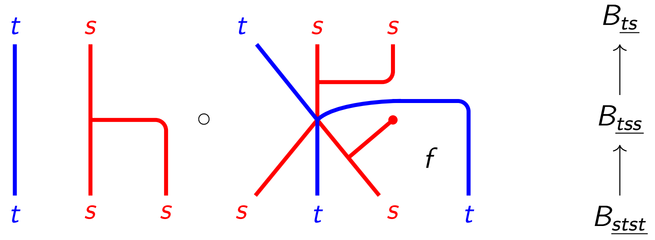

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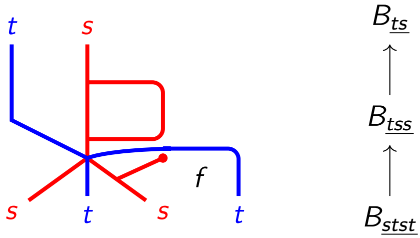

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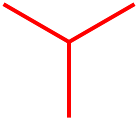

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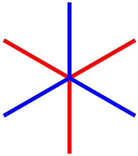

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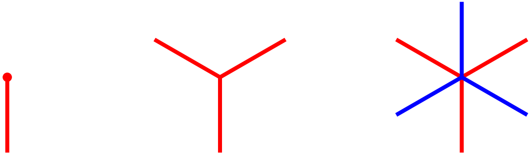

univalent (dot)	trivalent (fork)	$2m_{s,t}$-valent (braid)
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where $m_{s,t} < \infty$ is the order of st in W.

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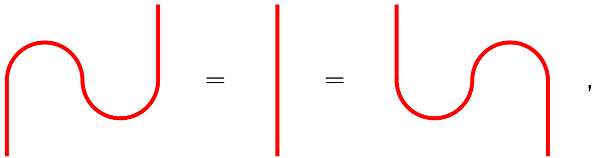

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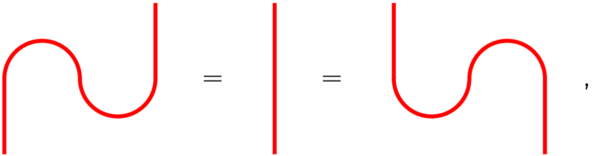

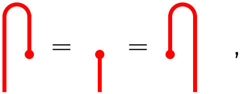

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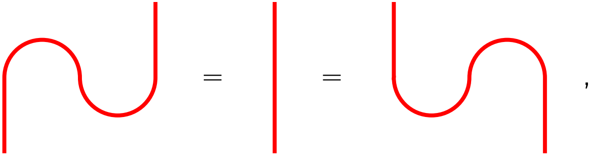

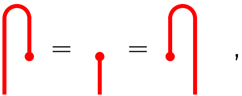

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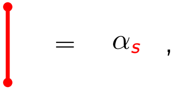

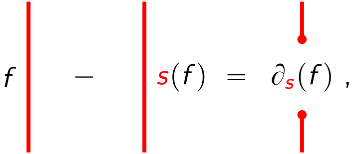

where

$$\begin{aligned}\partial_s : R &\longrightarrow R \\ f &\longmapsto \frac{f - s(f)}{\alpha_s}\end{aligned}$$

is the Demazure operator.

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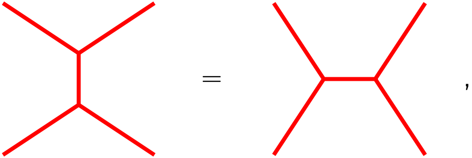

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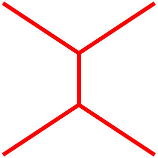
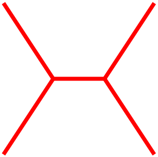
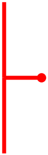
 =  ,

 = ,

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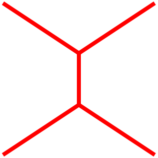
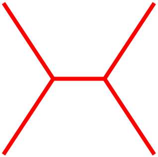
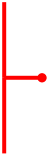
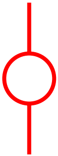
 =  ,

 = ,

 = 0.

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- Three color or Zamolodchikov relations, which involve lots of braids.

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and $[B_x]$ corresponds to an element of $H_x + \sum_{w<x} \mathbb{Z}_{\geqslant 0}[v^{\pm 1}]H_w$.

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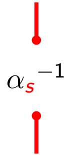

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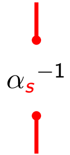

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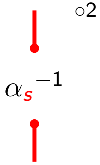

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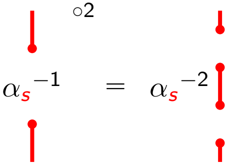

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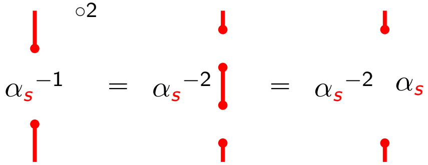

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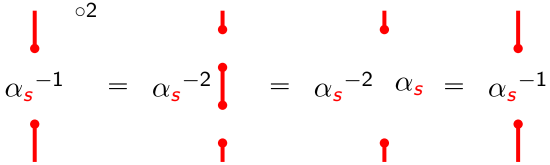

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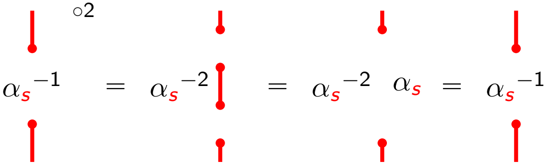

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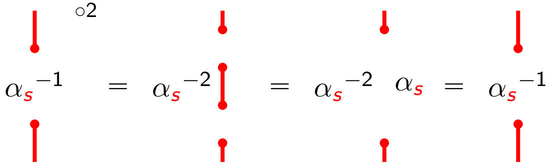

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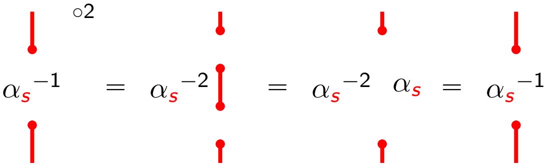

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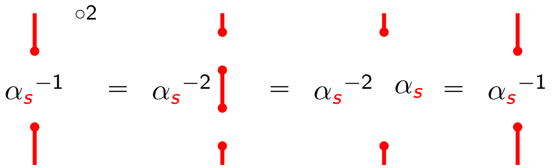

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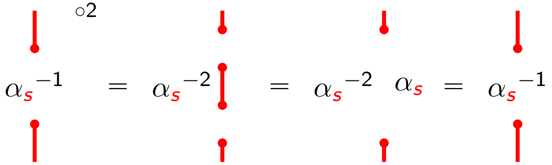

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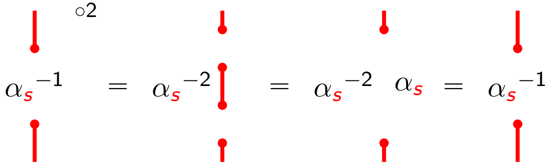

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Think of this as a kind of "higher order linkage principle" for tilting modules.

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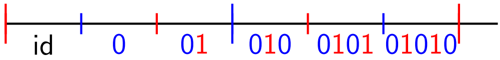

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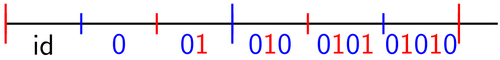

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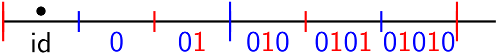

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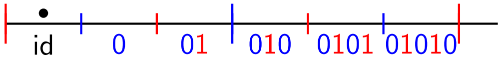

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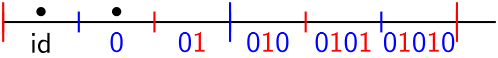

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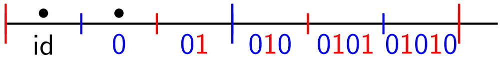

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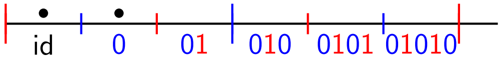

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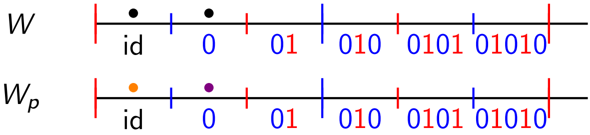

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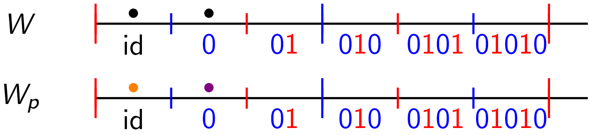

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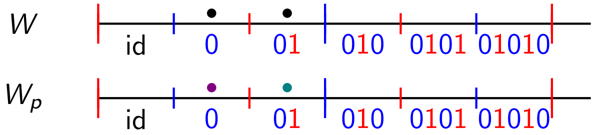

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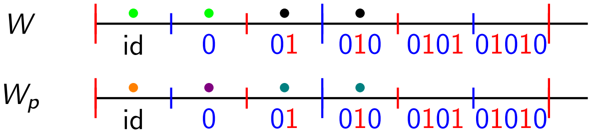

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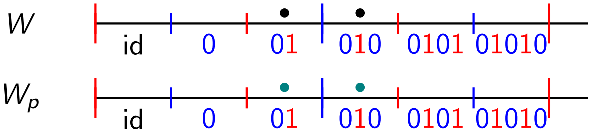

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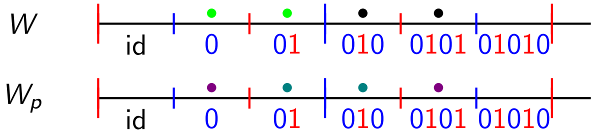

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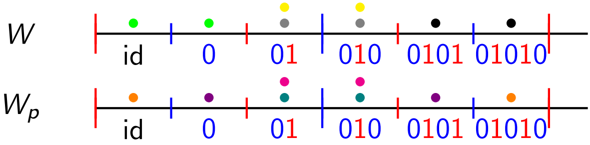

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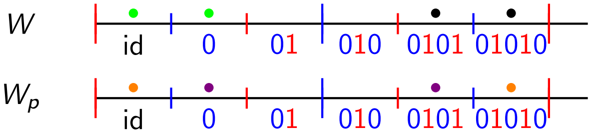

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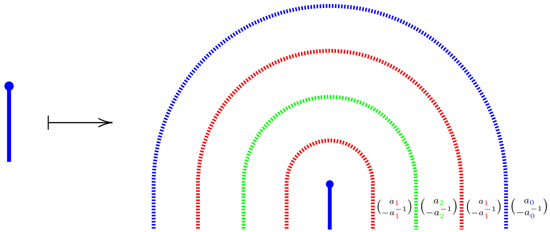

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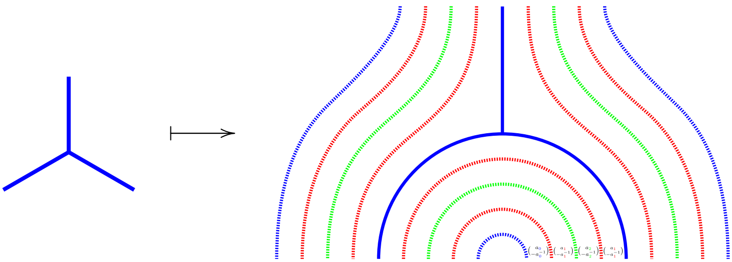

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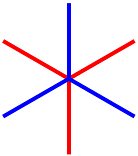

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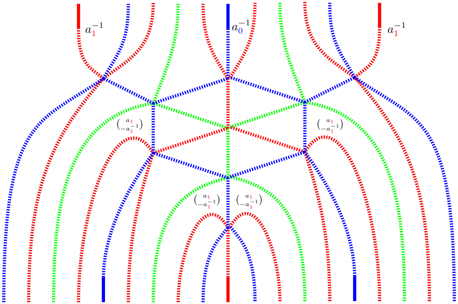

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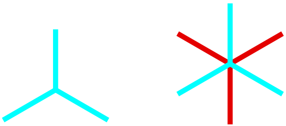

We will abbreviate the images of these vertices using strings colored cyan.

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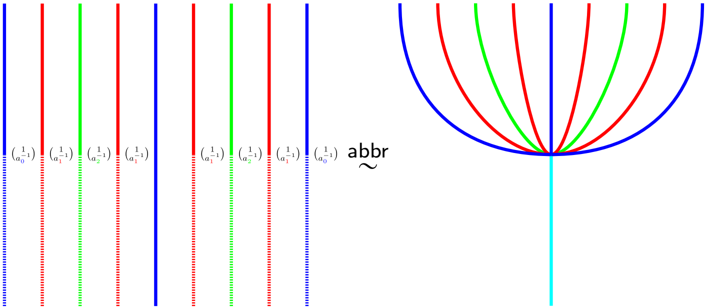

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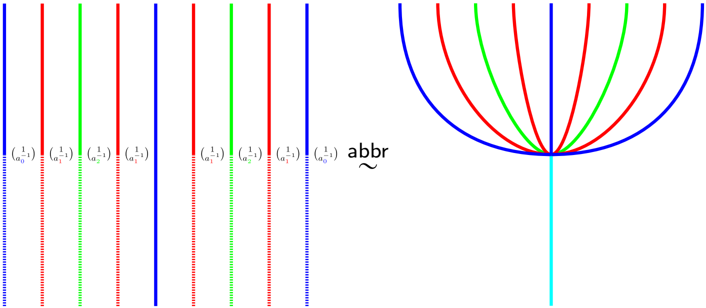

Definition

$\mathcal{D}_{\mathrm{BS},p|*}$ is the category generated from all these morphisms. The linkage category $\mathcal{D}_{p|*}$ is the Karoubi envelope of $\mathcal{D}_{\mathrm{BS},p|*}$.

Tensor actions on $\mathcal{D}_{p|*}$

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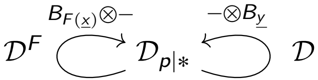

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The linkage functor

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from $\mathcal{D}^{\mathrm{ungr}}$ to left $\mathcal{D}^{F,\mathrm{ungr}} \otimes \hat{R}$-endofunctors of a direct sum category of $|W_p \backslash W|$ copies of $\mathcal{D}^{F,\mathrm{ungr}} \otimes \hat{R}$.

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The grading doesn't pass through because the block decomposition uses lots of fractions in $\hat{R}$.

Example (linkage functor)

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$W = \tilde{A}_1$, $p = 3$.

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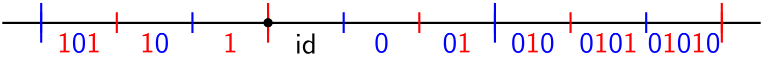

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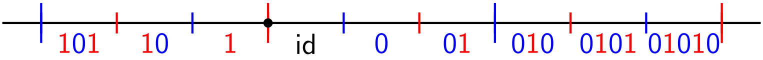

$$\mathrm{pr}(B_{\underline{010}}) = \mathrm{pr}(B_{\underline{0}})\,\mathrm{pr}(B_{\underline{1}})\,\mathrm{pr}(B_{\underline{0}})$$

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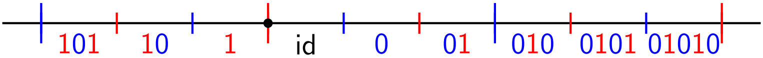

$$\begin{aligned}
\mathrm{pr}(B_{\underline{010}}) &= \mathrm{pr}(B_{\underline{0}})\,\mathrm{pr}(B_{\underline{1}})\,\mathrm{pr}(B_{\underline{0}}) \\
&= \begin{pmatrix} \hat{R} & \hat{R} & 0 \\ \hat{R} & \hat{R} & 0 \\ 0 & 0 & B_{\underline{0}_p} \end{pmatrix} \begin{pmatrix} B_{\underline{1}} & 0 & 0 \\ 0 & \hat{R} & \hat{R} \\ 0 & \hat{R} & \hat{R} \end{pmatrix} \begin{pmatrix} \hat{R} & \hat{R} & 0 \\ \hat{R} & \hat{R} & 0 \\ 0 & 0 & B_{\underline{0}_p} \end{pmatrix}
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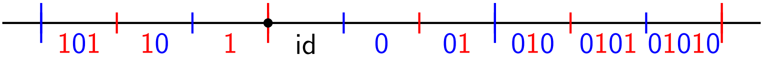

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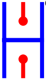
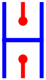
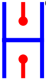
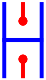
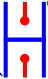
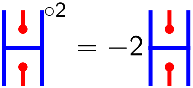
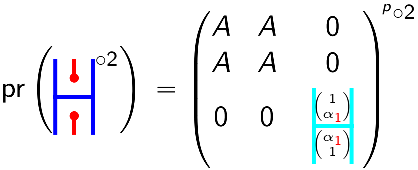
$$= \begin{pmatrix} B_{\underline{1}} \oplus \hat{R} & B_{\underline{1}} \oplus \hat{R} & B_{\underline{0}_p} \\ B_{\underline{1}} \oplus \hat{R} & B_{\underline{1}} \oplus \hat{R} & B_{\underline{0}_p} \\ B_{\underline{0}_p} & B_{\underline{0}_p} & B_{\underline{0}_p\underline{0}_p} \end{pmatrix}$$


$^{\circ 2} = -2$


 $^{\circ 2} = -2$ 

pr $\Big($  $^{\circ 2} \Big)$





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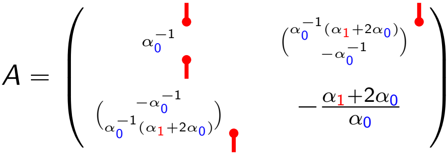

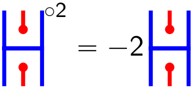

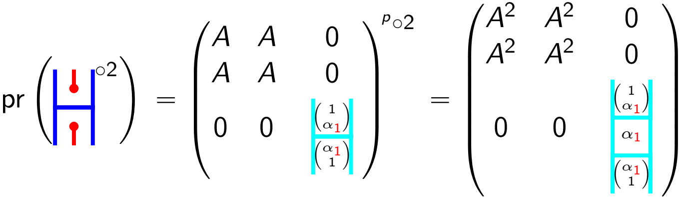

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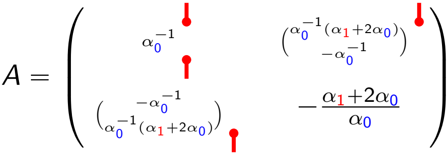

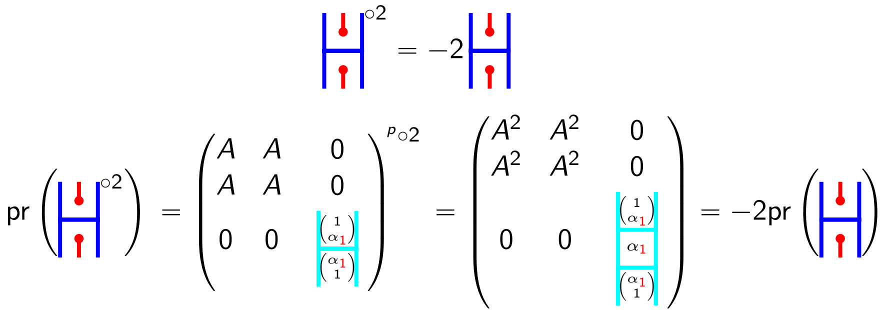

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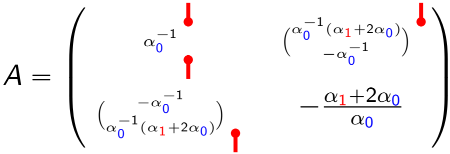

Decategorified linkage

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The result follows by reading the first row of $\mathrm{pr}(B_x)$. It is directly analogous to higher order linkage for tilting modules.

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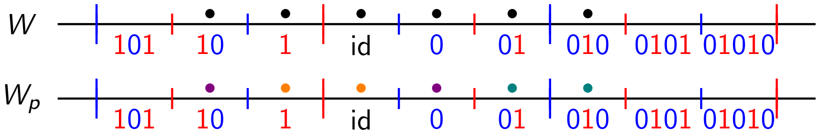

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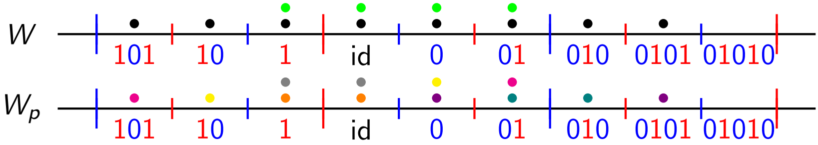

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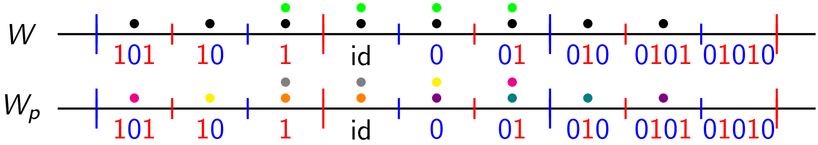

$$^3\underline{H}_{010} = \underline{H}_{010}$$
$$^3\underline{H}_{0101} = \underline{H}_{0101} + \underline{H}_{01}$$