Content systems and deformations of cyclotomic KLR algebras

Joint work with Anton Evseev

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Representation theory of symmetric groups and related algebras
December 2017

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Tableau combinatorics

A partition of n is a weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ of non-negative integers that sum to n. Identify λ with its Young diagram $[\lambda] = \{\, (r,c) \,|\, 1 \leq c \leq \lambda_r \,\}$, which is an array of boxes in the plane.

Let $\mathcal{P}_n^\Lambda$ be the set of partitions of n

Example The diagram of $(3,2)$ is

A λ-tableau with values in a set X is a function $\mathtt{t}\colon[\lambda]\longrightarrow X$, which we think of as a labelled diagram. A λ-tableau is standard if $X = \{1,\dots,n\}$ and the entries increase along rows and down columns.

Let $\operatorname{Std}(\lambda)$ be the set of standard λ-tableaux and $\operatorname{Std}(\mathcal{P}_n^\Lambda) = \bigcup_{\lambda\in\mathcal{P}_n^\Lambda}\operatorname{Std}(\lambda)$

Example The standard $(3,2)$-tableaux are:

$$\mathtt{t}^\lambda = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} = \mathtt{t}_\lambda$$

Remark We will also use standard $\boldsymbol{\lambda}$-tableaux where $\boldsymbol{\lambda} = (\lambda^{(1)}|\dots|\lambda^{(\ell)})$ is an ℓ-partition. These are ℓ-tuples of tableaux such that the entries increase along rows and along columns in each component

Content functions for symmetric groups

The content of a node (r,c) is $c-r$. If $\mathtt{t}$ is standard and $1 \leq m \leq n$ then the content of m in $\mathtt{t}$ is $c_m(\mathtt{t}) = c - r$, if $\mathtt{t}(r,c) = m$

Example If $\lambda = (4,3,3,2)$ then the contents in $[\lambda]$ are:

$$\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline -1&0&1\\\cline{1-3} -2&-1&0\\\cline{1-3} -3&-2\\\cline{1-2}\end{array}$$

Contents increase along rows, decrease down columns and are constant on the diagonals of λ. The addable nodes of λ have distinct contents

Lemma

Let $\mathtt{s} \in \operatorname{Std}(\lambda)$ and $\mathtt{t} \in \operatorname{Std}(\mu)$. Then $\mathtt{s} = \mathtt{t}$ if and only if $c_m(\mathtt{s}) = c_m(\mathtt{t})$ for $1 \leq m \leq n$. Consequently, if $1 \leq r < n$ then $c_m(\mathtt{t}) = c_m(\mathtt{t})$ for $r \neq m, m+1$ if and only if $\mathtt{s} = \mathtt{t}$ or $\mathtt{s} = s_r\mathtt{t}$

Proof Follows easily by induction because addable nodes have distinct contents

Seminormal forms

Let $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$ be the symmetric group, where $s_r = (r, r+1)$

Theorem (Young's seminormal form, 1901)

For each partition λ is there is a (unique) absolutely irreducible $\mathbb{Q}\mathfrak{S}_n$-module S^λ with basis $\{ v_{\mathtt t} \mid \mathtt t \in Std(\lambda) \}$ such that

$$s_k v_{\mathtt t} = \frac{1}{\rho_k(\mathtt t)} v_{\mathtt t} + \frac{1+\rho_k(\mathtt t)}{\rho_k} v_{s_k \mathtt t}$$

where $\rho_k(\mathtt t) = c_{k+1}(\mathtt t) - c_k(\mathtt t)$ and $v_{s_k\mathtt t} = 0$ if $s_k \mathtt t \notin Std(\lambda)$

Key point Let $\mathtt t \in \mathrm{Std}(\lambda)$ and $1 \le m \le n$. Then $L_m v_{\mathtt t} = c_m(\mathtt t) v_{\mathtt t}$

In fact, $L_k v_{\mathtt t} = c_k(\mathtt t) v_{\mathtt t} \quad \Longrightarrow \quad s_k v_{\mathtt t} = \frac{1}{\rho_k(\mathtt t)} v_{\mathtt t} + \beta_k(\mathtt t) v_{s_k \mathtt t}$
where the $\beta_k(\mathtt t)$ are scalars such that

$$\beta_k(\mathtt t)\beta_k(s_k\mathtt t) = \frac{(1+\rho_k(\mathtt t))(1+\rho_k(s_k\mathtt t))}{\rho_k(\mathtt t)\rho_k(s_k\mathtt t)}$$

$$\beta_k(\mathtt t)\beta_l(s_k\mathtt t) = \beta_l(\mathtt t)\beta_k(s_l\mathtt t)$$

$$\beta_k(\mathtt t)\beta_{k+1}(s_k\mathtt t)\beta_k(s_{k+1}s_k\mathtt t) = \beta_{k+1}(\mathtt t)\beta_k(s_{k+1}\mathtt t)\beta_{k+1}(s_k s_{k+1}\mathtt t)$$

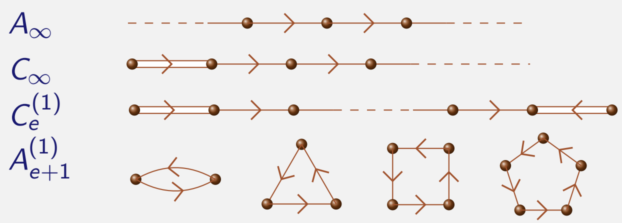
Symmetrizable quivers of affine type A and C

Let I be an index set and $\Bbbk$ a positively graded commutative ring
Let $C = (c_{ij})_{i,j \in I}$ be a generalised symmetrizable Cartan matrix, with DC symmetric for a diagonal matrix $D = \mathrm{diag}(d_i)_i \in I$
Fix homogeneous polynomials $P_{ij}(u,v) \in \Bbbk[u,v]$, with $P_{ii}(u,v) = 0$, and set $Q_{ij}(u,v) = P_{ij}(u,v)P_{ji}(v,u) = Q_{ji}(v,u)$

Assume that $Q_{ij}(u,v)$ is monic and homogeneous of degree $-c_{ij}$, for $i \ne j$

For example, we can take $Q_{ij}(u,v) = v - u^2$ if $i \Longrightarrow j$

We are mainly interested in the following Cartan types



We will freely use the corresponding Lie theoretic notation:
$P^+ = \sum_i \mathbb{N}\Lambda_i$, $Q^+ = \sum_i \mathbb{N}\alpha_i$, …

Quiver Hecke algebras

The symmetric group $\mathfrak{S}_n$ acts on I^n by place permutations:

$$w\mathbf{i} = (i_{w(1)}, \dots, i_{w(n)}), \text{ for } w \in \mathfrak{S}_n \text{ and } \mathbf{i} \in I^n$$

For $\alpha \in Q^+$ let $I^\alpha = \{ \mathbf{i} \in I^n \mid \alpha = \alpha_{i_1} + \dots + \alpha_{i_n} \}$, where $n = \mathrm{ht}(\alpha)$

Definition (Khovanov-Lauda, Rouquier 2008)

The quiver Hecke algebra, or KLR algebra, $\mathscr{R}_\alpha$ is the unital associative $\Bbbk$-algebra generated by $\{ 1_{\mathbf i} \mid \mathbf{i} \in I^\alpha \} \cup \{ \psi_r \mid 1 \le r < n \} \cup \{ y_s \mid 1 \le s \le n \}$ subject to the relations:

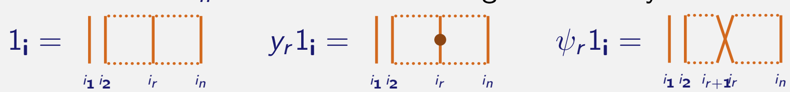
- $1_{\mathbf i} 1_{\mathbf j} = \delta_{\mathbf i,\mathbf j} 1_{\mathbf i}$, $\quad \sum_{\mathbf i \in I^\alpha} 1_{\mathbf i} = 1$, $\quad \psi_r 1_{\mathbf i} = 1_{s_r \mathbf i} \psi_r$,
- $y_r 1_{\mathbf i} = 1_{\mathbf i} y_r$, $\quad y_r y_t = y_t y_r$, $\quad \psi_r^2 1_{\mathbf i} = Q_{i_r, i_{r+1}}(y_r, y_{r+1}) 1_{\mathbf i}$
- $\psi_r y_t = y_t \psi_r$ if $s \ne r, r+1$, $\quad \psi_r \psi_t = \psi_t \psi_r$ if $|r-t| > 1$
- $(\psi_r y_{r+1} - y_r \psi_r) 1_{\mathbf i} = \delta_{i_r, i_{r+1}} 1_{\mathbf i} = (y_{r+1}\psi_r - \psi_r y_r) 1_{\mathbf i}$
- $(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r) 1_{\mathbf i} = \ \partial Q_{i_r, i_{r+1}, i_{r+2}}(y_r, y_{r+1}, y_{r+1}) 1_{\mathbf i}$

Let $\mathscr{R}_n = \bigoplus_{\alpha \in Q_n^+} \mathscr{R}_\alpha$, where $Q_n^+ = \{ \alpha \in \mathbb{Q}^+ \mid \mathrm{ht}(\alpha) = n \}$

Importantly, $\mathscr{R}_n$ is graded with the grading determined by
$\deg 1_{\mathbf i} = 0, \quad \deg y_r 1_{\mathbf i} = (\alpha_{i_r}, \alpha_{i_r}), \quad$ and $\quad \deg \psi_r 1_{\mathbf i} = -(\alpha_{i_r}, \alpha_{i_{r+1}})$

Diagrammatic presentation for $\mathscr{R}_n$

The elements of $\mathscr{R}_n$ can be described diagrammatically:



If D and E are diagrams then the diagram $D \circ E$ is zero if the residues of the strings do not match up and, when the residues coincide, $D \circ E$ is obtained by putting D on top of E and then rescaling using isotopy
The relations become "local" operations on the diagrams that describe how to move dots and strings past crossings.
For example, the relation $y_{r+1}\psi_r 1_{\mathbf i} = (\psi_r y_r + \delta_{i_r i_{r+1}}) 1_{\mathbf i}$ and the braid relation (in the simply laced case when $e \ne 2$), can be written as:

Definition (KL, R, Brundan-Stroppel, Brundan-Kleshchev)

Let $\Lambda \in P^+$ and $\alpha \in Q_n^+$. The cyclotomic quiver Hecke algebra, or cyclotomic KLR algebra, $\mathscr{R}_\alpha^\Lambda$ is the quotient of $\mathscr{R}_\alpha$ by the two-sided ideal generated by $\{ \kappa_{i_1}(y_1)\, 1_{\mathbf i} \mid \mathbf i \in I^\alpha \}$. Set $\mathscr{R}_n^\Lambda = \bigoplus_{\alpha \in Q_n^+} \mathscr{R}_\alpha^\Lambda$

Cyclotomic Hecke algebras of type A

Fix $\xi \in \Bbbk$ such that e is minimal with $1+\xi^2+\cdots+\xi^{2(e-1)}=0$
Fix integers $\kappa_1,\dots,\kappa_\ell$ with $\#\{\,1\le l\le \ell \mid \kappa_l \equiv i \pmod e\,\}=(h_i,\Lambda)$
For $m\in\mathbb{N}$ and define the ξ-quantum integer $[m]=[m]_\xi=\frac{\xi^{2m}-1}{\xi-\xi^{-1}}$

Definition (Ariki-Koike, Hu-M.)

The cyclotomic Hecke algebra of type A is the unital associative $\Bbbk$-algebra $\mathscr{H}_n^\Lambda=\mathscr{H}_n^\Lambda(\xi)$ with generators $T_1,\dots,T_{n-1}$, $L_1,\dots,L_n$ and relations

$$\prod_{l=1}^{\ell}(L_1-[\kappa_l])=0,\quad (T_r-\xi)(T_r+\xi^{-1})=0,\quad L_rL_t=L_tL_r$$
$$T_sT_{s+1}T_s=T_{s+1}T_sT_{s+1},\quad T_rT_s=T_sT_r \text{ if } |r-s|>1$$
$$T_rL_t=L_tT_r \text{ if } t\ne r,r+1,\quad L_{r+1}=T_rL_rT_r+T_r$$

When $\xi^2\ne$, $\mathscr{H}_n^\Lambda$ is an Ariki-Koike algebra, which is a deformation of the group algebra of $\mathbb{Z}/\ell\mathbb{Z}\wr\mathfrak{S}_n$. If $\xi^2=1$ then $\mathscr{H}_n^\Lambda$ is a degenerate Ariki-Koike algebra. If $\ell=1$ and $\xi^2=1$ then $\mathscr{H}_n^\Lambda\cong\Bbbk\mathfrak{S}_n$.

Theorem (Ariki-Koike)

The algebra $\mathscr{H}_n^\Lambda$ is free as a $\Bbbk$-module with basis

$$\{\,L_1^{a_1}\dots,L_n^{a_n}T_w \mid 0\le a_k<\ell \text{ and } w\in\mathfrak{S}_n\,\}$$

In particular, $\mathscr{H}_n^\Lambda$ is free of rank $\ell^n n!=\#(\mathbb{Z}/\ell\mathbb{Z}\wr\mathfrak{S}_n)$

The Brundan-Kleshchev graded isomorphism theorem

Theorem (Brundan-Kleshchev, Rouquier)

Suppose that $\Bbbk$ is a field and $\Lambda\in P^+$ and C is of type A_∞ or $A_e^{(1)}$. Then
$$\mathscr{H}_n^\Lambda\cong\mathscr{R}_n^\Lambda$$

Remarks

- This theorem is only true when $\Bbbk$ is a field. For example, both algebras are defined over $\mathbb{Z}[\xi]$ but in general the theorem is false over this ring
- Brundan and Kleshchev prove this by constructing two explicit maps $\mathscr{R}_n^\Lambda\longrightarrow\mathscr{H}_n^\Lambda$ and $\mathscr{H}_n^\Lambda\longrightarrow\mathscr{R}_n^\Lambda$ and then checking the relations on both sides: nice result, ugly proof
- As a consequence, $\mathscr{H}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra
- In the special case when $\ell=1$, $\mathscr{H}_n^\Lambda\cong\Bbbk\mathfrak{S}_n$. This gives a non-trivial $\mathbb{Z}$-grading on $\Bbbk\mathfrak{S}_n$ in the non-semisimple case
- In particular, over a field, $\mathscr{R}_n^\Lambda$ is free of dimension $\ell^n n!$

Seminormal forms for cyclotomic KLR algebras

If $\mathfrak{t}$ is a standard tableau the e-residue sequence of $\mathfrak{t}$ is the sequence
$$\mathbf{i}^{\mathfrak{t}}=(i_1^{\mathfrak{t}},\dots,i_n^{\mathfrak{t}}),\qquad \text{where } i_m^{\mathfrak{t}}=c_m(\mathfrak{t})+e\mathbb{Z}$$

Brundan and Kleshchev obtained the following analogue of the seminormal form for $\mathscr{R}_n^{\Lambda_0}\cong\Bbbk\mathfrak{S}_n$ ($e=p$) or, more generally, $\mathscr{R}_n^{\Lambda_0}\cong\mathscr{H}_\xi(\mathfrak{S}_n)$

Proposition (Brundan-Kleshchev)

Suppose that $\Bbbk$ is a field, $\ell=1$ and $e>n$. Then for each partition λ there is a (unique) irreducible graded $\mathscr{R}_n^\Lambda$-module, or $\Bbbk\mathfrak{S}_n$-module, $\mathbb{S}^\lambda$ with basis $\{\,v_{\mathfrak{t}}\mid \mathfrak{t}\in Std(\lambda)\,\}$ such that $\deg v_{\mathfrak{t}}=0$ for all $\mathfrak{t}\in Std(\lambda)$ and
$$1_{\mathbf{i}}v_{\mathfrak{t}}=\delta_{\mathbf{i}^{\mathfrak{t}}\mathbf{i}}v_{\mathfrak{t}},\quad y_rv_{\mathfrak{t}}=0\quad\text{and}\quad \psi_rv_{\mathfrak{t}}=v_{\mathfrak{s},\mathfrak{t}}$$

Kleshchev and Ram generalised this result to give seminormal forms for irreducible graded $\mathscr{R}_n^\Lambda$-modules that are concentrated in one degree, for C a Cartan matrix of finite type
All of these modules belong to semisimple blocks

We want to find seminormal forms that we can use to understand non-semisimple blocks of $\mathscr{R}_n^\Lambda$. Our starting point was a deformation of Brundan and Kleshchev's graded isomorphism theorem by Hu-M.

Content systems

In the most general set up, the cyclotomic quiver Hecke algebra depends on a choice of polynomials $\mathbf{Q}_I=\big(Q_{ij}(u,v)\big)$ and $\mathbf{K}_I=\big(\kappa_i(u)\big)$, so write
$$\mathscr{R}_n^\Lambda=\mathscr{R}_n^\Lambda(\mathbf{Q}_I,\mathbf{K}_I)$$
Fix $\rho\in I^\ell$ such that $\Lambda=\sum_{l=1}^{\ell}\Lambda_{\rho_l}$

Let Γ_ℓ be the quiver of type $A_\infty\times\cdots\times A_\infty$, with ℓ factors.
More explicitly, Γ_ℓ has vertex set $J_\ell=[1,\ell]\times\mathbb{Z}$ and edges
$$(l,a)\longrightarrow(l,a+1),\text{ for all } (l,a)\in J_\ell$$

Definition (Evseev-M.)

A content system for $\mathscr{R}_n^\Lambda(\mathbf{Q}_I,\mathbf{K}_I)$ is a pair of maps
$$\mathrm{r}\colon J_\ell\longrightarrow I\quad\text{and}\quad \mathrm{c}\colon J_\ell\longrightarrow\Bbbk\qquad\text{such that}$$

- $\mathrm{r}(l,0)=\rho_l$ and $\kappa_i(u)=\prod_{l\in[1,\ell],\rho_l\equiv i}(u-\mathrm{c}(l,0))$
- If $i=\mathrm{r}(k,a)$ then $Q_{ij}\big(\mathrm{c}(k,a),v\big)\simeq\prod_b\big(v-\mathrm{c}(k,b)\big)$, where $b=a\pm1$ and $\mathrm{r}(k,b)=j$
- If $(k,a),(l,b)\in J_\ell$ then $\mathrm{r}(k,a)=\mathrm{r}(l,b)$ and $\mathrm{c}(k,a)=\mathrm{c}(l,b)$ if and only if $(k,a)=(l,b)$
- Plus another technical condition

Examples of content systems

- If $\Gamma = A_\infty \sqcup \cdots \sqcup A_\infty$, so that $I = J_\ell$, then
$$\mathrm{r}(k,a) = (k,a) \quad \text{and} \quad \mathrm{c}(k,a) = 0$$
is a content system with coefficients in $\mathbb{Z}$
- If Γ is a quiver of type $A^{(1)}_{e+1}$ then a content system is given by:
$$\begin{array}{cccccccccc} \mathrm{r} & 0 & 1 & 2 & \dots & e & 0 & 1 & \dots \\ \mathrm{c} & 0 & x & 2x & \dots & ex & (e+1)x & (e+2)x & \dots \end{array}$$
- If Γ is a quiver of type $C^{(1)}_e$ then
$$\begin{array}{cccccccccccc} \mathrm{r} & 0 & 1 & \dots & e-1 & e & e-1 & \dots & 1 & 0 & 1 & \dots \\ \mathrm{c} & 0 & x & \dots & (e-1)x & (ex)^2 & -(e+1)x & \dots & -(2e-1)x & (2x)^2 & (2e+1)x & \dots \end{array}$$

Generically, content systems are defined over $\mathbb{Z}[x, x_1, \dots, x_\ell]$
All of the content systems above are defined over $\Bbbk[x]$ and there is a natural (homogeneous) specialisation map
$$\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I) \longrightarrow \mathscr{R}^\Lambda_n$$
given by tensoring with $\mathbb{Z}[x]/x\mathbb{Z}[x]$ — that is, specialising x to 0

Content systems corresponding to $\mathscr{R}^\Lambda_n$ under specialisation are not unique

Seminormal representations

Fix a content system (c, r) for $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)$

If $\mathfrak{t}$ is a standard tableau and m appears in row a, column b and component k of $\mathfrak{t}$ define $\mathrm{c}_m(\mathfrak{t}) = \mathrm{c}(k, b-a)$ and $r_m(\mathfrak{t}) = \mathrm{r}(k, b-a)$.

Set $\mathrm{c}(\mathfrak{t}) = (\mathrm{c}_1(\mathfrak{t}), \dots, \mathrm{c}_n(\mathfrak{t}))$ and $\mathrm{r}(\mathfrak{t}) = (\mathrm{r}_1(\mathfrak{t}), \dots, \mathrm{r}_n(\mathfrak{t}))$

Then $\mathfrak{s} = \mathfrak{t}$ if and only if $\mathrm{c}(s) = \mathrm{c}(\mathfrak{t})$ and $\mathrm{r}(\mathfrak{s}) = \mathrm{r}(\mathfrak{t})$, for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mathcal{P}^\Lambda_n)$

Proposition

Let $\mathbb{K}$ be the field of fractions of $\Bbbk$ and suppose that $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$. Then there exists a (unique) irreducible graded $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)_{\mathbb{K}}$-module $\mathbb{S}^{\boldsymbol{\lambda}}_{\mathbb{K}}$ with basis $\{\, v_\mathfrak{t} \mid \mathfrak{t} \in Std(\boldsymbol{\lambda}) \,\}$ such that
$$1_{\mathbf{i}} v_\mathfrak{t} = \delta_{\mathbf{i}, \mathbf{i}^\mathfrak{t}} v_\mathfrak{t}$$
$$y_k v_\mathfrak{t} = \mathrm{c}_k(\mathfrak{t}) v_\mathfrak{t}$$
$$\psi_k v_\mathfrak{t} = \beta_k(\mathfrak{t}) v_{s_k \mathfrak{t}} + \frac{\delta_{i^\mathfrak{t}_k, i^\mathfrak{t}_{k+1}}}{\mathrm{c}_{k+1}(\mathfrak{t}) - \mathrm{c}_k(\mathfrak{t})} v_\mathfrak{t}$$
where $\{\beta_k(\mathfrak{t})\}$ is a set of scalars that satisfy some natural conditions

Idea of proof Check the relations

Special case of symmetric groups

Theorem (Young's seminormal form, 1901)

For each partition λ is there is a (unique) absolutely irreducible $\mathbb{Q}\mathfrak{S}_n$-module S^λ with basis $\{\, v_\mathfrak{t} \mid \mathfrak{t} \in Std(\lambda) \,\}$ such that
$$s_k v_\mathfrak{t} = \frac{1}{\rho_k(\mathfrak{t})} v_\mathfrak{t} + \frac{1+\rho_k(\mathfrak{t})}{\rho_k} v_{s_k\mathfrak{t}} \quad \text{and} \quad L_k v_\mathfrak{t} = c_k(\mathfrak{t}) v_\mathfrak{t}$$
where $\rho_k(\mathfrak{t}) = c_{k+1}(\mathfrak{t}) - c_k(\mathfrak{t})$ and $v_{s_k\mathfrak{t}} = 0$ if $s_k\mathfrak{t} \notin Std(\lambda)$

Theorem (Homogeneous seminormal form, 2017)

For each partition λ is there is a (unique) absolutely irreducible $\mathbb{Q}(x)\mathfrak{S}_n$-module $\mathbb{S}^\lambda_{\mathbb{Q}(x)}$ with basis $\{\, v_\mathfrak{t} \mid \mathfrak{t} \in Std(\lambda) \,\}$ such that
$$\psi_k v_\mathfrak{t} = \frac{1}{x\rho_k(\mathfrak{t})} v_\mathfrak{t} + \delta_{i^\mathfrak{t}_r i^\mathfrak{t}_{r+1}} \frac{1+\rho_k(\mathfrak{t})}{\rho_k} v_{s_k\mathfrak{t}}$$
$$1_\mathbf{i} v_\mathfrak{t} = \delta_{\mathbf{i}^\mathfrak{t}\mathbf{i}} v_\mathfrak{t} \quad \text{and} \quad y_k v_\mathfrak{t} = x c_k(\mathfrak{t}) v_\mathfrak{t}$$

Semisimplicity and uniqueness

Theorem (Evseev-M.)

Suppose that $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over $\Bbbk$ and let $\mathbb{K}$ be the field of fractions of $\Bbbk$. Then $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)$ is a split semisimple graded $\mathbb{K}$-algebra that is canonically isomorphic to a cyclotomic quiver Hecke algebra for the quiver $A_\infty \sqcup \cdots \sqcup A_\infty$ with vertex set J_ℓ.

In particular, different content systems determine the same algebra over $\mathbb{K}$

The proof follows by splitting the idempotents $1_\mathbf{i} = \sum_{\mathfrak{t} \in \mathrm{Std}(\mathbf{i})} F_\mathfrak{t}$ and using this to construct an isomorphism

Specht modules

The basis $\{v_t\}$ of $\mathbb{S}^\lambda_\mathbb{K}$ is a "seminormal" basis. We can use $\mathbb{S}^\lambda_\mathbb{K}$ to construct an $\mathscr{R}^\Lambda_n$-module if we can specialise x to 0. That is, we need a $\Bbbk$-lattice

For $\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ define $d^\lhd_\mathtt{t}, d^\rhd_\mathtt{t} \in \mathfrak{S}_n$ by $d^\rhd_\mathtt{t}\mathtt{t}^\lambda = \mathtt{t} = d^\lhd_\mathtt{t}\mathtt{t}_\lambda$

Define $w^\rhd_\mathtt{t} = \psi_{d^\rhd_\mathtt{t}} v_{\mathtt{t}^\lambda}$ and $w^\lhd_\mathtt{t} = \psi_{d^\lhd_\mathtt{t}} v_{\mathtt{t}_\lambda}$

Let $\mathbb{S}^\lambda_\rhd = \sum_\mathtt{t} \Bbbk w^\rhd_\mathtt{t}$ and $\mathbb{S}^\lambda_\lhd = \sum_\mathtt{t} \Bbbk w^\lhd_\mathtt{t}$

Proposition

Let $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$. Then $\mathbb{S}^\lambda_\rhd$ and $\mathbb{S}^\lambda_\lhd$ are $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)_\Bbbk$-modules.

Corollary

For $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$ there exist "dual" $\mathscr{R}^\Lambda_n$-modules $S^\lambda_\rhd$ and $S^\lambda_\lhd$

- In type A this recovers the graded Specht modules of Brundan Kleshchev and Wang and the dual graded Specht modules of Hu-M.
- In type C_∞ this gives the graded Specht modules of Ariki-Park-Speyer
- In type $C^{(1)}_e$ this gives new Specht modules and proves the Ariki-Park-Speyer conjecture

Jantzen filtrations of Specht modules

The modules $\mathbb{S}^\lambda_\rhd$ and $\mathbb{S}^\lambda_\lhd$ come equipped with non-degenerate symmetric bilinear forms that are homogeneous of degree zero

Let $\mathtt{A} = \mathbb{Z}[x]_{(x)}$ be the localisation of $\mathbb{Z}[x]$ at the prime ideal (x)

The Jantzen filtration of an $\mathtt{A}$-module M_x with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ is

$$J_k(M_x) = \{\, m \in M \,|\, \langle m, a\rangle \in x^k\mathtt{A} \text{ for all } a \in M \,\}$$

Let M be the $\mathscr{R}^\Lambda_n$-module obtained by specialising x to 0.

The Jantzen filtration of M is given by

$$J_k(M) = \big(J_k(M) + xM_x\big)/xM_x, \text{ for } k \geq 0$$

By construction, $J_k(M)/J_{k+1}(M)$ has a non-degenerate homogeneous bilinear form of degree $-2k$

The aim is to find a Jantzen sum formula that explicitly describes $\sum_{k>0}[J_k(M)]$ in the Grothendieck group $\mathrm{Rep}(\mathscr{R}^\Lambda_n)$

In particular, we have Jantzen filtrations of the graded Specht modules $S^\lambda_\rhd$ and of the dual graded Specht modules $S^\lambda_\lhd$

Radical and socle filtrations

Write $d_{\lambda\mu}(q) = [S^\lambda_\rhd : D^\mu_\rhd]_q = \sum_{k\geq 0} d^{(k)}_{\lambda\mu} q^k$

Conjecture

Let $\boldsymbol{\lambda} \in \mathcal{P}^\Lambda_n$ and suppose that $\Bbbk = \mathbb{C}$ and that $\mathbb{C}$ is of type A_∞ or $A^{(1)}_e$.

Then $J_k(S^\lambda_\rhd)/J_{k+1}(S^\lambda_\rhd) = \bigoplus_\mu (D^\mu_\rhd\langle k\rangle)^{\oplus d^{(k)}_{\lambda\mu}}$

Corollary

Then the following filtrations of $S^\lambda_\rhd$ coincide:

1. *The Jantzen filtration:* $J_k(S^\lambda_\rhd)$
2. *The radical filtration:* $rad_k(S^\lambda_\rhd)$
3. *The socle filtration:* $soc_k(S^\lambda_\rhd)$
4. *The grading filtration:* $G_k(S^\lambda_\rhd)$

Cellular bases

Theorem (Evseev-M.)

Suppose that $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over $\Bbbk$
Then $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)$ is a split semisimple graded cellular algebra

Following Hu-M., there exist "integral elements" $\psi^\rhd_{\mathtt{st}}, \psi^\lhd_{\mathtt{st}} \in \mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)_\Bbbk$

Theorem (Evseev-M.)

Suppose that $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over $\Bbbk$. Then $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)_\Bbbk$ is a graded cellular $\Bbbk$-algebra with "dual" cellular bases $\{\psi^\rhd_{\mathtt{st}}\}$ and $\{\psi^\lhd_{\mathtt{st}}\}$.

The proof of this theorem is quite hard: the problem is in showing that these elements span $\mathscr{R}^\Lambda_n(\mathbf{Q}_I, \mathbf{K}_I)_\Bbbk$ — for this we need Webster algebras

Corollary (Evseev-M.)

Let $\mathscr{R}^\Lambda_n$ be a quiver Hecke algebra of type $C^{(1)}_e$
Then $\mathscr{R}^\Lambda_n$ is a graded cellular algebra

Loadings and Webster tableaux

Recall that $\Lambda \in P^+$ is a dominant weight of level ℓ.

A loading is a sequence $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{Z}^\ell$ such that

$$\theta_1 < \theta_2 < \cdots < \theta_\ell \text{ and } \theta_k \not\equiv \theta_l \pmod{\ell} \text{ for } 1 \le k < l \le \ell$$

Extend θ to the set of nodes by defining

$$\theta(l, r, c) = N\theta_l + L(c - r) + r + c - 1$$

where $N > (2n-1)$ and $L > (2n-1)\ell \implies \theta$ is injective on nodes

The loading of $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ is $L_\theta(\boldsymbol{\lambda}) = \{\, \theta(\alpha) \mid \alpha \in [\boldsymbol{\lambda}] \,\}$

Define the θ-dominance order on $\mathcal{P}_n^\Lambda$ by $\boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\mu}$ if for all nodes (l, r, c)

$$\#\{\, \alpha \in [\boldsymbol{\lambda}] \mid \theta(\alpha) \ge \theta(l, s, d) \,\} \ge \#\{\, (\alpha) \in [\boldsymbol{\mu}] \mid \theta(\alpha) \ge \theta(l, s, d) \,\}$$

A Webster $\boldsymbol{\lambda}$-tableau of type $\boldsymbol{\mu}$ is a bijection $\mathtt{T} : [\boldsymbol{\lambda}] \longrightarrow L_\theta(\boldsymbol{\mu})$ such that

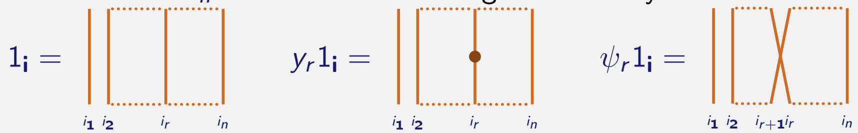
1. If $1 \le k \le \ell$ and $\lambda^{(k)} \ne (0)$ then $\mathtt{T}(k, 1, 1) \le N\theta_k$
2. If $(k, r-1, c), (k, r, c) \in \boldsymbol{\lambda}$ then $\mathtt{T}(k, r-1, c) < \mathtt{T}(k, r, c) + L$
3. If $(k, r, c-1), (k, r, c) \in \boldsymbol{\lambda}$ then $\mathtt{T}(k, r, c-1) < \mathtt{T}(k, r, c) - L$

Let $\mathrm{SStd}_\theta(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the set of Webster $\boldsymbol{\lambda}$-tableaux of type $\boldsymbol{\mu}$ and let $\mathrm{SStd}_\theta(\boldsymbol{\lambda}) = \bigcup_{\boldsymbol{\mu}} \mathrm{SStd}_\theta(\boldsymbol{\lambda}, \boldsymbol{\mu})$. Let $\boldsymbol{\omega}_n = (0|\dots|0|1^n)$.

Then $\mathrm{Std}_\theta(\boldsymbol{\lambda}) = \mathrm{SStd}_\theta(\boldsymbol{\lambda}, \boldsymbol{\omega}_n)$ is the set of standard Webster tableaux

Webster diagrams

The elements of $\mathscr{R}_n$ can be described diagrammatically:



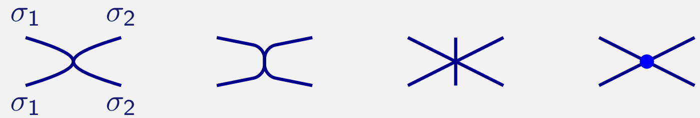
We want similar, but more complicated diagrams, to define an algebra $\mathscr{W}_n^{\theta,\Lambda}$

Webster diagrams have three types of strings:

- Thick red vertical strings with x-coordinates $N\theta_1, \dots, N\theta_\ell$
- Solid strings of residues $i_1, \dots, i_n$, for some $\mathbf{i} \in I^n$
- Dashed grey ghost strings that are translates, L-units to the left, of the solid strings. A ghost string has the same residue as the corresponding solid string

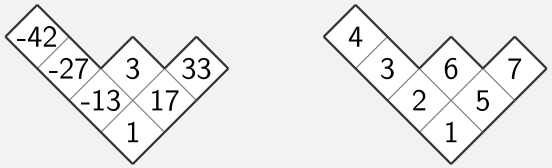
Diagrams are defined up to isotopy and solid strings can have dots

The following crossings are not allowed for red, solid or ghost strings):

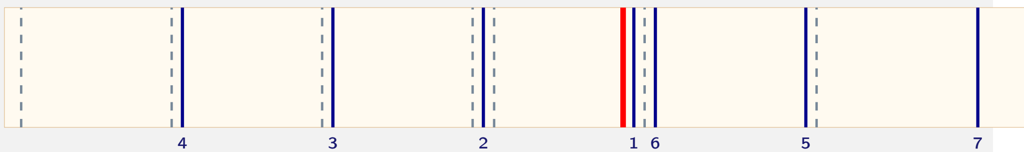


Examples of Webster diagrams

Example Let $\ell = 1$, $\theta = (0)$ and $\lambda = (4, 2, 1)$. Then $N = 15 = L$ and $\mathrm{SStd}_\theta(\lambda, \lambda)$ contains the tableaux:



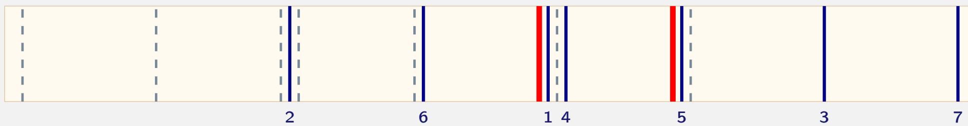
The corresponding Webster diagram 1_λ is:



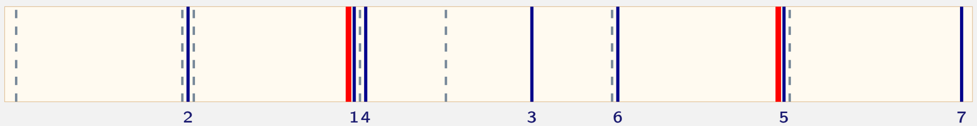
Examples of Webster diagrams II

Now let $\boldsymbol{\lambda} = (2^2|2, 1)$, so that $N = 15$ and $L = 30$.

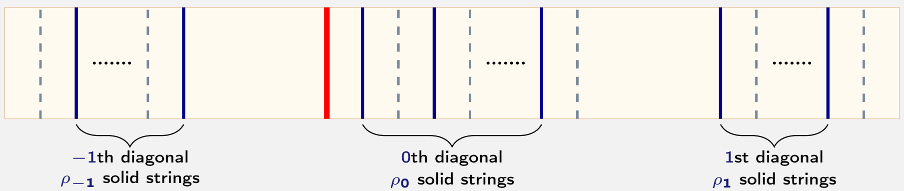
If $\theta = (0, 1)$ then $1_{\boldsymbol{\lambda}}^{\mathbf{i}}$ is the diagram



If $\theta = (0, 5)$ then $1_{\boldsymbol{\lambda}}^{\mathbf{i}}$ looks like:



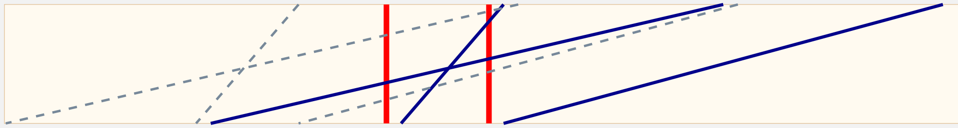
Strings in diagrams from ℓ-partitions "cluster" according to the diagonals:



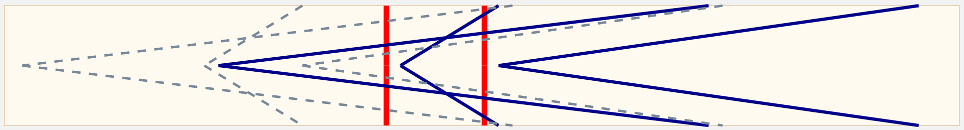
Composing Webster diagrams

We compose Webster diagrams in the usual way: if D and E are Webster diagrams then the diagram $D \circ E$ is 0 if their residues are different and when their residues are the same we put D on top of E and apply isotopy.

For example if D is the diagram



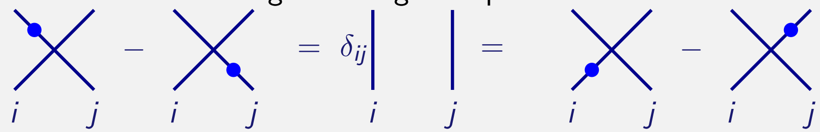
Let E be the diagram obtained by reflecting D in the line $y = 0$. Then $D \circ E$ is the diagram



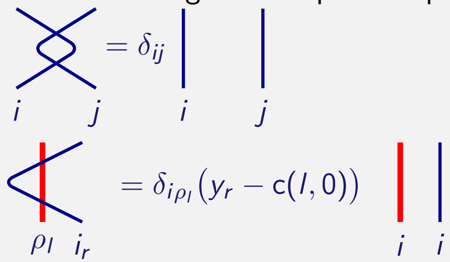
Relations for Webster algebras

The Webster algebra $\mathscr{W}_n^{\theta,\Lambda}$ is the $\Bbbk$-algebra spanned by isotopy classes of Webster diagrams with multiplication given composition and subject to the following bi-local relations:

1. If D intersects the region $(\infty, LN] \times [0, L]$ then $D = 0$ (unsteady)
2. Dots can move through crossings except for:

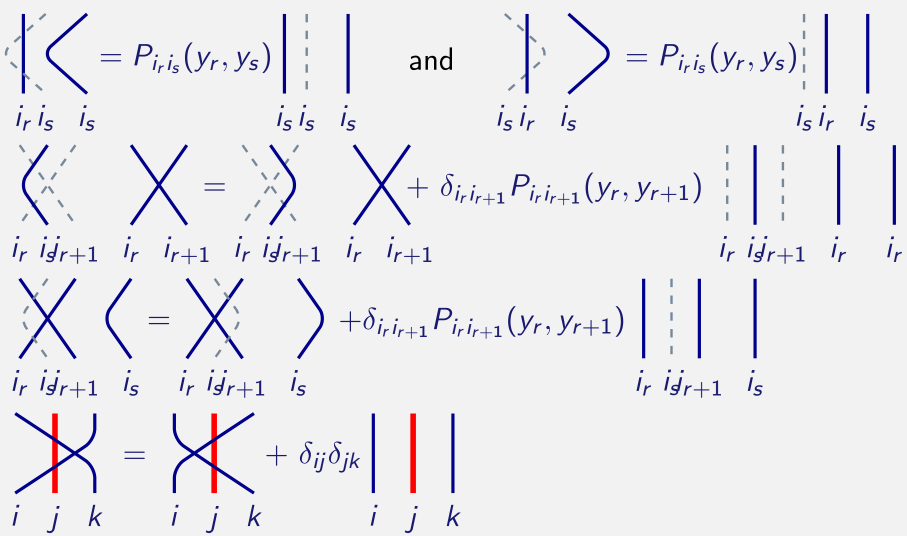


3. Double crossings can be pulled a part except for:



Relations for Webster algebras ...continued

4. Strings can be pulled through crossings except for:



A diagrammatic cellular basis

Inside $\mathscr{W}_n^{\theta,\Lambda}$, for $\mathtt{T} \in \mathrm{SStd}_\theta(\boldsymbol{\lambda}, \boldsymbol{\mu})$ define the diagram $C_\mathtt{T}$ to be a Webster diagram with a minimal number of crossings such that for each node $(l, r, c) \in [\boldsymbol{\lambda}]$ there is a solid string of residue $\kappa_l + c - r + e\mathbb{Z}$ that starts with x-coordinate $T(l, r, c) \in L_\theta(\boldsymbol{\mu})$ at the top of the diagram and that finishes with x-coordinate $\theta(l, r, c) \in L_\theta(\boldsymbol{\lambda})$ at the bottom of the diagram.

The diagram $C_\mathtt{T}$ is not unique, in general.

Let C_T^* be the diagram obtained from $C_\mathtt{T}$ by reflecting it in the line $y = 0$.

Define $C_{\mathtt{ST}}^\theta = C_\mathtt{S} C_\mathtt{T}^*$

Theorem (cf. Bowman, Webster)

The algebra $\mathscr{W}_n^{\theta,\Lambda}$ is spanned by the diagrams
$$\{\, C_{\mathtt{ST}}^\theta \mid \mathtt{S}, \mathtt{T} \in SStd_\theta(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$$

Idea of proof First push all strings to the left so that they are concave, turning at the equator. This shows that if D is a Webster diagram then $D \in \mathscr{W}_n^{\theta,\Lambda} 1_{\boldsymbol{\lambda}} \mathscr{W}_n^{\theta,\Lambda}$, for some $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

By resolving crossings it now follows that $\mathscr{W}_n^{\theta,\Lambda}$ is spanned by the $\{C_{\mathtt{ST}}^\theta\}$

Connection to KLR

Let $\boldsymbol{\omega}_n = (0|\dots|0|1^n)$ and that $\mathrm{Std}_\theta(\boldsymbol{\lambda}) = \mathrm{SStd}_\theta(\boldsymbol{\lambda}, \boldsymbol{\omega}_n)$

Theorem (cf. Bowman, Webster)

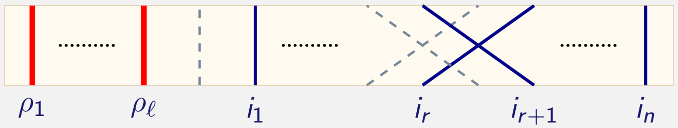
There is an isomorphism of graded algebras $\mathscr{R}_n^\Lambda \xrightarrow{\simeq} 1_{\omega_n} \mathscr{W}_n^{\theta,\Lambda} 1_{\omega_n}$

Idea of proof The isomorphism is given by:

$1_{\mathbf{i}} \mapsto$

$y_r 1_{\mathbf{i}} \mapsto$

$\psi_r 1_{\mathbf{i}} \mapsto$



Pull all strings to the right and check the relations