Restriction of characters to Sylow p-subgroups

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Introduction

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Conjecture (McKay; 1972)

Let G be a finite group, p prime. Then $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$.

Theorem (Malle, Spaeth; 2015)

Let G be a finite group, and $p = 2$. Then $|\mathrm{Irr}_{2'}(G)| = |\mathrm{Irr}_{2'}(\mathbf{N}_G(P))|$.

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Let S_n be the symmetric group and let $P_n \in \mathrm{Syl}_2(S_n)$.

Goal (2016)

Find a canonical bijection $\Phi : \mathrm{Irr}_{2'}(S_n) \longrightarrow \mathrm{Irr}_{2'}(\mathbf{N}_{S_n}(P_n))$

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Fact: $\mathbf{N}_{S_n}(P_n) = P_n$. Hence $\mathrm{Irr}_{2'}(\mathbf{N}_{S_n}(P_n)) = \mathrm{Lin}(P_n)$.

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Theorem A (G, 2016)

Let $\chi \in \mathrm{Irr}_{2'}(S_{2^k})$ *then:*

(i) *There exists a unique* $\chi^\star \in \mathrm{Lin}(P_{2^k})$ *such that* $\chi\downarrow_{P_{2^k}} = \chi^\star + \Delta$.

(Here Δ *is a sum of irreducible characters of even degree).*

(ii) *Moreover,* $\star : \mathrm{Irr}_{2'}(S_{2^k}) \longrightarrow \mathrm{Irr}_{2'}(\mathbf{N}_{S_{2^k}}(P_{2^k}))$ *is a bijection.*

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Theorem B (G, 2016)

Let $n \in \mathbb{N}$ *and* $\chi \in \mathrm{Irr}(S_n)$*, then:*

(i) *There always exists a* $\lambda \in \mathrm{Lin}(P_n)$ *such that* $\lambda \mid \chi \downarrow_{P_n}$.

(ii) λ *is unique if and only if* $n = 2^k$ *and* $\chi \in \mathrm{Irr}_{2'}(S_{2^k})$.

Theorem C (G, Kleshchev, Navarro, Tiep 2016)

There exists a combinatorially defined canonical bijection

$\Phi : \mathrm{Irr}_{2'}(S_n) \longrightarrow \mathrm{Irr}_{2'}(\mathbf{N}_{S_n}(P_n))$. *Moreover* $\Phi(\chi) \mid \chi\downarrow_{P_n}$, *for all* $\chi \in \mathrm{Irr}(S_n)$.

Restriction to Sylow p-subgroups

This is joint work with Gabriel Navarro.

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- *If $\chi \in \mathrm{Irr}_{p'}(G)$ then $|L_\chi| \neq 0$.*

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- *If $G = S_n$ and $p = 2$ then $|L_\chi| \neq 0$ for all χ.*

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- $P_{p^k} \cong \ C_p \wr \cdots \wr C_p \wr C_p = P_{p^{k-1}} \wr C_p = B \rtimes C_p$,
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Remark

Let $\lambda \in \mathrm{Irr}(P_{p^k})$. Then $\lambda(1) = 1$ if and only if there exists $\varphi \in \mathrm{Lin}(P_{p^{k-1}})$ such that $\varphi \times \varphi \times \cdots \times \varphi \mid \lambda\downarrow_B$.

...blackboard...

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Theorem B (The q-section of a character/partition)

Let $\chi \in \mathrm{Irr}(S_n)$. *Then, there exists* $\Delta(\chi) \in \mathrm{Irr}(S_m)$ *such that*
$\Delta(\chi) \times \Delta(\chi) \times \cdots \times \Delta(\chi) \mid \chi \downarrow_D$.

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What about arbitrary groups?

Conjecture C

Let $\chi \in \mathrm{Irr}(G)$ *be such that* $p \mid \chi(1)$. *If* $|L_\chi| \neq 0$ *then* $|L_\chi| \geq p$.

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Conjecture C holds for the following classes of groups:

- *Symmetric and Alternating groups. (Strong form).*
- *p-solvable groups.*
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Groups with abelian Sylow p-subgroups

Roughly speaking, **the same as above holds**.

- If B is the p-block of χ then $D \leq P$ is a *defect group* of B.

(Key tool: Green's theory of vertices and sources).

Future work: Prove Conjecture C, for all finite groups.....

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Suspect

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Permutation characters and Sylow p-subgroups

(A question of Alex Zalesski)

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Equivalent Question

Given $\lambda \vdash n$, is 1_{P_n} an irreducible constituent of $(\chi^\lambda)\downarrow_{P_n}$?

Theorem (G, Law; 2017)

Let p be an odd prime and let $n > 10$ be a natural number. Then the trivial character 1_{P_n} is a constituent of $(\chi^\lambda)\downarrow_{P_n}$ for all $\lambda \vdash n$, unless $n = p^k$ and $\lambda \in \{(p^k - 1, 1), (2, 1^{p^k - 2})\}$.

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- We determine the number of irreducible representations of the corresponding Hecke Algebra $\mathcal{H}(S_n, P_n, 1_{P_n})$.
- We obtain a similar characterization for Alternating groups.
- The situation is completely different, and more chaotic when $p = 2$.

Thank you very much!!