Invariants of Kazhdan–Lusztig cells

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19.12.2017

Coxeter groups

Let $S = \{s_1, \ldots, s_n\}$ be a finite non-empty set, and let W be a group with presentation

$$W = \langle\, s_1, \ldots, s_n \;:\; s_i^2 = e,\ (s_i s_j)^{m_{ij}} = e \,\rangle,$$

where $m_{ij} = m_{ji} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ if $i \neq j$.

Then we say that W is a Coxeter group with generating set S, and the ordered pair (W, S) is a Coxeter system.

Denote by $\ell : W \longrightarrow \mathbb{Z}_{\geqslant 0}$ the corresponding length function.

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A *weight function* is any map $\mathscr{L} : W \longrightarrow \mathbb{Z}$ such that:

$$\ell(yw) = \ell(y) + \ell(w) \;\Rightarrow\; \mathscr{L}(yw) = \mathscr{L}(y) + \mathscr{L}(w).$$

A weight function is determined by its values on S; we have

$$m_{ij} \text{ is odd } \;\Rightarrow\; \mathscr{L}(s_i) = \mathscr{L}(s_j).$$

Throughout this talk, we assume $\mathscr{L}(s) > 0$ for all $s \in S$.

Iwahori–Hecke algebras

The *Iwahori–Hecke algebra* $\mathcal{H} := \mathcal{H}(W, S, \mathscr{L})$ associated to a weighted Coxeter system is a deformation of the group algebra of W over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. It is an associative unital $\mathcal{A}$-algebra, with:

- basis: $\{T_w \,:\, w \in W\}$,
- identity: T_e,
- generators: $\{T_s \,:\, s \in S\}$,
- parameters: $\{v^{\mathscr{L}(s)} \,:\, s \in S\}$,
- relations:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (v^{\mathscr{L}(s)} - v^{-\mathscr{L}(s)})\, T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $\mathcal{K}$ be a the field of fractions of $\mathcal{A}$. If W is a finite Weyl group, then $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{H}$ is split semisimple, and isomorphic to $\mathcal{K}[W]$.

The Kazhdan–Lusztig basis

There exists a 'new' basis for $\mathcal{H}$ – the KL basis $\{C_w : w \in W\}$. Describing C_w in terms of the standard basis defines the Kazhdan–Lusztig polynomials $P_{y,w} \in \mathcal{A}$:

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We have multiplication rules for the KL basis.

$$C_s C_w = \begin{cases} C_{sw} + \sum\limits_{y:\, \substack{y<w \\ sy<y}} M^s_{y,w} C_y & \text{if } \ell(sw) > \ell(w), \\ \left(v^{\mathscr{L}(s)} + v^{-\mathscr{L}(s)}\right) C_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

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The *left elementary* relation $\leqslant_{L,E}$ defined by

$$y \leqslant_{L,E} w \quad \text{if} \quad \begin{cases} \text{there exists some } s \in S \text{ such that} \\ C_y \text{ occurs in } C_s C_w \end{cases}$$

can be extended to its reflexive, transitive closure – the Kazhdan–Lusztig preorder $\leqslant_L$.

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$$y \sim_L w \overset{\text{def.}}{\iff} y \leqslant_L w \quad \text{and} \quad w \leqslant_L y.$$

The resulting equivalence classes are called left cells. As the M-polynomials depend on $\mathscr{L}$, so does the partition of W into cells.

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An analogous preorder $\leqslant_R$ and equivalence relation $\sim_R$ exist, with equivalence classes called right cells.

Kazhdan–Lusztig cells

Finally, the two-sided preorder $\leqslant_{LR}$ arising from the relation

$$y \leqslant_{LR,E} w \overset{\text{def.}}{\iff} y \leqslant_{L,E} w \quad \text{or} \quad y \leqslant_{R,E} w$$

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- $y \sim_L w \iff y^{-1} \sim_R w^{-1}$.
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Let $\Gamma \subseteq W$ be a left cell, and $w \in \Gamma$. Then

$$I^{\Gamma}_{\leqslant} := \langle\, C_z \;:\; z \leqslant_L w \,\rangle_{\mathcal{A}}$$
$$I^{\Gamma}_{<} := \langle\, C_z \;:\; z \leqslant_L w,\, z \not\sim_L w \,\rangle_{\mathcal{A}}$$

are two left ideals of $\mathcal{H}$, and $[\Gamma] := I^{\Gamma}_{\leqslant}/I^{\Gamma}_{<}$ is a $\mathcal{H}$-module.

So, $[\Gamma]_{\mathcal{K}} := \mathcal{K} \otimes_{\mathcal{A}} [\Gamma]$ is a $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{H}$-module.

Cells of the symmetric group

$$\mathfrak{S}_n \cong W(A_{n-1})$$

For each $w \in \mathfrak{S}_n$, we may associate a pair of standard tableaux $(P(w), Q(w))$, both of shape $\mathrm{sh}(w)$ via the Robinson–Schensted correspondence.

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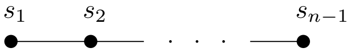

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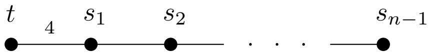
Theorem (Kazhdan–Lusztig, 1979)

Let $\mathcal{H}$ be the Iwahori–Hecke algebra of $\mathfrak{S}_n$, and let $y, w \in \mathfrak{S}_n$.

- $y \sim_L w \iff Q(y) = Q(w) \iff y \approx_\tau w$.
- $y \sim_R w \iff P(y) = P(w) \iff y \overset{*}{\longleftrightarrow} w$.
- $y \sim_{LR} w \iff \mathrm{sh}(y) = \mathrm{sh}(w)$.
- Every left cell module is irreducible.
- Every irreducible $\mathcal{H}$-module is isomorphic to a left cell module.
- $[\Gamma] \cong [\Gamma']$ if and only if Γ, Γ' lie in the same two-sided cell.
- $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{H} \cong \bigoplus_{\Gamma \subseteq \mathfrak{S}_n} [\Gamma]_{\mathcal{K}}$.

The Coxeter group of type B_n

$W_n := W(B_n)$



Let $\mathscr{L} : W_n \longrightarrow \mathbb{Z}$ be a weight function; it suffices to describe its values on S, so denote $b := \mathscr{L}(t)$, $a := \mathscr{L}(s_i)$.

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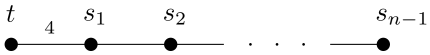

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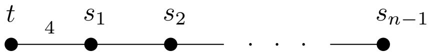

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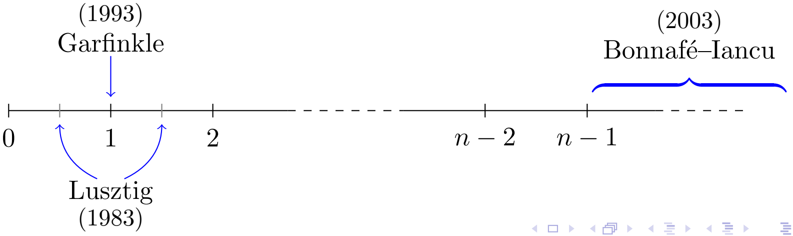
- The cells of W_n are independent of the exact value of b/a provided it is sufficiently large (with respect to n); this situation is known as the 'asymptotic case', and occurs precisely when $b/a > n-1$.
- We know the cells for W_n with respect to $\mathscr{L}$ if b/a is equal to...



The Coxeter group of type B_n

For each $w \in W_n$, we may associate a pair of standard bitableaux $(A(w), B(w))$, both of shape $\mathrm{sh}(w)$, via a generalised Robinson–Schensted correspondence.

Let $\mathcal{H}$ be the Iwahori–Hecke algebra of $(W_n, S, \mathscr{L})$, and let $y, w \in W_n$.

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Theorem (Bonnafé–Iancu, Bonnafé, 2003)

$\mathscr{L}$ is an asymptotic weight function if and only if $b/a > n-1$. Suppose that we are in the asymptotic case. Then:

- $y \sim_L w \iff B(y) = B(w)$.
- $y \sim_R w \iff A(y) = A(w)$.
- $y \sim_{LR} w \iff \mathrm{sh}(y) = \mathrm{sh}(w)$.
- Every left cell module is irreducible.
- Every irreducible $\mathcal{H}$-module is isomorphic to a left cell module.
- $[\Gamma] \cong [\Gamma']$ if and only if Γ, Γ' lie in the same two-sided cell.
- $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{H} \cong \bigoplus_{\Gamma \subseteq W_n} [\Gamma]_{\mathcal{K}}$.

Invariants of cells

"The problem seems to have two parts: the use of algebraic methods to reduce to questions about Weyl groups, and then combinatorics to study these questions." – Vogan

Let $(W, S, \mathscr{L})$ be an arbitrary weighted Coxeter system, and let the *right descent set* of $w \in W$ be $\mathcal{R}(w) := \{\, s \in S \,:\, \ell(ws) < \ell(w)\,\}$. Then $y \sim_L w \;\Rightarrow\; \mathcal{R}(y) = \mathcal{R}(w)$.

$$W \;=\; \bigsqcup_{I \subseteq S} \{\, w \in W \,:\, \mathcal{R}(w) = I\,\}.$$

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Bonnafé–Geck generalise this as follows.

Let $S^{\mathscr{L}} := S \cup \{ sts : \mathscr{L}(t) > \mathscr{L}(s) \}$,

$$\mathcal{R}^{\mathscr{L}}(w) := \{ \sigma \in S^{\mathscr{L}} : \ell(w\sigma) < \ell(w) \}.$$

Then $y \sim_L w \Rightarrow \mathcal{R}^{\mathscr{L}}(y) = \mathcal{R}^{\mathscr{L}}(w)$. If W is of type $I_2(m)$, then the converse holds too.

Invariants of cells

Let $\overline{S}^{\mathscr{L}} := S \cup \left\{ s_k \cdots s_1 t s_1 \cdots s_k \ : \ \begin{array}{l} \mathscr{L}(t) > k \cdot \mathscr{L}(s_i) \ \text{ and} \\ m_{i,i+1} = 3 \text{ for } 1 \leqslant i \leqslant k-1 \end{array} \right\},$

$$\overline{\mathcal{R}}^{\mathscr{L}}(w) := \{ \sigma \in \overline{S}^{\mathscr{L}} \ : \ \ell(w\sigma) < \ell(w) \}.$$

Proposition (H., 2017)

- Let $(W, S, \mathscr{L})$ be a finite weighted Coxeter system. Then $y \sim_L w \ \Rightarrow \ \overline{\mathcal{R}}^{\mathscr{L}}(y) = \overline{\mathcal{R}}^{\mathscr{L}}(w)$.
- Consider $(W_n, S, \mathscr{L})$ with $b/a \in (k, k+1] \subseteq (1, n]$, with $k \in \mathbb{Z}$. Then $\overline{\mathcal{R}}^{\mathscr{L}}$ partitions W_n into $2^{n-k} \cdot 3^k$ non-empty subsets.

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Suppose that we are in the asymptotic case; that is, $b/a > n-1$.

	W_3	W_4	W_5	W_6	W_7
$\mathcal{R}$	8	16	32	64	128
$\overline{\mathcal{R}}^{\mathscr{L}}$	18	54	162	486	1456
cells	20	76	312	1384	6512

Parabolic subgroups

Consider $(W, S, \mathscr{L})$ and let $I \subseteq S$ be non-empty. Then $W_I := \langle I \rangle$ is a subgroup called a parabolic subgroup. Aspects of its structure can be realised in the group W, for instance, the Kazhdan–Lusztig cells of W_I (with respect to the weight function $\mathscr{L}|_{W_I}$).

If $\varphi : W_I \longrightarrow W_I$ is any map, then φ (left) extends to a map $\varphi^L : W \longrightarrow W$ in a natural way.

$*$-operations

Consider some $(W, S, \mathscr{L})$ and let $I = \{s_i, s_j\} \subseteq S$ with $m_{ij} = 3$. The parabolic subgroup $W_I \cong \mathfrak{S}_3$ has six elements.

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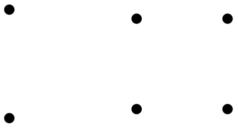

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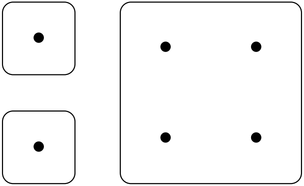

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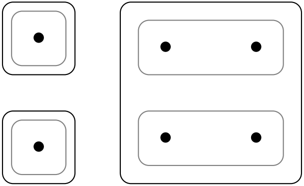

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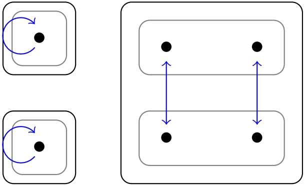
Define $\sigma : W_I \longrightarrow W_I$ by:



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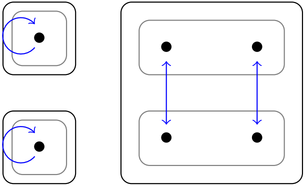

Note that $\sigma : W_I \longrightarrow W_I$ has the following properties:

- Γ is a left cell of $W_I \iff \sigma(\Gamma)$ is a left cell of W_I.
- $u \sim_{R,I} \sigma(u) \quad \forall\, u \in W_I$.

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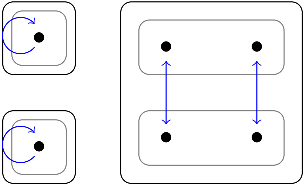

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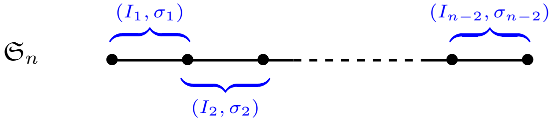
It can be seen that the map $\sigma^L : W \longrightarrow W$ is such that:

- Γ is a left cell of $W \iff \sigma^L(\Gamma)$ is a left cell of W.
- $w \sim_R \sigma^L(w) \quad \forall\, w \in W$.

The map σ^L is a $*$-operation for W (Vogan, Kazhdan–Lusztig).

Generalised τ-invariant



For $(W, S, \mathscr{L})$, these involutive maps σ^L permute the elements of W, so $\mathcal{P}(*) := \langle\, \sigma^L \,:\, \sigma^L$ is a $*$-operation for $W \,\rangle$ is a permutation group.

The $*$-operations, together with $\mathcal{R}$, are used to define the generalised τ-invariant (Vogan, 1979).

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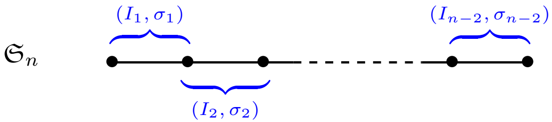

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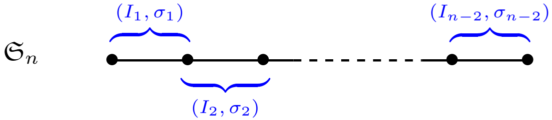
- Let $y, w \in W$. If $\mathcal{R}(y) = \mathcal{R}(w)$, continue.

Generalised τ-invariant



For $(W, S, \mathscr{L})$, these involutive maps σ^L permute the elements of W, so $\mathcal{P}(*) := \langle \sigma^L : \sigma^L$ is a $*$-operation for $W \rangle$ is a permutation group.

The $*$-operations, together with $\mathcal{R}$, are used to define the generalised τ-invariant (Vogan, 1979).

- Let $y, w \in W$. If $\mathcal{R}(y) = \mathcal{R}(w)$, continue.
- Apply a $*$-operation σ^L to y and w. If $\mathcal{R}(\sigma^L(y)) = \mathcal{R}(\sigma^L(w))$, continue.

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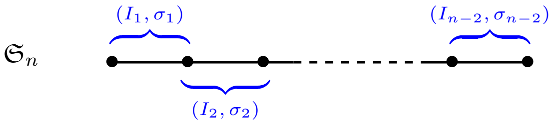

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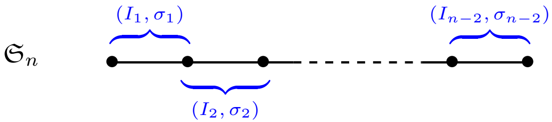
Generalised τ-invariant



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- $y \sim_L w \;\Rightarrow\; y \approx_\tau w$.

KL-admissible pairs

Let $I \subseteq S$ and $\delta : W_I \longrightarrow W_I$ be such that the following are satisfied:

(A1) If $\Gamma \subseteq W_I$ is a left cell, then so is $\delta(\Gamma)$.

(A2) The map δ induces a $\mathcal{H}_I$-module isomorphism $[\Gamma] \cong [\delta(\Gamma)]$.

(A3) We have $u \sim_{R,I} \delta(u)$ for all $u \in W_I$.

Then the pair (I, δ) is called a *strongly KL-admissible pair*.

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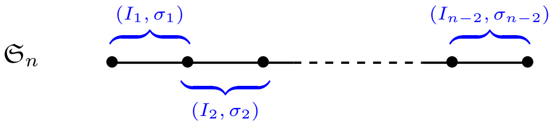

Vogan classes

Consider a weighted Coxeter system $(W, S, \mathscr{L})$, and let:

- Δ be a collection of strongly KL-admissible pairs for $(W, S, \mathscr{L})$,
- ρ be an invariant of the left cells of W,
- $\mathcal{P}(\Delta) := \langle\, \delta^L \;:\; (I, \delta) \in \Delta \,\rangle$.

We say that $y, w \in W$ are in the same left Vogan (Δ, ρ)-class if:

$$\rho(\nu(y)) \;=\; \rho(\nu(w)) \qquad \forall\, \nu \in \mathcal{P}(\Delta).$$

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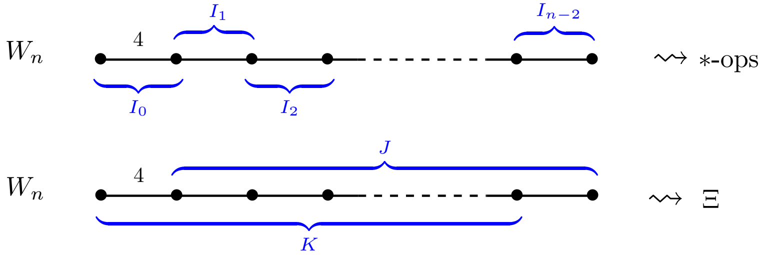
If $y \sim_L w$ then $y \approx^{\Delta,\rho} w$.

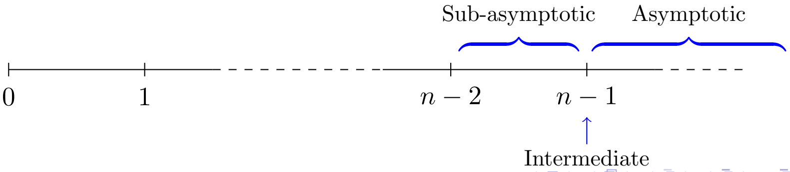
Write $y \overset{\Delta}{\longleftrightarrow} w$ if $w = \nu(y)$ for some $\nu \in \mathcal{P}(\Delta)$, and say that y and w lie in the same Δ-orbit. Then we have

- $y \overset{\Delta}{\longleftrightarrow} w \;\Rightarrow\; y \sim_R w$ and $y \sim_L w \iff y^{-1} \sim_R w^{-1}$, so:
- $y^{-1} \overset{\Delta}{\longleftrightarrow} w^{-1} \;\Rightarrow\; y \sim_L w \;\Rightarrow\; y \approx^{\Delta,\rho} w.$

Improving the maps



We have $W_K = W_{n-1}$ and $W_J \cong \mathfrak{S}_n$. The cells of $\mathfrak{S}_n$ are understood. If $b/a > n-2$, then we are in the asymptotic case for W_K, and its cells are known as well. **From now on, we assume** $b/a > n-2$, putting us in one of the following cases:



Improving the maps

Let (I, δ) be a strongly KL-admissible pair. We say that (I, δ) is *maximally KL-admissible* if it additionally satisfies the condition:

(A4) If $u \sim_{R,I} v$, then $\exists\, k \in \mathbb{Z}_{\geqslant 0}$ such that $u = \delta^k(v)$.

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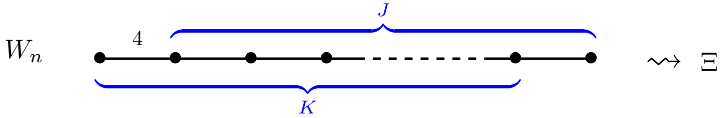

There exist maps $\varepsilon : W_J \longrightarrow W_J$ and $\psi : W_K \longrightarrow W_K$ such that both (J, ε) and (K, ψ) are maximally KL-admissible pairs.

Set $\Xi := \{(J, \varepsilon), (K, \psi)\}$ and $\mathcal{P}(\Xi) := \langle \varepsilon^L, \psi^L \rangle$.

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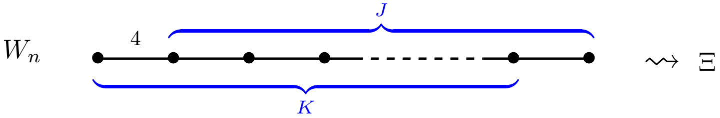

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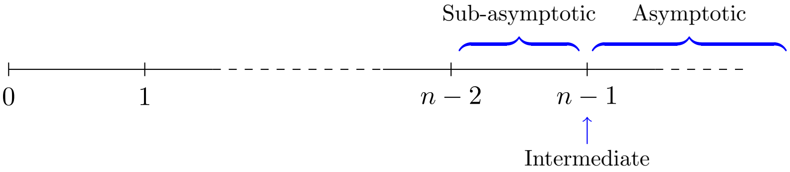
Proposition (H., 2017)

- The group $\mathcal{P}(\Xi)$ is independent of the choices made during the construction of the maps ε and ψ.
  Therefore, the left Vogan (Ξ, ρ)-classes are well-defined.
- We have $\mathcal{P}(\Delta) \leqslant \mathcal{P}(\Xi)$. Therefore, for all $y, w \in W_n$, we have $y \approx^{\Xi,\rho} w \;\Rightarrow\; y \approx^{\Delta,\rho} w$.

Left Vogan $(\Xi, \overline{\mathcal{R}}^{\mathscr{L}})$-classes



Orbits of $\mathcal{P}(\Xi)$ partition W_n; right cells of W_n are unions of Ξ-orbits.

	W_3	W_4	W_5	W_6	W_7

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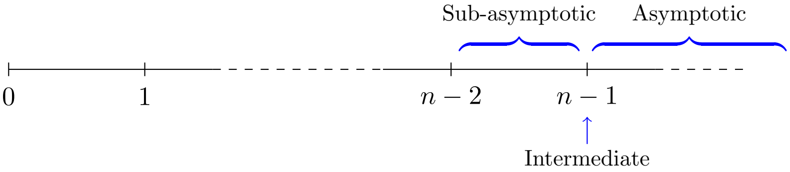

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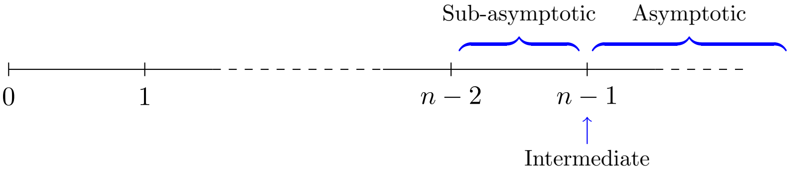

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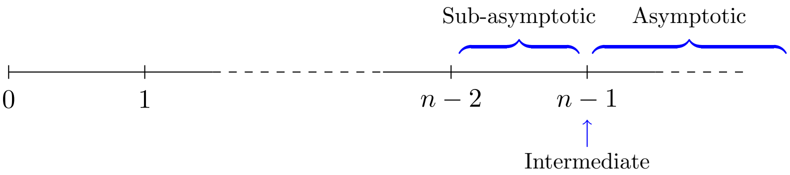

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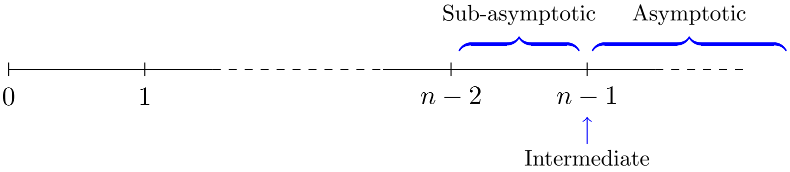

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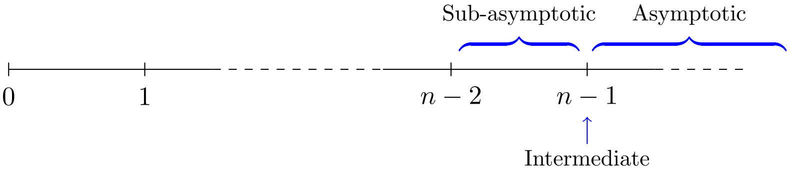

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Theorem (H., 2017)

Consider $(W_n, S, \mathscr{L})$ with $b/a \geqslant n-1$. Then:

$$y \sim_L w \iff y \approx^{\Xi, \overline{\mathcal{R}}^{\mathscr{L}}} w.$$

Thank you for your attention!