Weight 2 blocks of double covers of symmetric groups

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Part I: The symmetric groups

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Want to know decomposition numbers $[\mathrm{S}^\lambda : \mathrm{D}^\mu]$.

Basic result: $[\mathrm{S}^\mu : \mathrm{D}^\mu] = 1$, and $[\mathrm{S}^\lambda : \mathrm{D}^\mu] > 0$ only if $\lambda \trianglerighteq \mu$.

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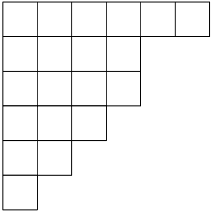

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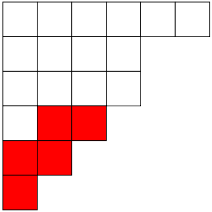

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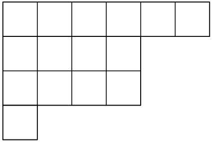

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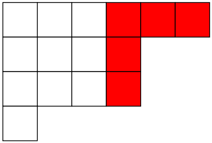

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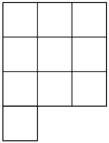

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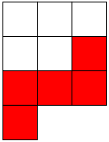

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Brauer–Robinson Theorem 1947: D^λ and D^μ (or S^λ and S^μ) lie in the same block of $\mathbb{F}\mathfrak{S}_n$ if and only if λ and μ have the same p-core (and hence the same p-weight).

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Look at blocks of small weight ...

Blocks of weight 1

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$p = 7$	$(4,2,1^7)$	$(4,2^2,1^5)$	$(4,3^2,1^3)$	$(4^2,3,1^2)$	$(5^2,3)$	$(8,5)$
$(4,2,1^7)$	1	·	·	·	·	·
$(4,2^2,1^5)$	1	1	·	·	·	·
$(4,3^2,1^3)$	·	1	1	·	·	·
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Richards 1996: Combinatorial formula for decomposition numbers.

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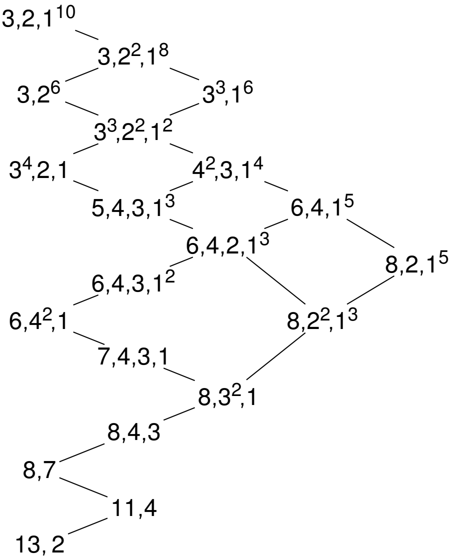

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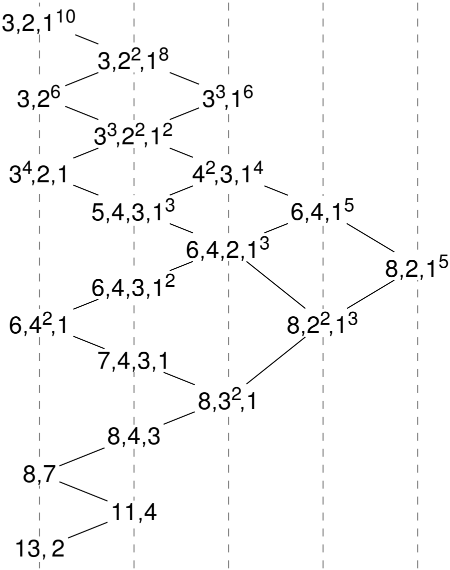

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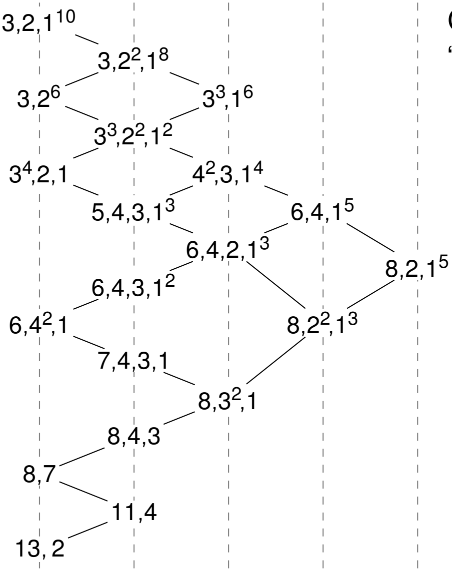

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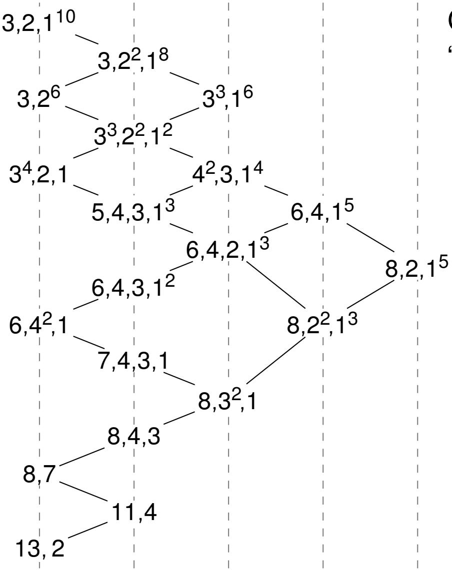

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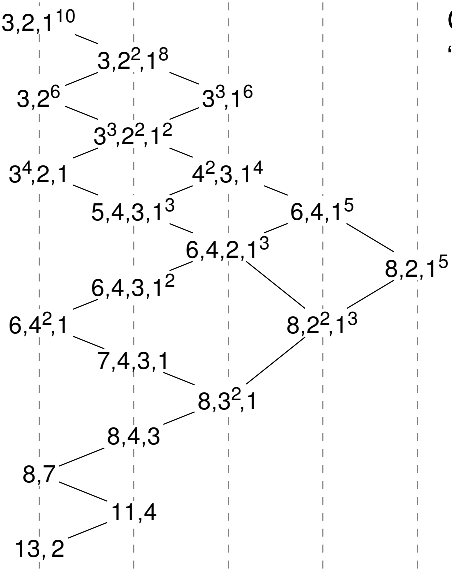

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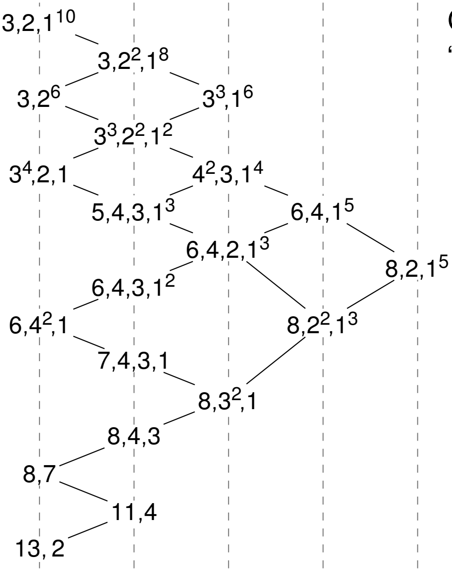

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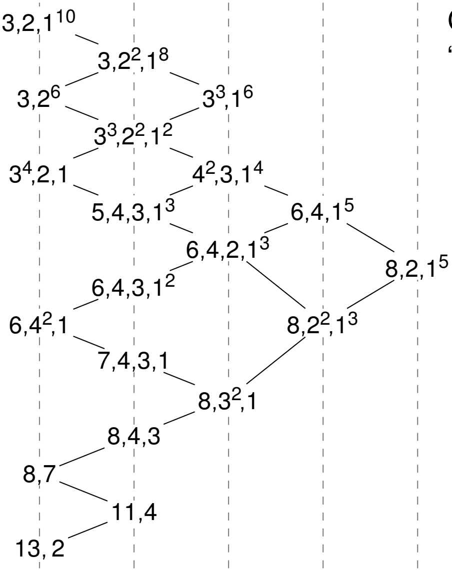

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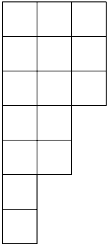

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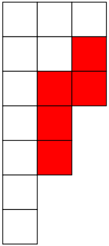

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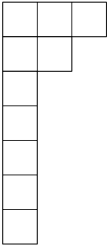

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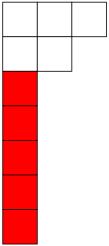

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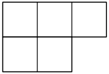

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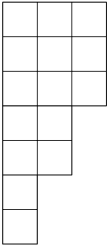

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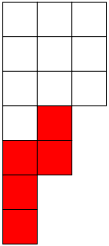

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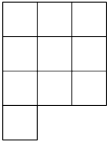

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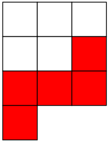

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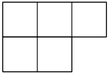
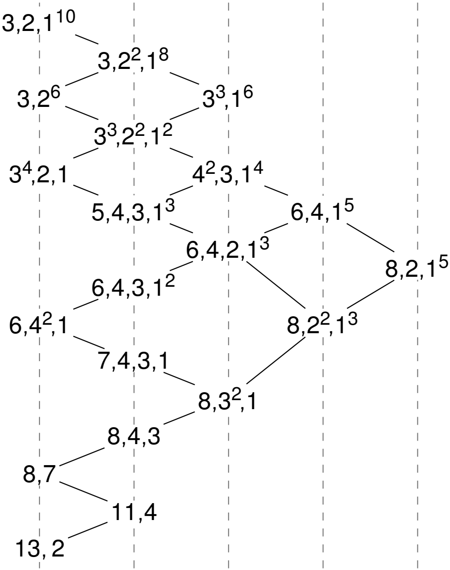


$\partial\lambda = |3 - 2| = 1$

Blocks of weight 2: $p = 5$, core $(3, 2)$



Weight 2 blocks

B a block of weight 2, core ν.

If λ in *B*, then we remove two rim *p*-hooks to reach ν from λ.

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- If *p* odd, then $[\mathrm{S}^\lambda : \mathrm{D}^\mu] = 1$ if:
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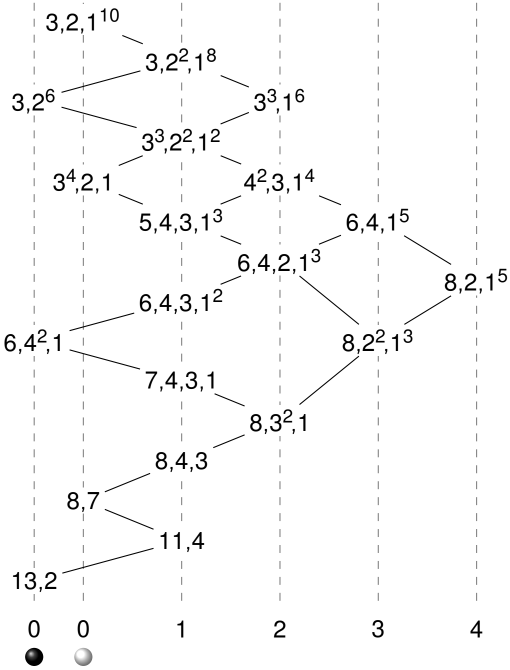
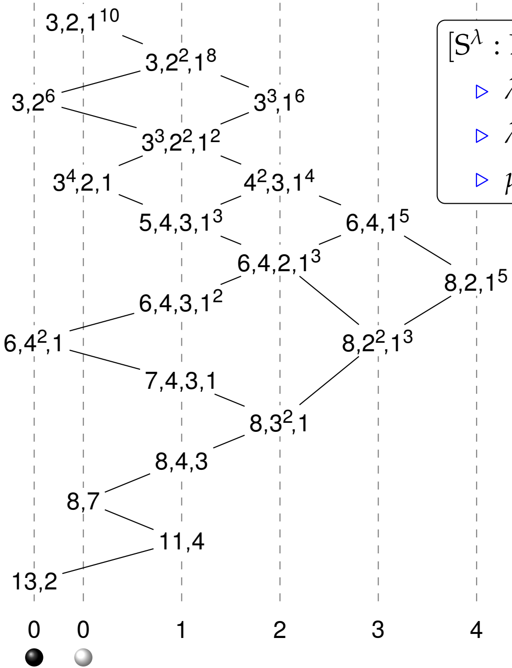


3,2,1^10
3,2^2,1^8
3,2^6
3^3,1^6
3^3,2^2,1^2
3^4,2,1
4^2,3,1^4
5,4,3,1^3
6,4,1^5
6,4,2,1^3
8,2,1^5
6,4,3,1^2
6,4^2,1
8,2^2,1^3
7,4,3,1
8,3^2,1
8,4,3
8,7
11,4
13,2
0 0 1 2 3 4



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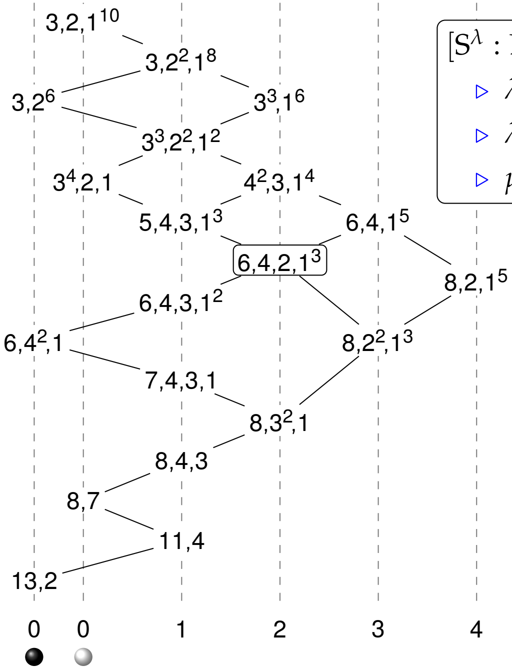

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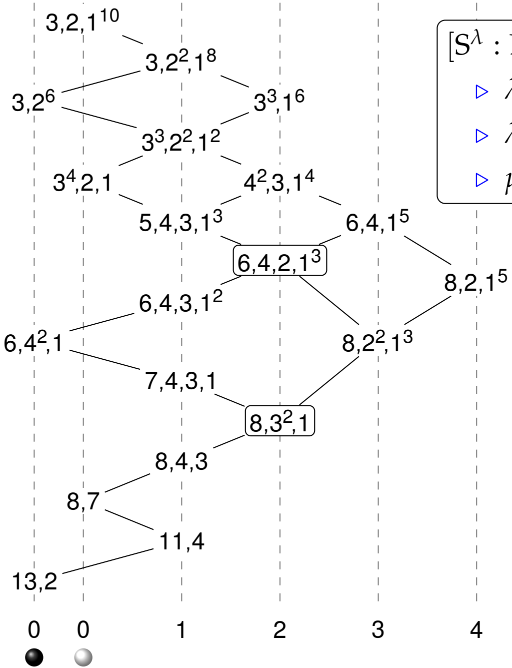

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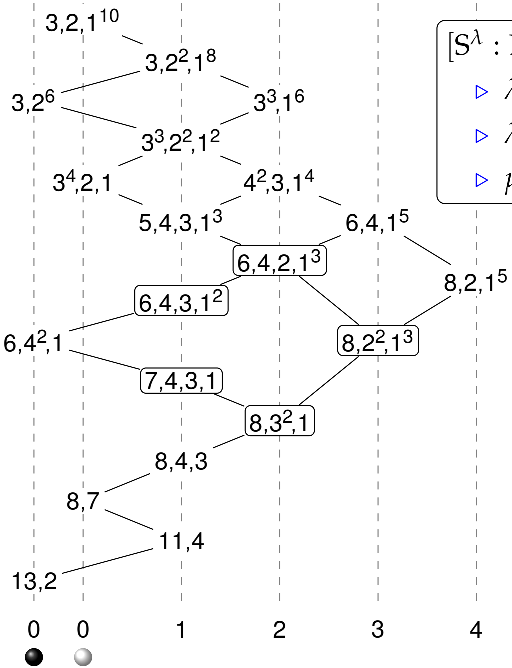

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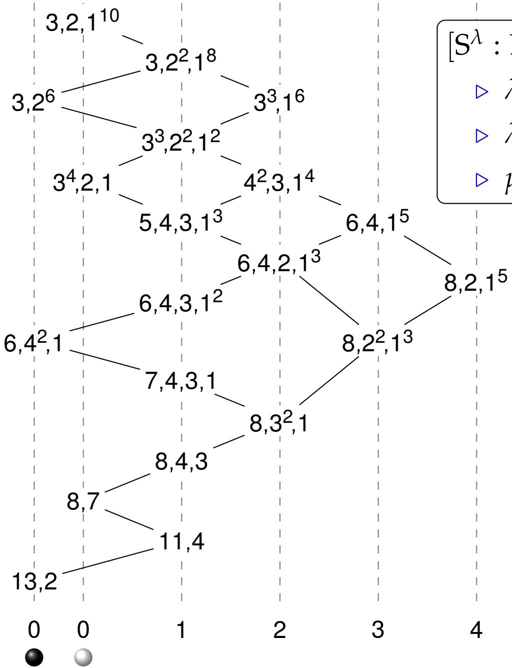

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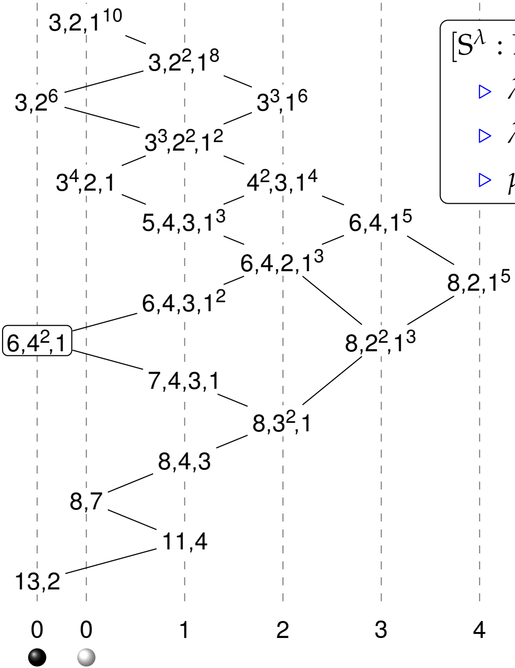

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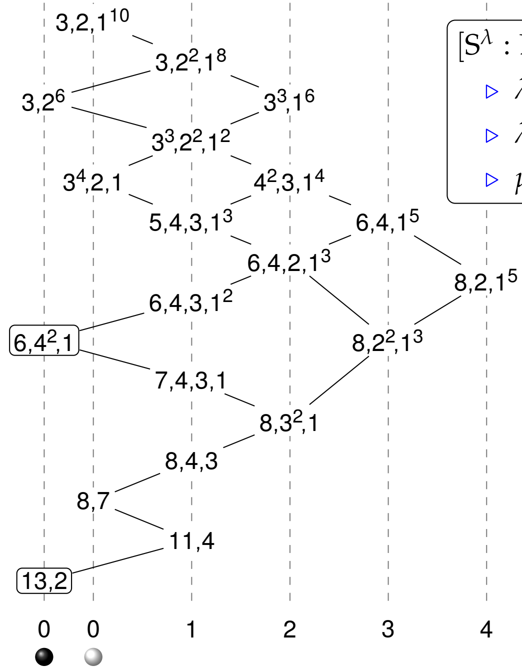

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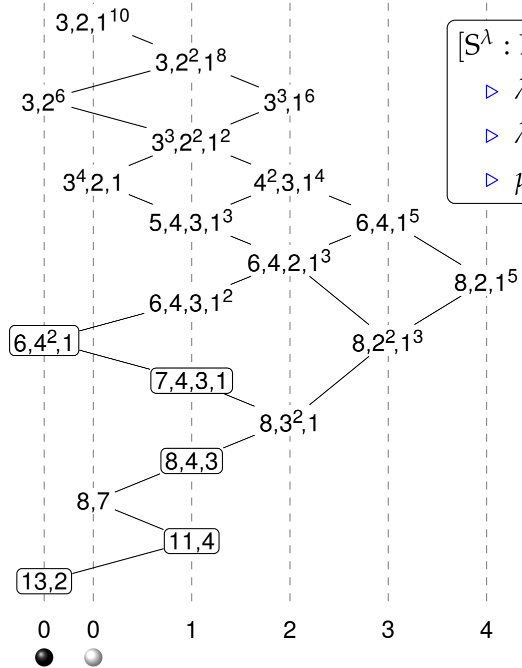

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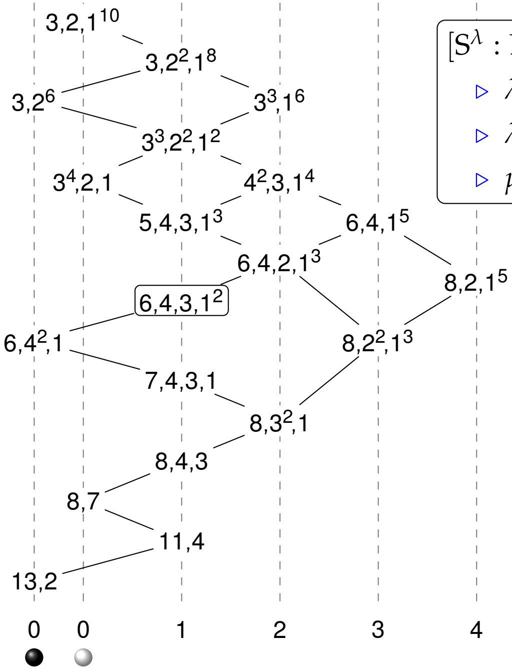

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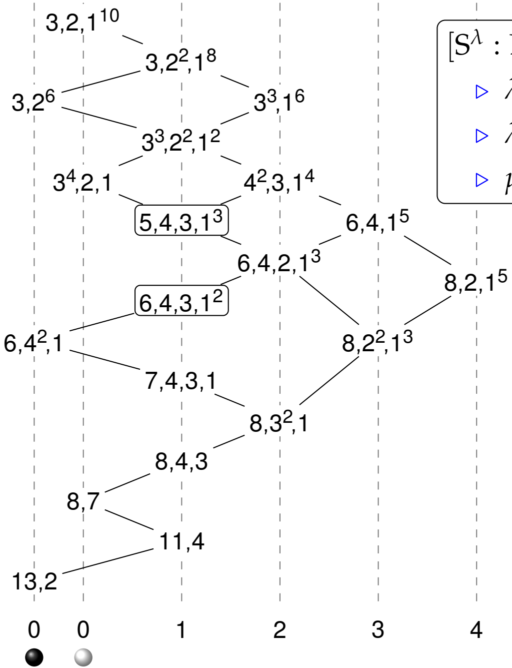

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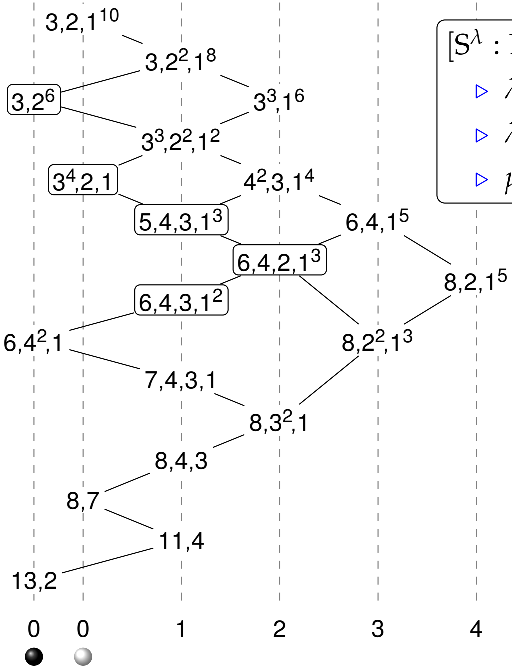

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Part II: Double covers

A double cover of $\mathfrak{S}_n$

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p odd from now on.

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Brundan–Kleshchev 2002: $\mathbb{F} = \bar{\mathbb{F}}$, characteristic p odd.
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$n = 9$, $p = 5$	$(4,3,2)$	$(5,3,1)$	$(5,4)$	$(6,2,1)$	$(6,3)$	$(7,2)$
$(4,3,2)$	1	·	·	·	·	·
$(5,3,1)$	·	1	·	·	·	·
$(5,4)$	1	·	1	·	·	·
$(6,2,1)$	·	·	·	1	·	·
$(6,3)$	·	1	·	·	1	·
$(7,2)$	·	·	·	·	·	1
$(8,1)$	·	·	·	·	1	·
(9)	·	·	1	·	·	·

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Brundan–Kleshchev:
$[\hat{\mathrm{S}}^\mu : \hat{\mathrm{D}}^\mu] = 1$, and $[\hat{\mathrm{S}}^\lambda : \hat{\mathrm{D}}^\mu] > 0$ only if $\lambda \trianglerighteq \mu$.

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(9)	·	·	1	·	·	·

Brundan–Kleshchev:
$[\hat{\mathrm{S}}^\mu : \hat{\mathrm{D}}^\mu] = 1$, and $[\hat{\mathrm{S}}^\lambda : \hat{\mathrm{D}}^\mu] > 0$ only if $\lambda \trianglerighteq \mu$.
Now look at individual blocks . . .

p-bars

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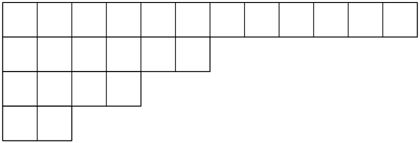

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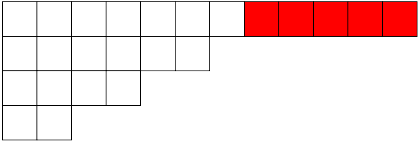

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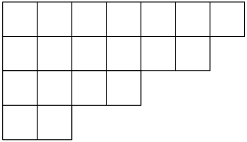

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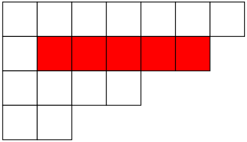

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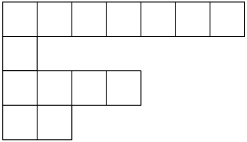

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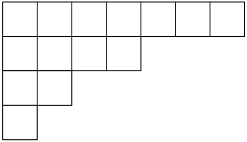

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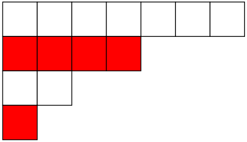

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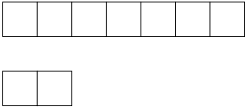

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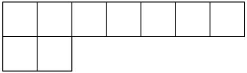

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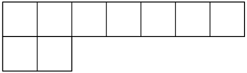

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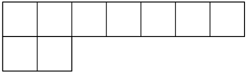

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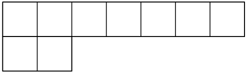

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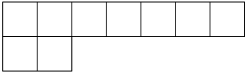

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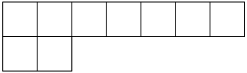

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Look at blocks of small weight . . .

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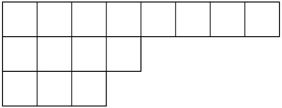

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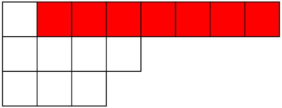

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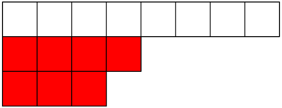

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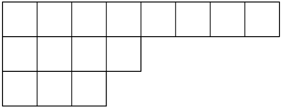

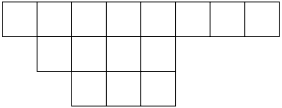

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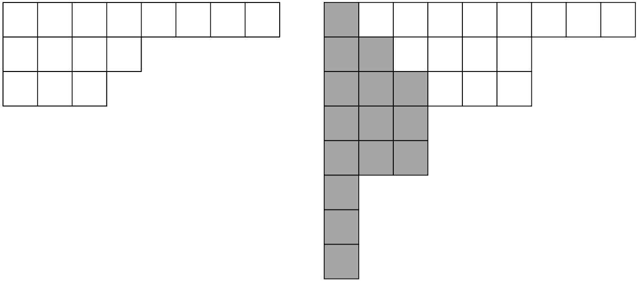

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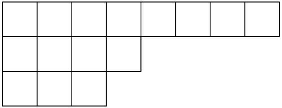

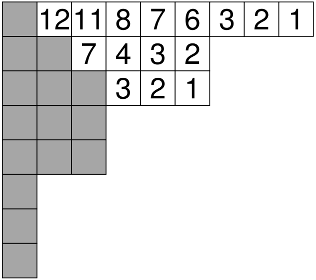

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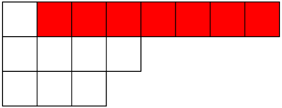

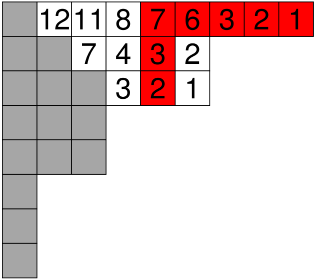

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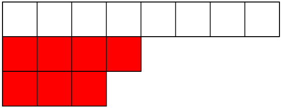

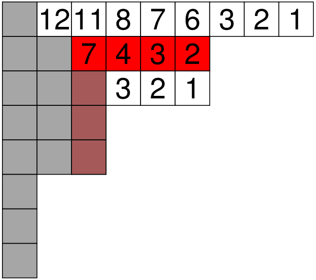

$\longrightarrow$ leg length of a removable p-bar (Hoffman–Humphreys).

Spin blocks of weight 1

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$p = 7$	$(8,4,3,1)$	$(8,5,2,1)$	$(8,7,1)$
$(8,4,3,1)$	1	·	·
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$(15,1)$	·	·	1

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Instead, we use the Fock space ....

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Assign spin residues to nodes:

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0	1	2	3	2	1	0	0	1	2	3	2	1	0	0	1	2		
0	1	2	3	2	1	0	0	1	2									
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0	1	2	3	2	1	0	0	1	2									
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Then

$$\begin{aligned} t_i\lambda &\in \langle \lambda \rangle \\ f_i\lambda &\in \langle \mu \mid \mu = \lambda \cup \text{an } i\text{-node} \rangle \\ e_i\lambda &\in \langle \mu \mid \mu = \lambda \setminus \text{an } i\text{-node} \rangle . \end{aligned}$$

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(Kashiwara–Miwa–Petersen–Yung 1996, Leclerc–Thibon 1997.)

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Hope this is true for blocks of weight 2, and calculate ...

Blocks of weight 2

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$p = 5$

5-bar-core $(6, 1)$

	$(6,5,3,2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$
$(6,5,3,2,1)$	1	·	·	·	·
$(7,6,3,1)$	2	2	1	·	·
$(8,6,2,1)$	1	2	1	1	·
$(10,6,1)$	1	1	·	1	1
$(11,3,2,1)$	·	·	·	1	·
$(11,5,1)$	·	1	·	1	1
$(11,6)$	·	·	·	·	1
$(16,1)$	·	1	·	·	·

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$(6,5,3,2,1)$	1	·	·	·	·
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$(8,6,2,1)$	1	2	1	1	·
$(10,6,1)$	1	1	·	1	1
$(11,3,2,1)$	·	·	·	1	·
$(11,5,1)$	·	1	·	1	1
$(11,6)$	·	·	·	·	1
$(16,1)$	·	1	·	·	·

Let's try a Richards formula anyway ...

A spin Richards formula?

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B weight 2 spin block of $\hat{\mathfrak{S}}_n$ with p-bar-core ν.

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Two special partitions:

- ♔ $= (\dots, p^2, \dots)$;
- ♕ $=$ unique minimal partition with ♕ $\rhd$ ♔.

A spin Richards formula

A spin Richards formula

	∂		$(6,5,3,2,1)$	$(6,5^2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$
	1	$(6,5,3,2,1)$	1	·	·	·	·	·
	0	$(6,5^2,1)$ ♔	1	1	1	·	·	·
	1	$(7,6,3,1)$ ♕	1	1	1	1	·	·
	1	$(8,6,2,1)$	·	1	1	1	1	·
	0	$(10,6,1)$	·	1	·	·	1	1
	2	$(11,3,2,1)$	·	·	·	·	1	·
	1	$(11,5,1)$	·	·	1	·	1	1
	0	$(11,6)$	·	·	·	·	·	1
	0	$(16,1)$	·	·	1	·	·	·

A spin Richards formula

"Richards matrix"

	∂		$(6,5,3,2,1)$	$(6,5^2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$
	1	$(6,5,3,2,1)$	1	·	·	·	·	·
	0	$(6,5^2,1)$ ♔	1	1	1	·	·	·
	1	$(7,6,3,1)$ ♕	1	1	1	1	·	·
	1	$(8,6,2,1)$	·	1	1	1	1	·
	0	$(10,6,1)$	·	1	·	·	1	1
	2	$(11,3,2,1)$	·	·	·	·	1	·
	1	$(11,5,1)$	·	·	1	·	1	1
	0	$(11,6)$	·	·	·	·	·	1
	0	$(16,1)$	·	·	1	·	·	·

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	∂		$(6,5,3,2,1)$	$(6,5^2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$
○	1	$(6,5,3,2,1)$	1	·	·	·	·	·
◑	0	$(6,5^2,1)$ ♔	1	1	1	·	·	·
◑	1	$(7,6,3,1)$ ♕	1	1	1	1	·	·
◑	1	$(8,6,2,1)$	·	1	1	1	1	·
●	0	$(10,6,1)$	·	1	·	·	1	1
	2	$(11,3,2,1)$	·	·	·	·	1	·
◑	1	$(11,5,1)$	·	·	1	·	1	1
●	0	$(11,6)$	·	·	·	·	·	1
○	0	$(16,1)$	·	·	1	·	·	·

Now take all columns starting with ♔ or ending with ♕ and add together in adjacent pairs.

A spin Richards formula

“Richards matrix”

	∂		$(6,5,3,2,1)$	$(6,5^2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$	$(6,5,3,2,1)$ (right)	$(6,5^2,1)$ (right)	$(7,6,3,1)$ (right)	$(8,6,2,1)$ (right)	$(10,6,1)$ (right)
	1	$(6,5,3,2,1)$	1	·	·	·	·	·	1	·	·	·	·
	0	$(6,5^2,1)$ ♔	1	1	1	·	·	·	2	2	·	·	·
	1	$(7,6,3,1)$ ♕	1	1	1	1	·	·	2	2	1	·	·
	1	$(8,6,2,1)$	·	1	1	1	1	·	1	2	1	1	·
	0	$(10,6,1)$	·	1	·	·	1	1	1	1	·	1	1
	2	$(11,3,2,1)$	·	·	·	·	1	·	·	·	·	1	·
	1	$(11,5,1)$	·	·	1	·	1	1	·	1	·	1	1
	0	$(11,6)$	·	·	·	·	·	1	·	·	·	·	1
	0	$(16,1)$	·	·	1	·	·	·	·	1	·	·	·

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A spin Richards formula

"Richards matrix"

∂		(6, 5, 3, 2, 1)	$(6, 5^2, 1)$	$(6, 5^2, 1)$	(7, 6, 3, 1)	(8, 6, 2, 1)	(10, 6, 1)
1	(6, 5, 3, 2, 1)	1	·	·	·	·	·
0	$(6, 5^2, 1)$ ♔	1	1	1	·	·	·
1	(7, 6, 3, 1) ♕	1	1	1	1	·	·
1	(8, 6, 2, 1)	·	1	1	1	1	·
0	(10, 6, 1)	·	1	·	·	1	1
2	(11, 3, 2, 1)	·	·	·	·	1	·
1	(11, 5, 1)	·	·	1	·	1	1
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0	(16, 1)	·	·	1	·	·	·

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1	(6, 5, 3, 2, 1)	1	·	·	·	·
0	$(6, 5^2, 1)$ ♔					
1	(7, 6, 3, 1) ♕	2	2	1	·	·
1	(8, 6, 2, 1)	1	2	1	1	·
0	(10, 6, 1)	1	1	·	1	1
2	(11, 3, 2, 1)	·	·	·	1	·
1	(11, 5, 1)	·	1	·	1	1
0	(11, 6)	·	·	·	·	1
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Now take all columns starting with ♔ or ending with ♕ and add together in adjacent pairs. Delete the ♔ row.

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	∂		$(6,5,3,2,1)$	$(6,5^2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$		$(6,5,3,2,1)$	$(6,5^2,1)$	$(7,6,3,1)$	$(8,6,2,1)$	$(10,6,1)$
	1	$(6,5,3,2,1)$	1	·	·	·	·	·		1	·	·	·	·
	0	$(6,5^2,1)$ ♔	1	1	1	·	·	·						
	1	$(7,6,3,1)$ ♕	1	1	1	1	·	·		2	2	1	·	·
	1	$(8,6,2,1)$	·	1	1	1	1	·		1	2	1	1	·
	0	$(10,6,1)$	·	1	·	·	1	1		1	1	·	1	1
	2	$(11,3,2,1)$	·	·	·	·	1	·		·	·	·	1	·
	1	$(11,5,1)$	·	·	1	·	1	1		·	1	·	1	1
	0	$(11,6)$	·	·	·	·	·	1		·	·	·	·	1
	0	$(16,1)$	·	·	1	·	·	·		·	1	·	·	·

Now take all columns starting with ♔ or ending with ♕ and add together in adjacent pairs. Delete the ♔ row.

Conjecture: This produces the decomposition matrix of B.

How to prove it

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Step 1: Prove it for the $d_{\lambda\mu}(q)$.

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Final question: what is the Richards matrix?

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Step 1: Prove it for the $d_{\lambda\mu}(q)$.
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Final question: what is the Richards matrix?

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & 2 & 1 & \cdot & \cdot \\ 1 & 2 & 1 & 1 & \cdot \\ 1 & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix} \times \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$