Schur-Weyl duality and dominant dimension

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Notation

Let k be a field and E be an n-dimensional k-vector space. The symmetric group Σ_r acts on the r-fold tensor product $E^{\otimes r}$ via:

$$(e_1 \otimes \cdots \otimes e_r)\sigma = e_{1\sigma} \otimes \cdots \otimes e_{r\sigma}, \qquad \forall\, e_1, \ldots, e_r \in E, \sigma \in \Sigma_r$$

The Schur algebra $S_k(n,r)$ is known to be $\mathrm{End}_{\Sigma_r}(E^{\otimes r})$, and the Schur functor $\mathcal{F}$ is:

$$\mathrm{Hom}_{S_k(n,r)}(E^{\otimes r}, -) : S_k(n,r)\text{-mod} \longrightarrow k\Sigma_r\text{-mod}$$

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Theorem (Schur-Weyl duality)

The canonical morphism $k\Sigma_r \to \mathrm{End}_{S_k(n,r)}(E^{\otimes r})$ is always an epimorphism of k-algebras, and is an isomorphism if $n \geq r$.

Motivation

Around 2000, Doty,Hemmer,Kleschev and Nakano compared the cohomologies of general linear and symmetric groups and obtained: If $n \geq r$ and $p = \mathrm{char}(k) > 0$, then the Schur functor $\mathcal{F}$ induces

$$\mathrm{Ext}^i_{S_k(n,r)}(M, N) \cong \mathrm{Ext}^i_{\Sigma_r}(\mathcal{F}(M), \mathcal{F}(N)) \qquad 0 \leq i \leq p-3$$

for any M and N with N filtered by dual Weyl modules.

Question: Is $p-3$ the best upper bound? What is its meaning?

In 2004, Doty,Erdmann and Nakano set up a general framework to study the cohomological behavior of general Schur functors: Given an idempotent e in a finite dimensional k-algebra A, how are the extension groups preserved under the Schur functor

$$\mathrm{Hom}_A(Ae, -) : A\text{-mod} \longrightarrow eAe\text{-mod}\,.$$

Question: Can we recover the result above for Schur algebras? What about other algebras?

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In 2001, Koenig-Slungård-Xi gave a computation free proof of the Schur-Weyl duality by showing that the Schur algebra $S_k(n, r)$ has dominant dimension at least two if $n \geq r$.

Question: What happens if the dominant dimension is large? What is the dominant dimension of $S_k(n, r)$?

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Remark. Each gendo-symmetric algebra A contains a unique (up to conjugate) idempotent $e \in A$ such that Ae is a projective, injective and faithful left A-module, and any other faithful left A modules must contain Ae as a direct summand. Therefore, we also denote by (A, e) a gendo-symmetric algebra.

Definition (Koenig-F 2010)

Let class $\mathcal{A}$ denote the set of all finite dimensional k-algebras A satisfying

(1) *A is quasi-hereditary;*

(2) *A is gendo-symmetric;*

(3) *A has an anti-involution.*

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Examples: (quantized) Schur algebras $S_k(n, r)$ for $n \geq r$ and their blocks; cyclotomic Schur algebras and block algebras of BGG category $\mathcal{O}$, etc.

Definition (Nakayama,Tachikawa)

The dominant dimension of a finite dimensional k*-algebra* A*, denoted by* domdim A*, is the largest number* t *or* ∞ *such that in a minimal injective resolution of* ${}_AA$

$$0 \to {}_AA \to I^0 \to I^1 \to \cdots$$

all injective modules $I^0, \ldots, I^{t-1}$ *are projective.*

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Examples: The k-algebra $k[x,y]/(x^2, xy, yx, y^2)$ has dominant dimension zero;

The algebra of upper triangular 2×2-matrices has dominant dimension one.

Self-injective algebras, and in particular semisimple algebras have dominant dimension ∞.

Dominant dimension encodes information about abundance of projective-injective modules:

a) If domdim $A \geq 2$, then A has a unique (up to conjugate) idempotent e such that Ae is projective, injective and faithful and any other faithful A modules contain Ae as a direct summand. Moreover,

$$\mathrm{End}_A(Ae) \cong eAe, \qquad \mathrm{End}_{eAe}(Ae) \cong A.$$

b) Nakayama conjecture: domdim $A = \infty$ if and only if A is self-injective.

Cohomological behavior of Schur functors

Theorem (Koenig-F, 2010)

Let (A, e) *be an algebra in the class* $\mathcal{A}$. *Then*

(1) $\operatorname{domdim} A$ *is a positive even number;*

(2) $\operatorname{domdim} A = \operatorname{domdim} R(A)$, *the Ringel dual of A;*

(3) *Standard A-modules are torsionless, i.e., are submodules of projective A-modules.*

(4) *The Schur functor* $eA \otimes_A - : A\text{-mod} \to eAe\text{-mod}$ *induces*

$$\operatorname{Ext}^i_A(M, N) \cong \operatorname{Ext}^i_{eAe}(eM, eN), \qquad 0 \leq i \leq \operatorname{domdim} A/2 - 2$$

for any M, N *with N filtered by standard A-modules, and the bound* $\operatorname{domdim} A/2 - 2$ *is optimal.*

Theorem (Miyachi-F)

For an algebra (A, e) in the class $\mathcal{A}$, the Schur functor induces:

$$\mathrm{HH}^i(A) \cong \mathrm{HH}^i(eAe), \qquad 0 \leq i \leq \text{domdim}\, A - 2$$

and the best upper bound is either domdim $A - 2$ *or* domdim $A - 1$*. Moreover, any homogeneous generator of the reduced Hochschild cohomology ring* $\overline{HH}^*(eAe)$ *has degree* 0 *or at least* domdim $A - 1$*.*

Dominant dimension when $n \geq r$

Theorem (Keonig-F, 2010)

If $n \geq r$ and $p = \mathrm{char}(k)$, then

$$\mathrm{domdim}\, S_k(n,r) = \begin{cases} 2(p-1), & 0 < p \leq r; \\ \infty, & \textit{else.} \end{cases}$$

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Theorem (Miyachi-F)

If $n \geq r$ and ℓ is the quantum characteristic, then

$$\text{domdim}\, S_q(n,r) = \begin{cases} 2(\ell-1), & \ell \leq r; \\ \infty, & \textit{else}. \end{cases}$$

Theorem (Hu,Miyachi,Koenig,F)

If $n \geq r$ and q is a root of one in the field k of characteristic p. Then for each block $B_{\tau,w}$ of $S_q(n,r)$,

$$\mathrm{gldim} B_{\tau,w} = 2(\ell w - d_{\ell,p}(\ell w)),$$

$$\text{domdim } B_{\tau,w} = \begin{cases} 2(\ell - 1), & w \neq 0; \\ \infty, & w = 0. \end{cases}$$

Dominant dimension when $n < r$

Theorem (Hu-F)

For all n and r, the endomorphism ring $\mathrm{End}_{S_k(n,r)}(E^{\otimes r})$ is a gendo-symmetric algebra, hence has dominant dimension at least two. For any partition λ and μ of r, the dimension of $\mathrm{Hom}_{S_k(n,r)}(S^\lambda E, D^\mu E)$ is independent of the characteristic of the ground field.

Theorem (F,2014)

For all n and r, there exists a $S_k(n,r)$-bimodule morphism

$$\Theta : A_k(n,r) \otimes A_k(n,r) \to A_k(n,r)$$

defined over $\mathbb{Z}$, which defines an associative multiplication on $A_k(n,r)$. Moreover, if $p = \mathrm{char}(k) > 0$, then the following are equivalent

(1) *Θ is an epimorphism;*

(2) *$A_k(n,r) = D_p(n,r)$;*

(3) *$r \leq n(p-1)$.*

In particular, if $r \leq n(p-1)$, then $S_k(n,r)$ is gendo-symmetric, hence has dominant dimension at least two, and given by

$$\text{domdim}\, S_k(n,r) = \max\{d \mid \mathrm{H}_i(\mathcal{B}_\bullet) = 0, 0 \leq i \leq d\} + 1$$

where $\mathcal{B}_\bullet$ is the bar complex associated to $A_k(n,r)$ and Θ.

Dominant dimension at least two

Theorem (Kerner-Yamagata-F,2017)

Let A be a finite dimensional k-algebra, and $\mathrm{D}(A)$ the k-dual of A. Let $V = \mathrm{Hom}_A(\mathrm{D}(A), A)$. Then $\mathrm{domdim}\, A \geq 2$ if and only if

$$\mathrm{D}(A) \otimes_A V \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$$

as A-bimodules, and A is an endomorphism algebra of a generator over some self-injective algebra if and only if $\mathrm{domdim}\, A \geq 2$ and $\mathrm{D}(A) \otimes_A V \cong V \otimes_A \mathrm{D}(A)$ as A-bimodules.

Characterization of dominant dimension

Theorem (Koenig-F, 2011,2016)

Let A be a finite dimensional k-algebra and $\mathrm{D}(A)$ be the k-dual of A. Then A is a gendo-symmetric algebra if and only if $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ as A-bimodules. Moreover let μ be the induced multiplication on $\mathrm{D}(A)$ and $\mathcal{C}_\bullet$ be the associated bar complex, then

$$\mathrm{domdim}\, A = \max\{d \mid \mathrm{H}_i(\mathcal{C}_\bullet) = 0, 0 \leq i \leq d\} + 1.$$

Derived invariance

Theorem (Hu-Koenig-F)

Let A be a k-algebra with an anti-automorphism fixing simples. Then for any algebra B derived equivalent to A,

$$\mathrm{gldim} B \geq \mathrm{gldim} A.$$

In particular, if both A and B have anti-automorphisms fixing simples, then $\mathrm{gldim} A = \mathrm{gldim} B$.
If furthermore, both A and B have dominant dimension at least one, then $\text{domdim}\, A = \text{domdim}\, B$.