Monoidal categories associated with strata of flag manifolds

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16 Dec 2017

This is a joint work with
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1.1. Quantum groups

- I := an index set
- $\mathrm{A} := (a_{ij})_{i,j\in I}$ (generalized Cartan matrix)
- $\mathrm{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (root lattice)
- $\mathrm{Q}^+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ (positive root lattice)
- P := weight lattice
- $\mathrm{P}^\vee$:= dual weight lattice
- $\Lambda_i \in \mathrm{P}^+$: fundamental weight (i.e., $\Lambda_i(h_j) = \delta_{ij}$)

Definition The quantum group $U_q(\mathfrak{g})$ associated with A is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by e_i, f_i ($i \in I$) and q^h ($h \in \mathrm{P}^\vee$) satisfying certain defining relations.

We consider the unipotent quantum coordinate ring

$$A_q(\mathfrak{n}) = \bigoplus_{\beta \in \mathsf{Q}_-} A_q(\mathfrak{n})_\beta, \quad \text{where } A_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q)).$$

Note $A_q(\mathfrak{n}) \simeq U_q^-(\mathfrak{g})$ as a $\mathbb{Q}(q)$-algebra.

For $\Lambda \in \mathsf{P}_+$ and $\mu, \zeta \in \mathsf{W}\Lambda$ with $\mu \preceq \zeta$,

$D(\mu, \zeta) :=$ unipotent quantum minor associated with μ, ζ

It was known that

- $D(\mu, \zeta)$ is a member of the upper global basis (dual canonical basis) of $A_q(\mathfrak{n})$.
- Let $w, v \in \mathsf{W}$ with $v \le w$. Then

$$D(w\Lambda, v\Lambda) D(w\Lambda', v\Lambda') = q^{-(v\Lambda, v\Lambda' - w\Lambda')} D(w(\Lambda+\Lambda'), v(\Lambda+\Lambda')).$$

- If $n := \langle h_i, \mu \rangle \geq 0$, then

$$\varepsilon_i(D(\mu,\zeta)) = 0 \text{ and } e_i^{(n)} D(s_i\mu,\zeta) = D(\mu,\zeta).$$

- If $\langle h_i, \mu \rangle \leq 0$ and $s_i\mu \preceq \zeta$, then $\varepsilon_i(D(\mu,\zeta)) = -\langle h_i, \mu \rangle$.
- If $m := -\langle h_i, \zeta \rangle \geq 0$, then

$$\varepsilon_i^*(D(\mu,\zeta)) = 0 \text{ and } e_i^{*(m)} D(\mu, s_i\zeta) = D(\mu,\zeta).$$

- If $\langle h_i, \zeta \rangle \geq 0$ and $\mu \preceq s_i\zeta$, then $\varepsilon_i^*(D(\mu,\zeta)) = \langle h_i, \zeta \rangle$.

1.2. Quiver Hecke algebras

For $\alpha \in \mathsf{Q}^+$ with $|\alpha| = m$, let

$$I^\alpha = \{\nu = (\nu_1, \dots, \nu_m) \in I^m \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_m} = \alpha\}$$

For $i, j \in I$, we take a homogeneous polynomial

$$\mathcal{Q}_{i,j}(u,v) = \begin{cases} \sum_{-2(\alpha_i|\alpha_j)-2\mathsf{d}_i p-2\mathsf{d}_j q=0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

such that $\mathcal{Q}_{i,j}(u,v) = \mathcal{Q}_{j,i}(v,u)$ and $t_{i,j;-a_{ij},0} \in \mathbf{k}_0^\times$.

Let $\alpha \in \mathrm{Q}^+$ with height m.

Definition The quiver Hecke algebra $R(\alpha)$ is the associative graded **k**-algebra generated by

$$e(\nu)\ (\nu \in I^\alpha),\ x_k\ (1 \le k \le m),\ \tau_t\ (1 \le t \le m-1)$$

satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu),\ \sum_{\nu \in I^\alpha} e(\nu) = 1,\ x_k e(\nu) = e(\nu)x_k,\ x_k x_l = x_l x_k,$$

$$\tau_t e(\nu) = e(s_t(\nu))\tau_t,\ \tau_t\tau_s = \tau_s\tau_t \text{ if } |t-s|>1,$$

$$\tau_t^2 e(\nu) = \begin{cases} 0 & \text{if } \nu_t = \nu_{t+1}, \\ \mathcal{Q}_{\nu_t,\nu_{t+1}}(x_t, x_{t+1})e(\nu) & \text{if } \nu_t \neq \nu_{t+1}, \end{cases}$$

$$(\tau_t x_k - x_{s_t(k)}\tau_t)e(\nu) = \begin{cases} -e(\nu) & \text{if } k = t \text{ and } \nu_t = \nu_{t+1}, \\ e(\nu) & \text{if } k = t+1 \text{ and } \nu_t = \nu_{t+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{t+1}\tau_t\tau_{t+1} - \tau_t\tau_{t+1}\tau_t)e(\nu)$$
$$= \begin{cases} \overline{\mathcal{Q}}_{\nu_t,\nu_{t+1}}(x_t, x_{t+1}, x_{t+2})e(\nu) & \text{if } \nu_t = \nu_{t+2} \neq \nu_{t+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\overline{\mathcal{Q}}_{i,j}(u, v, w) := \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u - w}.$$

The $\mathbb{Z}$-grading on $R(\alpha)$ is given by

$$\deg(e(\nu)) = 0, \ \deg(x_k e(\nu)) = (\alpha_{\nu_k}|\alpha_{\nu_k}), \ \deg(\tau_t e(\nu)) = -(\alpha_{\nu_t}|\alpha_{\nu_{t+1}}).$$

Convolution product

$$M \circ N := R(\beta+\beta')e(\beta,\beta') \otimes_{R(\beta)\otimes R(\beta')} (M \otimes N)$$

Functor E_i and F_i ($i \in I$)

$$E_i : R(\beta+\alpha_i)\text{-Mod} \to R(\beta)\text{-Mod}$$
$$F_i : R(\beta)\text{-Mod} \to R(\beta+\alpha_i)\text{-Mod}$$

defined by $E_i(N) = e(\alpha_i,\beta)N$ and $F_i(M) = R(\alpha_i) \circ M$.

We set

$R(\beta)$-proj := category of f. g. projective graded $R(\beta)$-modules

$R(\beta)$-mod := category of f. d. graded $R(\beta)$-modules

- [Khovanov-Lauda, Rouquier, Kang-Kashiwara, Webster]
  $[R\text{-proj}] \simeq U^-_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g}), \quad [R\text{-mod}] \simeq A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$
  $[R^\Lambda\text{-proj}] \simeq V_{\mathbb{Z}[q,q^{-1}]}(\Lambda), \quad [R^\Lambda\text{-mod}] \simeq V_{\mathbb{Z}[q,q^{-1}]}(\Lambda)^*$
- [Brundan-Kleshchev, Rouquier]
  In type A, $R^\Lambda(n) \simeq$ cyclotomic Hecke algebras.
- [Brundan, Hu, Kleshchev, Mathas, Ram, Wang, ...] In type A, Specht modules, graded cellular bases for R^Λ,
- [Ariki-P.-Speyer] Specht modules for type C,
- [Brundan, Kleshchev, McNamara, Ram, Tingley, Webster, ...] PBW bases theory, Convex orders, Cuspudal modules theory
- [Kang-Kashiwara-Kim-Oh] Quantum cluster algebras
- many results $\cdots$

Remark Quiver Hecke algebras are a vast generalization of Hecke algebras in the direction of categorification.

- [Khovanov-Lauda, Rouquier, Kang-Kashiwara, Webster]
  $[R\text{-proj}] \simeq U^-_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$, $[R\text{-mod}] \simeq A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$
  $[R^\Lambda\text{-proj}] \simeq V_{\mathbb{Z}[q,q^{-1}]}(\Lambda)$, $[R^\Lambda\text{-mod}] \simeq V_{\mathbb{Z}[q,q^{-1}]}(\Lambda)^*$
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1.3. Convex preorders

We review results in [Tingley-Webster, MV ploytopes and KLR algebras, Compos. Math. 152 (2016)].

Definition A face is a decomposition of a subset X of V into three disjoint subsets $X = A_- \sqcup A_0 \sqcup A_+$ such that

$$(\mathrm{span}_{\mathbb{R}_{\geq 0}} A_+ + \mathrm{span}_{\mathbb{R}} A_0) \cap \mathrm{span}_{\mathbb{R}_{\geq 0}} A_- = \{0\},$$
$$(\mathrm{span}_{\mathbb{R}_{\geq 0}} A_- + \mathrm{span}_{\mathbb{R}} A_0) \cap \mathrm{span}_{\mathbb{R}_{\geq 0}} A_+ = \{0\}.$$

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Definition Let X be a subset of $V \setminus \{0\}$.

(i) A convex preorder $\preceq$ on X is a total preorder on X such that, for any $\preceq$-equivalence class $\mathcal{C}$, the triple $(\{x \in X \mid x \prec \mathcal{C}\}, \mathcal{C}, \{x \in X \mid x \succ \mathcal{C}\})$ is a face.

(ii) A convex preorder $\preceq$ on X is called a convex order if every $\preceq$-equivalence class is of the form $X \cap l$ for some line l.

Note $\preceq$: a convex preorder on $X \subset V \setminus \{0\}$.

- If $\alpha, \beta, \gamma \in X$ with $\alpha + \beta = \gamma$ and $\alpha \prec \gamma$, then $\gamma \prec \beta$.
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We set

$$\underline{w} := s_{i_1} s_{i_2} \cdots s_{i_l} \in \mathrm{W},$$
$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \text{ for } k = 1, \ldots, l.$$

Then we have $\Delta_+ \cap w\Delta_- = \{\beta_1, \ldots, \beta_l\}$.

Proposition There is a convex preorder $\preceq$ on Δ_+ such that

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_l \prec \gamma$$

for any $\gamma \in \Delta_+ \cap w\Delta_+$.

Notation $\preceq^{\underline{w}}$:= a convex order which refines the above convex preorder

Example type A_3, $1 - 2 - 3$

- $\mathbf{j} = s_2 s_1 s_3 s_2 s_3 s_1 \in \mathsf{W}$

$$\alpha_3 \succ^{\mathbf{j}} \alpha_1 \succ^{\mathbf{j}} \alpha_1 + \alpha_2 + \alpha_3 \succ^{\mathbf{j}} \alpha_2 + \alpha_3 \succ^{\mathbf{j}} \alpha_1 + \alpha_2 \succ^{\mathbf{j}} \alpha_2$$

- $\mathbf{i} = s_3 s_2 s_1 s_3 s_2 s_3 \in \mathsf{W}$

$$\alpha_1 \succ^{\mathbf{i}} \alpha_1 + \alpha_2 \succ^{\mathbf{i}} \alpha_2 \succ^{\mathbf{i}} \alpha_1 + \alpha_2 + \alpha_3 \succ^{\mathbf{i}} \alpha_2 + \alpha_3 \succ^{\mathbf{i}} \alpha_3$$

2. Categories $\mathscr{C}_{w,v}$

We assume that A is arbitrary

Definition For $M \in R(\beta)\text{-}\mathrm{Mod}$, we define

$$\mathrm{W}(M) := \{\gamma \in \mathrm{Q}_+ \cap (\beta - \mathrm{Q}_+) \mid e(\gamma, \beta - \gamma)M \neq 0\},$$
$$\mathrm{W}^*(M) := \{\gamma \in \mathrm{Q}_+ \cap (\beta - \mathrm{Q}_+) \mid e(\beta - \gamma, \gamma)M \neq 0\}.$$

Then we have

- $\mathrm{W}^*(M) = \beta - \mathrm{W}(M)$
- For R-modules M and N,

$$\mathrm{W}(M \circ N) = \mathrm{W}(M) + \mathrm{W}(N), \quad \mathrm{W}^*(M \circ N) = \mathrm{W}^*(M) + \mathrm{W}^*(N).$$

We fix a convex order $\preceq$ on

$$\mathbb{Z}_{>0}\Delta_+ := \{k\beta \mid k \in \mathbb{Z}_{>0}, \beta \in \Delta_+\}.$$

Definition A simple $R(\beta)$-module L is $\preceq$-cuspidal if

(a) $\beta \in \mathbb{Z}_{>0}\Delta_+$,

(b) $\mathrm{W}(L) \subset \mathrm{span}_{\mathbb{R}_{\geq 0}}\{\gamma \in \Delta_+ \mid \gamma \preceq \beta\}$.

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It was shown in [Tingley-Webster] that, for a simple R-module L, there exists a unique sequence $(L_1, L_2, \dots, L_h)$ of $\preceq$-cuspidal modules (up to isomorphisms) such that

- $-\mathrm{wt}(L_1) \succ -\mathrm{wt}(L_2) \succ \cdots \succ -\mathrm{wt}(L_h)$,
- $L \simeq \mathrm{hd}(L_1 \circ L_2 \circ \cdots \circ L_h)$.

The sequence

$$\mathfrak{d}(L) := (L_1, L_2, \dots, L_h)$$

is called the $\preceq$-cuspidal decomposition of L.

Example type A_3, $1 - 2 - 3$

- $\mathbf{j} = s_2 s_1 s_3 s_2 s_3 s_1$

$$\alpha_3 \succ^{\mathbf{j}} \alpha_1 \succ^{\mathbf{j}} \alpha_1 + \alpha_2 + \alpha_3 \succ^{\mathbf{j}} \alpha_2 + \alpha_3 \succ^{\mathbf{j}} \alpha_1 + \alpha_2 \succ^{\mathbf{j}} \alpha_2$$

$\preceq^{\mathbf{j}}$-cuspidal modules corresponding to Δ_+

$L(3)$ $L(1)$ $L(213)$ $L(23)$ $L(21)$ $L(2)$

- $\mathbf{i} = s_3 s_2 s_1 s_3 s_2 s_3$

$$\alpha_1 \succ^{\mathbf{i}} \alpha_1 + \alpha_2 \succ^{\mathbf{i}} \alpha_2 \succ^{\mathbf{i}} \alpha_1 + \alpha_2 + \alpha_3 \succ^{\mathbf{i}} \alpha_2 + \alpha_3 \succ^{\mathbf{i}} \alpha_3$$

$\preceq^{\mathbf{i}}$-cuspidal modules corresponding to Δ_+

$L(1)$ $L(21)$ $L(2)$ $L(321)$ $L(32)$ $L(3)$

(Here, $L(ijk) := \tilde{f}_i \tilde{f}_j \tilde{f}_k 1 \in R\text{-mod}$)

Definition $w, v \in \mathsf{W}$.

- $\mathscr{C}_w :=$ full subcategory of R-mod whose objects M satisfy

$$\mathrm{W}(M) \subset \mathrm{span}_{\mathbb{R}_{\geq 0}}(\Delta_+ \cap w\Delta_-)$$

- $\mathscr{C}_{*,v} :=$ full subcategory of R-mod whose objects N satisfy

$$\mathrm{W}^*(N) \subset \mathrm{span}_{\mathbb{R}_{\geq 0}}(\Delta_+ \cap v\Delta_+)$$

- $\mathscr{C}_{w,v} = \mathscr{C}_w \cap \mathscr{C}_{*,v}$

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Remark $\mathscr{C}_w$, $\mathscr{C}_{*,v}$ and $\mathscr{C}_{w,v}$ are stable under taking subquotients, extensions, convolution products and grading shifts. In particular, $K_0(\mathscr{C}_w)$, $K_0(\mathscr{C}_{*,v})$ and $K_0(\mathscr{C}_{w,v})$ are $\mathbb{Z}[q, q^{-1}]$-algebras.

$\underline{w} := s_{i_1} s_{i_2} \cdots s_{i_\ell}$
$\preceq^{\underline{w}}$:= a convex order on Δ_+ associated with $\underline{w}$
L := a simple R-module with

$$\mathfrak{d}(L) := (L_1, L_2, \ldots, L_h), \quad \gamma_k := -\mathrm{wt}(L_k) \quad \text{for } k = 1, \ldots, h.$$

$$\beta_\ell := s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell})$$

Proposition

- $L \in \mathscr{C}_w \iff \beta_\ell \succeq \gamma_k$ for any k,
- $L \in \mathscr{C}_{*,w} \iff \gamma_k \succ \beta_\ell$ for any k.

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- $L \in \mathscr{C}_{*,w} \iff \gamma_k \succ \beta_\ell$ for any k.

Note $\quad K_0(\mathscr{C}_w) = A_q(\mathfrak{n}(w))_{\mathbb{Z}[q,q^{-1}]}$.

i.e. $\mathscr{C}_w =$ the monoidal category giving a monoidal categorification of $A_q(\mathfrak{n}(w))_{\mathbb{Z}[q,q^{-1}]}$ given by Kang-Kashiwara-Kim-Oh.

Example type A_3, $w = s_2 s_1 s_3 s_2 s_3$, $v = s_3 s_2$

- $\mathbf{j} = s_2 s_1 s_3 s_2 s_3 s_1$, $\beta_5 = s_2 s_1 s_3 s_2(\alpha_3) = \alpha_1$

$$\alpha_3 \succ^{\mathbf{j}} \alpha_1 \succ^{\mathbf{j}} \alpha_1 + \alpha_2 + \alpha_3 \succ^{\mathbf{j}} \alpha_2 + \alpha_3 \succ^{\mathbf{j}} \alpha_1 + \alpha_2 \succ^{\mathbf{j}} \alpha_2$$

$$L \in \mathscr{C}_w \iff \mathfrak{d}(L) = (L(1)^{\circ t_1}, L(213)^{\circ t_2}, L(23)^{\circ t_3}, L(21)^{\circ t_4}, L(2)^{\circ t_5})$$

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- $\mathbf{i} = s_3 s_2 s_1 s_3 s_2 s_3$ $\beta_2 = s_3(\alpha_2) = \alpha_2 + \alpha_3$

$$\alpha_1 \succ^{\mathbf{i}} \alpha_1 + \alpha_2 \succ^{\mathbf{i}} \alpha_2 \succ^{\mathbf{i}} \alpha_1 + \alpha_2 + \alpha_3 \succ^{\mathbf{i}} \alpha_2 + \alpha_3 \succ^{\mathbf{i}} \alpha_3$$

$$L \in \mathscr{C}_{*,v} \iff \mathfrak{d}(L) = (L(1)^{\circ t_1}, L(21)^{\circ t_2}, L(2)^{\circ t_3}, L(321)^{\circ t_4})$$

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- $\mathscr{C}_{w,v} = \mathscr{C}_w \cap \mathscr{C}_{*,v}$

We revisit the unipotent quantum coordinate ring $A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$.

Definition

- A_w := the $\mathbb{Z}[q,q^{-1}]$-linear subspace of $A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ spanned by $x \in A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ such that

$$e_{i_1} \cdots e_{i_l} x = 0$$

  for any sequence $(i_1, \ldots, i_l) \in I^\beta$ with $\beta \in \mathsf{Q}_+ \cap w\mathsf{Q}_+ \setminus \{0\}$,
- $A_{*,v}$:= the $\mathbb{Z}[q,q^{-1}]$-linear subspace of $A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ spanned by $x \in A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ such that

$$e^*_{i_1} \cdots e^*_{i_l} x = 0$$

  for any sequence $(i_1, \ldots, i_l) \in I^\beta$ with $\beta \in \mathsf{Q}_+ \cap v\mathsf{Q}_- \setminus \{0\}$,
- $A_{w,v} := A_w \cap A_{*,v} \subset A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$

Remark

G : reductive group over $\mathbb{C}$
N and N^- : unipotent radicals of B and B^-
$\mathfrak{n}$ and $\mathfrak{n}^-$: Lie algebras of N and N^-
$w, v \in \mathsf{W}$

Note that $\mathbb{C}[N]$ is isomorphic to the dual of $U(\mathfrak{n})$. We set

$$N'(w) = N \cap (wNw^{-1}) \text{ and } N(v) := N \cap (vN^-v^{-1}).$$

Then the doubly-invariant algebra

$$\begin{aligned} {}^{N'(w)}\mathbb{C}[N]^{N(v)} &= \{f \mid f(nxm) = f(x), x \in N, m \in N'(w), n \in N(v)\} \\ &= \{f \mid U(\mathfrak{n})_\beta\, f\, U(\mathfrak{n})_\gamma = 0, \\ &\qquad \text{for all } \beta \in \mathsf{Q}_+ \cap w\mathsf{Q}_+ \setminus \{0\}, \gamma \in \mathsf{Q}_+ \cap v\mathsf{Q}_- \setminus \{0\}\} \end{aligned}$$

Therefore, $A_{w,v}$ = a quantum deformation of ${}^{N'(w)}\mathbb{C}[N]^{N(v)}$.

Theorem Under the categorification, for $w, v \in \mathsf{W}$, we have

- $K_0(\mathscr{C}_w) = A_w$,
- $K_0(\mathscr{C}_{*,v}) = A_{*,v}$,
- $K_0(\mathscr{C}_{w,v}) = A_{w,v}$,

(Idea of Proof) Key properties

Cuspidal decomposition $\rightsquigarrow$ unmixed

$M \in \mathscr{C}_w \iff \mathsf{W}(M) \cap \Delta_+ \subset \Delta_+ \cap w\Delta_-$

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(Idea of Proof) Key properties

Cuspidal decomposition $\rightsquigarrow$ unmixed

$M \in \mathscr{C}_w \iff \mathsf{W}(M) \cap \Delta_+ \subset \Delta_+ \cap w\Delta_-$

Corollary The subspaces A_w, $A_{*,v}$ and $A_{w,v}$ are subalgebras of $A_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$.

3. Determinantial modules

We assume that A is arbitrary. We take

$\Lambda \in \mathrm{P}_+$
$w, v \in \mathrm{W}$ with $w \geq v$
$\underline{w} = s_{i_1} \cdots s_{i_p}$ and $\underline{v} = s_{j_1} \cdots s_{j_q}$

Definition Determinantial modules

$$\mathrm{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_p}^{m_p} 1, \quad \text{where } m_k = \langle h_k, s_{i_{k+1}} \cdots s_{i_p} \Lambda \rangle,$$
$$\mathrm{M}(w\Lambda, v\Lambda) := \widetilde{E}_{j_1}^{*\,\max} \cdots \widetilde{E}_{j_q}^{*\,\max} \mathrm{M}(w\Lambda, \Lambda).$$

Note that

- $\mathrm{M}(w\Lambda, v\Lambda)$ is a self-dual simple module.
- It does not depend on the choice of reduced expressions.

Let $\Lambda, \Lambda' \in \mathsf{P}_+$, $w, v \in \mathsf{W}$ with $w \geq v$ and $\lambda, \mu \in \mathsf{W}\Lambda$ with $\lambda \preceq \mu$.

Proposition

- $[\mathsf{M}(\lambda, \mu)] =$ unipotent quantum minor $D(\lambda, \mu)$. Thus, it's in the upper global basis (dual canonical basis).
- $\mathsf{M}(w\Lambda, v\Lambda) \circ \mathsf{M}(w\Lambda', v\Lambda') \simeq q^{-(v\Lambda, v\Lambda' - w\Lambda')} \mathsf{M}(w(\Lambda + \Lambda'), v(\Lambda + \Lambda'))$ Thus, $\mathsf{M}(\lambda, \mu)$ is real.
- Let $\lambda, \lambda', \lambda'' \in \mathsf{W}\Lambda$ with $\lambda \preceq \lambda' \preceq \lambda''$. Then there is an epimorphism

$$\mathsf{M}(\lambda, \lambda') \circ \mathsf{M}(\lambda', \lambda'') \twoheadrightarrow \mathsf{M}(\lambda, \lambda'').$$

Lemma $\Lambda \in \mathsf{P}_+$, $\lambda, \mu \in \mathsf{W}\Lambda$ with $\lambda \preceq \mu$
$u_\lambda :=$ the extremal weight vector of weight λ in $V(\Lambda)$.

- If $\beta \in \mathrm{W}(\mathrm{M}(\lambda, \mu))$, then

$$\lambda + \beta \in \mathrm{wt}(U_q^+(\mathfrak{g})u_\lambda) \subset \mathrm{wt}(V(\Lambda)).$$

- If $\gamma \in \mathrm{W}^*(\mathrm{M}(\lambda, \mu))$, then

$$\mu - \gamma \in \mathrm{wt}(U_q^+(\mathfrak{g})u_\lambda) \subset \mathrm{wt}(V(\Lambda)).$$

Note $U_q^+(\mathfrak{g})u_\lambda =$ Demazure module associated with λ in $V(\Lambda)$

$w, v \in \mathsf{W}$ with $v \le w$

Fix $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_l}$ of $w \in \mathsf{W}$. For $i = 1, \ldots, l$, we set

$$w_{\le k} := s_{i_1} \cdots s_{i_k}, \qquad w_{\ge k+1} := s_{i_{k+1}} \cdots s_{i_l}.$$

We also define

$$\text{(i) } v_{\ge k} = (v_{\le k-1})^{-1} v,$$

$$\text{(ii) } v_{\le k} = \begin{cases} v_{\le k-1} s_{i_k} & \text{if } s_{i_k} v_{\ge k} < v_{\ge k}, \\ v_{\le k-1} & \text{if } s_{i_k} v_{\ge k} > v_{\ge k}. \end{cases}$$

Here we set $w_{\le 0} = v_{\le 0} = \mathrm{id} \in \mathsf{W}$.

Note

- $w_{\le l} = w$ and $v_{\le l} = v$
- $\ell(v) = \ell(v_{\le k-1}) + \ell(v_{\ge k})$
- $v_{\le k} \le w_{\le k}$ and $v_{\ge k} \le w_{\ge k}$

$\Lambda, \Lambda' \in \mathrm{P}_+$, $w, v \in \mathrm{W}$ with $v \le w$, $\underline{w} := s_{i_1} \cdots s_{i_l}$.

Theorem For $k = 0, 1, \ldots, l$,

$$\mathrm{M}(w_{\le k}\Lambda, v_{\le k}\Lambda) \in \mathscr{C}_{w,v}$$

(Idea of Proof) properties of wt(Demazure module)

$\Lambda, \Lambda' \in \mathrm{P}_+$, $w, v \in \mathrm{W}$ with $v \leq w$, $\underline{w} := s_{i_1} \cdots s_{i_l}$.

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(Idea of Proof) properties of wt(Demazure module)

Theorem Suppose that A is symmetric and R is symmetric.

- For $1 \leq k \leq j \leq l$,

$$\mathrm{M}(w_{\leq j}\Lambda, v_{\leq j}\Lambda) \circ \mathrm{M}(w_{\leq k}\Lambda', v_{\leq k}\Lambda') \text{ is simple,}$$

- For $1 \leq k \leq j \leq l$,

$$\Lambda(\mathrm{M}(w_{\leq k}\Lambda, v_{\leq k}\Lambda), \mathrm{M}(w_{\leq j}\Lambda', v_{\leq j}\Lambda')) = (w_{\leq k}\Lambda + v_{\leq k}\Lambda, v_{\leq j}\Lambda' - w_{\leq j}\Lambda'),$$

where $\Lambda(M, N) =$ degree of the R-matrix : $M \circ N \to N \circ M$.

(Idea of Proof) properties of R-matrices

Remark Results on cluster algebras in
[Leclerc, Cluster str. on strata of flag varieties, Adv. Math., 2016]

G: *ADE* type, $w, v \in \mathsf{W}$ with $v \le w$, $\underline{w} = s_{i_1} \cdots s_{i_l}$.

$R_{w,v}$ = Schubert cell $C_w \cap$ opposite Schubert cell C^v

- $\mathbb{C}[R_{w,v}] \simeq$ a certain localization of ${}^{N'(w)}\mathbb{C}[N]^{N(v)}$
- $^{\exists}$ Frobenious subcategory $C_{w,v}$ of $\mathrm{mod}(\Lambda)$ such that $\mathrm{Span}_{\mathbb{C}}\{\varphi_M \mid M \in C_{w,v}\} = {}^{N'(w)}\mathbb{C}[N]^{N(v)}$ where $\varphi_M \in \mathbb{C}[N]$: cluster character of M
- $^{\exists}$ subalgebra of ${}^{N'(w)}\mathbb{C}[N]^{N(v)}$ having a cluster structure with an initial cluster coming from $D(w_{\le k}(\varpi_{k+1}), v_{\le k}(\varpi_{k+1}))$.

Note A quantization of $\mathbb{C}[R_{w,v}]$ was studied by Lenagan-Yakimov in the aspect of quantum cluster algs.

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Conjecture $\mathscr{C}_{w,v}$ gives a monoidal categorification of a quantization of the cluster algebra arising from $R_{w,v}$.

4. Finite *ADE* types

We assume that A is of finite *ADE* type and **k** is of characteristic 0.

It was known that

- any quiver Hecke algebra is isomorphic to a symmetric quiver Hecke algebra.
- [Rouquier, Varagnolo-Vasserot] simple R-modules correspond to the upper global basis (dual canonical basis).

We define

$_iR\text{-mod} :=$ the full subcategory of $R\text{-mod}$ whose objects M satisfy $\varepsilon_i(M) = 0$

$R_i\text{-mod} :=$ the full subcategory of $R\text{-mod}$ whose objects M satisfy $\varepsilon_i^*(M) = 0$

It was proved by S. Kato that there exist reflection functors

$$\mathcal{T}_i : R_i(\beta)\text{-mod} \xrightarrow{\sim} {}_iR(s_i\beta)\text{-mod},$$
$$\mathcal{T}_i^* : {}_iR(\beta)\text{-mod} \xrightarrow{\sim} R_i(s_i\beta)\text{-mod},$$

which are equivalences of categories.

The functors $\mathcal{T}_i$ and $\mathcal{T}_i^*$ are counterparts of the Saito crystal reflections

$$T_i : B_i(\infty) \longrightarrow {}_iB(\infty), \qquad b \mapsto \tilde{f}_i^{*\,\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)} b,$$
$$T_i^* : {}_iB(\infty) \longrightarrow B_i(\infty), \qquad b \mapsto \tilde{f}_i^{\varphi_i^*(b)} \tilde{e}_i^{*\,\varepsilon_i^*(b)} b.$$

Note $\mathcal{T}_i(L(b)) \simeq L(T_i(b))$ and $\mathcal{T}_i^*(L(b')) \simeq L(T_i^*(b'))$
where $L(b)$ is the simple R-module corresponding to b.

$\underline{w_0} := s_{i_1} \cdots s_{i_\ell}$ of the longest $w_0 \in W$,
$\preceq^{\underline{w_0}}$:= a convex order corresponding to $\underline{w_0}$
L_k := $\preceq^{\underline{w_0}}$-cuspidal module cor. to $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$

Then, we have

$$L_k \simeq L(T^*_{i_1} T^*_{i_2} \cdots T^*_{i_{k-1}}(f_{i_k})) \simeq \mathcal{T}^*_{i_1} \mathcal{T}^*_{i_2} \cdots \mathcal{T}^*_{i_{k-1}} L(i_k)$$

Lemma Let $i \in I$, $w \in$ W, and let M be a simple R-module.

(i) Suppose that $M \in \mathscr{C}_w$. Then we have
  (a) if $s_i w < w$ and $\varepsilon^*_i(M) = 0$, then $\mathcal{T}_i(M) \in \mathscr{C}_{s_i w}$,
  (b) if $s_i w > w$, then $\mathcal{T}^*_i(M) \in \mathscr{C}_{s_i w}$.

(ii) Suppose that $M \in \mathscr{C}_{*,w}$. Then we have
  (a) if $s_i w < w$, then $\mathcal{T}_i(M) \in \mathscr{C}_{*,s_i w}$,
  (b) if $s_i w > w$ and $\varepsilon_i(M) = 0$, then $\mathcal{T}^*_i(M) \in \mathscr{C}_{*,s_i w}$.

Theorem Let $w, v \in \mathsf{W}$ with $v \le w$, $s_i w > w$ and $s_i v > v$. Then the restrictions of the functors

$$\mathcal{T}_i|_{\mathscr{C}_{s_i w, s_i v}} : \mathscr{C}_{s_i w, s_i v} \xrightarrow{\sim} \mathscr{C}_{w,v}, \qquad \mathcal{T}_i^*|_{\mathscr{C}_{w,v}} : \mathscr{C}_{w,v} \xrightarrow{\sim} \mathscr{C}_{s_i w, s_i v}$$

give equivalences of the categories.

Theorem Let $w, v \in \mathrm{W}$ with $v \le w$, $s_i w > w$ and $s_i v > v$. Then the restrictions of the functors

$$\mathcal{T}_i|_{\mathscr{C}_{s_iw,s_iv}} : \mathscr{C}_{s_iw,s_iv} \xrightarrow{\sim} \mathscr{C}_{w,v}, \qquad \mathcal{T}_i^*|_{\mathscr{C}_{w,v}} : \mathscr{C}_{w,v} \xrightarrow{\sim} \mathscr{C}_{s_iw,s_iv}$$

give equivalences of the categories.

Corollary Let $w, u, v \in \mathrm{W}$ with $w = vu$, $\ell(w) = \ell(v) + \ell(u)$. Then there is an equivalence of the categories

$$\mathscr{C}_{w,v} \simeq \mathscr{C}_u.$$

Note [Kang-Kashiwara-Kim-Oh]
$\mathscr{C}_u$ gives a monoidal categorification of $A_q(\mathfrak{n}(u))_{\mathbb{Z}[q,q^{-1}]}$.

Conjecture $\mathscr{C}_{w,v}$ gives a monoidal categorification of $A_{w,v}$.

Note If $\mathcal{T}_i$ and $\mathcal{T}_i^*$ are monoidal functors, then it implies that the conjecture is true.

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- S. Kato extended the reflection functors $\mathcal{T}_i$ and $\mathcal{T}_i^*$ to symmetric cases and proved the monoidality (arXiv:1711.09085).
- P. McNamara also proved the monoidality independently (arXiv:1712.00173).

Thus, the conjecture is proved. Thanks!!

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Further Study (work in progress)

We expect that $\mathscr{C}_{w,v}$ gives a monoidal categorification of a quantization of the cluster algebra arising from $R_{w,v}$

THANK YOU