On block algebras of Hecke algebras of classical type

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Hecke algebras

Let K be an algebraically closed field of odd characteristic p. We fix $1 \neq q \in K^\times$ and consider the Hecke algebra $H_W(q)$ associated with an irreducible Weyl group W. It is the K-algebra generated by $\{T_s \mid s \in S\}$ and the generators are subject to the braid relations and the quadratic relations

$$(T_s - q)(T_s + 1) = 0 \quad (s \in S)$$

Theorem 1.1 (Geck 2007)

Suppose further that $p \neq 2, 3$ for $W(E_{6,7})$, $W(F_4)$, $W(G_2)$, $p \neq 2, 3, 5$ for $W(E_8)$. Then $H_W(q)$ and its block algebras are cellular algebras.

In this talk, we focus on classical types A, B, D. For type B, we consider the unequal parameter case as well. Let $Q \in K^\times$ be another parameter and replace the quadratic relation for the generator T_0 corresponding to the short simple root with $(T_0 - Q)(T_0 + 1) = 0$.

Parametrization of blocks in type A

Let e be the quantum characteristic, the multiplicative order of $q \neq 1$.

Theorem 1.2 (James-Mathas 1997, based on Dipper-James 1987)

If $W = W(A_{n-1}) = S_n$, then blocks of $H_W(q)$ are labeled by

$$\{e\text{-core partition } \kappa \mid \mathrm{wt}(\kappa) := \frac{n - |\kappa|}{e} \in \mathbb{Z}_{\geq 0}\}.$$

We may view this result as follows. Let $\mathfrak{g} = \mathfrak{g}(A^{(1)}_{e-1})$ be the Kac-Moody Lie algebra of type $A^{(1)}_{e-1}$, P its weight lattice,

$$V(\Lambda_0) = \oplus_{\mu \in P} V(\Lambda_0)_\mu$$

the weight decomposition of the basic module. Here, the weight Λ_0 is the fundamental weight corresponding to the node 0 added the finite Dynkin diagram to obtain the affine Dynkin diagram.

Parametrization of blocks in type A (cont'd)

Then, $V(\Lambda_0)$ is realized on the vector space with basis consisting of e-restricted partitions and the labeling set may be interpreted as

$$\{\mu \in P \mid V(\Lambda_0)_\mu \neq 0, \mathrm{ht}(\Lambda_0 - \mu) = n\}.$$

Note that $\Lambda_0 - \mu$ is a nonnegative integral linear combination of simple roots and the height is the sum of the coefficients.

Remark 1.3

The set of e-core partitions in this interpretation is the affine Weyl group orbit $W(A^{(1)}_{e-1})\Lambda_0(\subseteq P)$ through Λ_0. If $\mu \in P$ satisfies $V(\Lambda_0)_\mu \neq 0$ then there exists $\kappa \in W(A^{(1)}_{e-1})\Lambda_0$ such that

$$\mu = \kappa - \mathrm{wt}(\kappa)\delta,$$

where δ is the null root, which is the sum of simple roots in type $A^{(1)}_{e-1}$.

Parametrization of blocks in type B

We denote the Hecke algebra in the unequal parameter case by $H_n(q,Q)$. It is generated by T_0 and $T_1,\dots,T_{n-1}$, and the quadratic relations are

$$(T_0-Q)(T_0+1)=0, \quad (T_i-q)(T_i+1)=0 \ (1\le i\le n-1).$$

The braid relations are $(T_0T_1)^2=(T_1T_0)^2$, $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$, for $1\le i\le n-2$, and $T_iT_j=T_jT_i$, for $j\ne i\pm 1$.

Theorem 1.4 (Dipper-James, 1992)

If $-Q\notin q^{\mathbb{Z}}$, then $H_n(q,Q)$ is Morita equivalent to

$$\bigoplus_{k=0}^{n} H_{S_k}(q)\boxtimes H_{S_{n-k}}(q).$$

In particular, blocks of $H_n(q,Q)$ are labeled by pairs of e-cores or

$$\{(\mu,\nu)\in P\oplus P \mid V(\Lambda_0)\mu\ne 0, V(\Lambda_0)_\nu\ne 0, \mathrm{ht}(\Lambda_0-\mu)+\mathrm{ht}(\Lambda_0-\nu)=n\}.$$

Parametrization of blocks in type B (cnt'd) and type D

Theorem 1.5 (Lyle-Mathas, 2007)

If $-Q = q^s$, *for* $0 \le s \le e-1$, *then blocks of* $H_n(q,Q)$ *are labeled by*

$$\{\mu \in P \mid V(\Lambda_0+\Lambda_s)_\mu \neq 0, \mathrm{ht}(\Lambda_0+\Lambda_s-\mu)=n\}.$$

For $W = W(D_n)$, we consider $H_n(q,1)$ and the commuting involutions

$$\sigma : T_0 \mapsto -T_0, \quad T_i \mapsto T_i \ (i \neq 0),$$
$$\tau : T_1 \mapsto T_0 T_1 T_0, \quad T_i \mapsto T_i \ (i \neq 1).$$

Then, $H_W(q) = H_n(q,1)^\sigma$ and, for a block B of $H_n(q,1)$, we have either

(i) B^σ is a single block b of $H_W(q)$, or

(ii) B^σ is the direct sum of two blocks b' and b'' of $H_W(q)$.

We have $\tau(b) = b$ in case (i) and τ swaps b' and b'' in case (ii). We say that B covers b, b' and b''.

Representation type

Representation type is a concept to tell the possibility of classifying indecomposable objects. Let A be a finite dimensional algebra over an algebraically closed field K.

- If there are finitely many indecomposable A-modules, we say that A is finite. Otherwise, A has infinite representation type.
- Suppose that A has infinite representation type. If there are finitely many $(A, K[T])$-modules $M_{d,1}, M_{d,2}, \ldots, M_{d,n_d}$ which are free of finite rank over $K[T]$, for each $d \in \mathbb{N}$, such that all d-dimensional indecomposable A-modules but finitely many exceptions are of the form $M_{d,i} \otimes K[T]/(T-\lambda)$, we say that A is (infinite-)tame.
- If there is an $(A, K\langle X, Y\rangle)$-module which is free of finite rank over $K\langle X, Y\rangle$ such that the exact functor $M \otimes_{K\langle X,Y\rangle} -$ respects indecomposability and isomorphism classes, we say that A is wild.

Trichotomy theorem

The following is the famous trichotomy theorem.

Theorem 2.1 (Drozd, 1979, Crawley-Bowvey, 1988)

Over an algebraically closed field, every finite dimensional algebra has finite, (infinite-)tame or wild representation type, and these types are mutually exclusive.

Example 2.2

Let $A = KQ$ be the path algebra of an acyclic quiver Q.

- If $Q = A_n, D_n, E_{6,7,8}$ then A is finite.
- If $Q = A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$ then A is tame.

Example 2.3

For a block algebra of KG, for a finite group G, the representation type is determined by its defect group.

Brauer tree algebras

Let $G = (V, E)$ be a finite graph, where V is the set of vertices and E is the set of undirected edges. We assume that there are no loops and no multiple edges. It is called a tree if $|V| = |E| + 1$. A Brauer tree is a tree with the following additional data:

(i) For each $v \in V$, a cyclic ordering is given on the set

$$\{e \in E \mid v \text{ is an endpoint of } e\}.$$

(ii) There is at most one vertex which is assigned an integer $m \geq 2$.

The vertex from (ii) is called the exceptional vertex and the integer m is called its multiplicity. Each Brauer graph gives rise a Morita equivalence class of symmetric algebras, where simple modules are labeled by E and the data (i) and (ii) determine the radical series of indecomposable projective modules.

Main result (corollary)

Let T be a Brauer tree without exceptional vertex. If the underlying tree is the straight line with n edges, the corresponding Brauer tree algebra has n simple modules $S_1, \ldots, S_n$, say. Let P_i be the projective cover of S_i. Then, the following shape of P_i, for $1 \le i \le n$, characterizes the algebra.

(a) $\mathrm{Rad}(P_1)/\mathrm{Soc}(P_1) = 0$ if $n = 1$.

(b) $\mathrm{Rad}(P_1)/\mathrm{Soc}(P_1) = S_2$ and $\mathrm{Rad}(P_n)/\mathrm{Soc}(P_n) = S_{n-1}$ if $n \ge 2$.

(c) $\mathrm{Rad}(P_i)/\mathrm{Soc}(P_i) = S_{i-1} \oplus S_{i+1}$, for $2 \le i \le n-1$, if $n \ge 3$.

Theorem 2.4 (A, 2017)

Let B be a block of a Hecke algebra of classical type over an algebraically closed field of odd characteristic. If B is finite then it is a Brauer tree algebra whose Brauer tree is a straight line without exceptional vertex.

The proof starts with determining representation type of Hecke algebras of classical type blockwise, which I am going to explain.

Representation type in type A and D

Theorem 2.5 (Erdmann-Nakano, 2002)

Suppose that $W = S_n$ and let B_κ be the block labeled by an e-core κ. Then, B_κ is

(i) *simple if and only if* $\mathrm{wt}(\kappa) = 0$.

(ii) *not semisimple and finite if and only if* $\mathrm{wt}(\kappa) = 1$.

(iv) *(infinite-)tame if and only if* $e = 2$ *and* $\mathrm{wt}(\kappa) = 2$.

(v) *wild otherwise.*

Further, if B_κ has finite type then it is a Brauer tree algebra whose tree is a straight line without exceptional vertex. For type D, we may reduce the question to type B.

Proposition 2.6

Suppose that $W = W(D_n)$ and b is a block of $H_W(q)$. If b is covered by a block B of $H_n(q, 1)$, then b and B have the same representation type.

Representation type in type B

It is easy to show the following proposition.

Proposition 2.7

Suppose that $-Q \notin q^{\mathbb{Z}}$ and let $B_{\kappa,\rho}$ be the block labeled by a pair of e-cores κ and ρ. Then, $B_{\kappa,\rho}$ is

(i) *simple if and only if $\mathrm{wt}(\kappa) = 0$ and $\mathrm{wt}(\rho) = 0$.*

(ii) *not semisimple and finite if and only if $\{\mathrm{wt}(\kappa), \mathrm{wt}(\rho)\} = \{0, 1\}$.*

(iv) *(infinite-)tame if and only if $e = 2$ and $\mathrm{wt}(\kappa) = 1$, $\mathrm{wt}(\rho) = 1$.*

(v) *wild otherwise.*

The crucial case is the remaining case that $-Q = q^s$, for $0 \le s \le e - 1$. Let $B(\beta)$ be the block of $H_n(q, Q)$ labeled by $\mu = \Lambda_0 + \Lambda_s - \beta \in P$. Here, β is a sum of n simple roots.

Theorems of Rickard and Krause

As the blocks of $H_n(q, Q)$ are symmetric algebras, the theorems below reduce the problem of determining the representation type to that for representatives of derived equivalence classes of blocks.

Theorem 2.8 (Rickard, 1989)

Let A and B be two finite dimensional selfinjective algebras over K. If they are derived equivalent then they are stably equivalent.

Theorem 2.9 (Krause, 1997)

Let A and B be two finite dimensional algebras over K. If they are stably equivalent then they have the same representation type.

These general results may be applied to Hecke algebras by the sl_2 categorification theorem by Chuang and Rouquier.

Theorem by Chuang and Rouquier

Chuang and Rouquier proved the sl_2 categorification theorem, and applied it to cyclotomic Hecke algebras. In particular, it gives derived equivalence among blocks of $H_n(q, Q)$, if $-Q \in q^{\mathbb{Z}}$.

Let $-Q = q^s$ and $\Lambda = \Lambda_0 + \Lambda_s$ as before, and we denote by $B(\beta)$ the block of $H_n(q, Q)$ labeled by $\Lambda - \beta \in P$.

Theorem 2.10 (Chuang-Rouquier, 2008)

Let $W = \langle s_0, s_1, \ldots, s_{e-1} \rangle$ be the affine symmetric group. If

$$\Lambda - \gamma = w(\Lambda - \beta),$$

for some $w \in W$, then $B(\beta)$ and $B(\gamma)$ are derived equivalent.

Hence we look at representatives of W-orbits on the set of weights of $V(\Lambda)$, which is a combinatorial problem.

Derived equivalence representatives

Let $\Lambda = \Lambda_0 + \Lambda_s$ as before. We set $\lambda_0^s = 0$ and define

$$\lambda_i^s = \sum_{k=0}^{s} i\alpha_k + \sum_{k=1}^{i-1}(i-k)\alpha_{s+k} + \sum_{k=1}^{i-1} k\alpha_{\ell-i+1+k},$$
$$\mu_i^s = \sum_{k=0}^{i-1}(i-k)\alpha_k + \sum_{k=1}^{i-1} k\alpha_{s-i+k} + \sum_{k=1}^{\ell-s+1} i\alpha_{s-1+k}.$$

The null root is $\delta = \alpha_0 + \cdots + \alpha_{e-1}$. Then we have the following.

Lemma 2.11

Representatives under the Chuang-Rouquier derived equivalence are $B(\beta)$ such that β are of the following form.

$$\lambda_i^s + k\delta \;\; (0 \le i \le \frac{e-s}{2}, k \ge 0), \;\; \mu_i^s + k\delta \;\; (1 \le i \le \frac{s}{2}, k \ge 0).$$

$\lambda_i^s + k\delta$ is enough

Lemma 2.12

We have the following isomorphism of algebras

$$B(\mu_i^s + k\delta) \simeq B(\lambda_i^{\ell-s+1} + k\delta), \quad \textit{for } 1 \le i \le \frac{s}{2} \textit{ and } k \in \mathbb{Z}_{\ge 0}.$$

For the proof, we switch to cyclotomic quiver Hecke algebras. Recall that the Cartan matrix of Lie type $A_{e-1}^{(1)}$ is

$$A = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

Cyclotomic quiver Hecke algebras

The cyclotomic quiver Hecke algebras $R^{\Lambda}(\beta)$ are defined by generators

- $e(\nu)$, for $\nu=(\nu_1,\ldots,\nu_n)\in I^n$ with $\sum_{k=1}^n \alpha_{\nu_k}=\beta$
- $x_1,\ldots,x_n$, $\psi_1,\ldots,\psi_{n-1}$

and the following relations, where $s_k\nu=(\nu_1,\ldots,\nu_{k+1},\nu_k,\ldots,\nu_n)$.

$$e(\nu)e(\nu')=\delta_{\nu,\nu'}e(\nu),\ \ \sum_{\nu\in I^n} e(\nu)=1$$

$$x_k e(\nu)=e(\nu)x_k,\ \ x_k x_l=x_l x_k$$

$$\psi_l e(\nu)=e(s_l(\nu))\psi_l,\ \ \psi_k\psi_l=\psi_l\psi_k \text{ if } |k-l|>1$$

$$(\psi_k x_l - x_{s_k(l)}\psi_k)e(\nu)=\begin{cases} -e(\nu) & \text{if } l=k \text{ and } \nu_k=\nu_{k+1}\\ e(\nu) & \text{if } l=k+1 \text{ and } \nu_k=\nu_{k+1}\\ 0 & \text{otherwise}\end{cases}$$

$$x_1^{\langle \alpha_{\nu_1}^{\vee},\Lambda\rangle}e(\nu)=0$$

Cyclotomic quiver Hecke algebras (cont'd)

and if we define $Q_{ii}(u,v)=0$, and if $i\neq j$ we define

$$Q_{ij}(u,v)=r_{ij}(u-v)^{-a_{ij}},\ \text{where } r_{ij}=\pm 1 \text{ with } r_{ji}=r_{ij}(-1)^{a_{ij}},$$

for the Cartan matrix A of type $A^{(1)}_{e-1}$, then the remaining relations are

$$\psi_k^2 e(\nu)=Q_{\nu_k\nu_{k+1}}(x_k,x_{k+1})e(\nu)$$

and $(\psi_{k+1}\psi_k\psi_{k+1}-\psi_k\psi_{k+1}\psi_k)e(\nu)$ is equal to

$$\begin{cases} \dfrac{Q_{\nu_k\nu_{k+1}}(x_k,x_{k+1})-Q_{\nu_k\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k-x_{k+2}}e(\nu) & \text{if } \nu_k=\nu_{k+2} \\ 0 & \text{otherwise} \end{cases}$$

The Brundan-Kleshchev isomorphism theorem

Theorem 2.13 (Brundan-Kleshchev, 2009)

We have an isomorphism of algebras $B(\beta) \simeq R^{\Lambda}(\beta)$.

One advantage of $R^{\Lambda}(\beta)$ is that it has many natural idempotents $e(\nu)$: if we find an appropriate ν such that we prove $e(\nu)R^{\Lambda}(\beta)e(\nu)$ is wild, then $R^{\Lambda}(\beta)$ is wild. Another result we use is the following.

Proposition 2.14 (Erdmann-Nakano, 2002)

Let A and B be finite dimensional K-algebras. If there exist a constant $C > 0$ *and functors* $F : A\text{-mod} \to B\text{-mod}$, $E : B\text{-mod} \to A\text{-mod}$ *such that, for any A-module M,*

(i) *M is a direct summand of $EF(M)$ as an A-module,*

(ii) $\dim F(M) \leq C \dim M$.

Then, the wildness of A implies the wildness of B.

The cyclotomic categorification theorem

To apply the proposition to our setting, we use the induction and restriction functors for cyclotomic quiver Hecke algebras:

$$F_i : \; R^\Lambda(\beta)\text{-mod} \to R^\Lambda(\beta+\alpha_i)\text{-mod},$$
$$E_i : \; R^\Lambda(\beta)\text{-mod} \to R^\Lambda(\beta-\alpha_i)\text{-mod}.$$

We state the following result by Kang and Kashiwara in ungraded form. Note that if $l_i > 0$ then the conditions (i) and (ii) are satisfied.

Theorem 2.15 (Kang-Kashiwara, 2012)

Set $l_i = \langle \alpha_i^\vee, \Lambda - \beta \rangle$, for $i \in I$. Then one of the following isomorphisms of endofuctors on the category of finitely generated $R^\Lambda(\beta)$-modules holds.

1. *If $l_i \geq 0$, then $F_iE_i \oplus |l_i|\mathrm{id} \xrightarrow{\sim} E_iF_i$.*
2. *If $l_i \leq 0$, then $F_iE_i \xrightarrow{\sim} E_iF_i \oplus |l_i|\mathrm{id}$.*

Minimal wild block algebras

We start with finding the border line of the area of wild $R^\Lambda(\beta)$'s in (i,k)-coordinates. (Recall that we consider $\beta = \lambda_i^s + k\delta$.)
To prove that they are wild, we show that $e(\nu)R^\Lambda(\beta)e(\nu)$ is wild for an appropriate ν. For this, we may utilize the fact that $R^\Lambda(\beta)$ is graded:

$$\deg(e(\nu)) = 0, \ \ \deg(x_k e(\nu)) = (\alpha_{\nu_k}|\alpha_{\nu_k})$$
$$\deg(\psi_l e(\nu)) = -(\alpha_{\nu_l}|\alpha_{\nu_{l+1}})$$

Lemma 2.16

If $e = e_1 + e_2$ with $e_1^2 = e_1 \neq 0$, $e_2^2 = e_2 \neq 0$, $e_1 e_2 = e_2 e_1 = 0$ and

$$\dim_q e_i R^\Lambda(\beta) e_j - \delta_{ij} - c_{ij} q^2 \in q^3 \mathbb{Z}_{\geq 0}[q],$$

for $i, j = 1, 2$, then the quiver of $eR^\Lambda(\beta)e$ has two vertices 1 and 2, c_{ii} loops on the vertex i, for $i = 1, 2$, and there are at least c_{12} arrows and c_{21} reverse arrows between 1 and 2.

Graded dimension formula

The lemma may be used in the following way.

If there are at least two loops on one vertex and it is connected to the other vertex, then $e(\nu)R^{\Lambda}(\beta)e(\nu)$ is wild.

Remark 2.17

We may compute the graded dimensions in the lemma by the following graded dimension formula.

$$\dim_q e(\nu')R^{\Lambda}(\beta)e(\nu) = \sum_{\lambda \vdash n} K_q(\lambda, \nu')K_q(\lambda, \nu),$$

where $K_q(\lambda, \nu)$ is the sum of $q^{\deg(T)}$ over $T \in \mathrm{ST}(\lambda)$ with $\mathrm{res}(T) = \nu$. In particular, the formula tells us when $e(\nu)$ is nonzero.

Strategy to determine representation type

Now we state the strategy to determine representation type.

- Recall that it is enough to consider $\beta = \lambda_i^s + k\delta$.
- We find minimal wild $R^\Lambda(\beta)$'s and show that they are wild by using the previous lemma or an explicit computation of $e(\nu)R^\Lambda(\beta)e(\nu)$ for an appropriate ν.
- We show that all the other wild $R^\Lambda(\beta)$'s are indeed wild by using the induction and restriction functors in the proposition. This reduces to simple calculation of scalar products of roots $l_i = \langle \alpha_i^\vee, \Lambda - \beta \rangle$ to check that they are positive.
- We analyze the remaining small number of $R^\Lambda(\beta)$'s, and determine whether they are finite or tame.

Representation type for blocks of $H_n(q, -q^s)$

We state the representation type for blocks of $H_n(q, -q^s)$ when $s \neq 0$.

Theorem 2.18 (A, 2017)

Let $\Lambda = \Lambda_0 + \Lambda_s$, for $1 \le s \le \ell$, and $\beta = \lambda_i^s + k\delta$, for $0 \le i \le \frac{\ell - s + 1}{2}$ and $k \in \mathbb{Z}_{\ge 0}$. Then $R^\Lambda(\beta)$ is

(i) *simple if $i = 0$ and $k = 0$.*

(ii) *of finite representation type if $i = 1$ and $k = 0$.*

(iii) *of tame representation type if $i = 0$, $k = 1$ and $\ell = 1$.*

(iv) *of wild representation type otherwise.*

If $R^\Lambda(\beta)$ is finite, we may compute the radical series of indecomposable projective modules by utilizing the grading. For $s = 0$, my student Kakei has computed the representation type in his master thesis.

Remark on the generic degree

Finite type appears when $i = 1$ and $k = 0$. In this case, the labels of the ordinary irreducible characters in the block are $(s+1-m, 1^m)$, for $0 \le m \le n = s+1$. Their Lusztig symbols are

$$\phi = \begin{pmatrix} \lambda_1, \dots, \lambda_{m+1} \\ \mu_1, \dots, \mu_m \end{pmatrix} = \begin{pmatrix} 0, 1, \dots, m-1, s+1 \\ 1, 2, \dots, m \end{pmatrix}.$$

The generic degree D_ϕ is given by the following formula.

$$\frac{q^{m+\binom{m}{2}} \prod_{i=1}^{n}(q^i-1)(q^{i-1}Q+1) \prod_{i<j}(q^{\lambda_j}-q^{\lambda_i}) \prod_{i<j}(q^{\mu_j}-q^{\mu_i}) \prod_{i,j}(q^{\lambda_i-1}Q+q^{\mu_j})}{q^{\binom{2m-1}{2}+\binom{2m-3}{2}+\cdots} Q^{\binom{m}{2}}(q+Q)^m \prod_{i=1}^{m+1}\prod_{k=1}^{\lambda_i}(q^k-1)(q^{k-1}Q+1) \prod_{j=1}^{m}\prod_{k=1}^{\mu_j}(q^k-1)(q^{k+1}Q^{-1}+1)}$$

Suppose that $e \ge 4$ is even and consider the equal parameter case. Then, we must set $Q = q$ in D_ϕ but choose $-Q = q^{s=\frac{e}{2}-1}$ in order that the number of times $\Phi_e(q)$ divides P_W/D_ϕ, where $P_W = \sum_{w \in W(B_n)} q^{\ell(w)}$, is constant on this block. Then, this block becomes a Φ_e-defect one block.

Representation type for blocks of $H_n(q, -q^s)$ (cont'd)

Theorem 2.19 (Kakei, 2016)

Let $\Lambda = 2\Lambda_0$ and $\beta = \lambda_i^0 + k\delta$, for $0 \le i \le \frac{e}{2}$ and $k \in \mathbb{Z}_{\ge 0}$. Then $R^\Lambda(\beta)$ is

(i) *simple if $i = 0$ and $k = 0$.*

(ii) *of finite representation type if $i = 1$ and $k = 0$.*

(iii) *of tame representation type if $\ell = 1, i = 0, k = 1$ or $\ell \ge 2, i = 0, k = 1, \lambda \ne (-1)^{\ell+1}$, or $\ell \ge 3, i = 2, k = 0, \mathrm{char} K \ne 2$.*

(iv) *of wild representation type otherwise.*

Corollary 2.20

Let $\Lambda = \Lambda_0 + \Lambda_s$, for $0 \le s \le e-1$, and $\beta = \lambda_i^s + k\delta$, for $0 \le i \le \frac{e-s}{2}$ and $k \in \mathbb{Z}_{\ge 0}$. If $R^\Lambda(\beta)$ is finite then it is a Brauer tree algebra whose Brauer tree is a straight line without exceptional vertex.

Cellularity and Ohmatsu's theorem

We have proved that the representatives of blocks of finite type under Chuang-Rouquier derived equivalence are Brauer tree algebras whose Brauer tree are straight lines without exceptional vertex.

Theorem 2.21 (Ohmatsu, 2014)

A symmetric cellular algebra of finite type is a Brauer tree algebra whose Brauer tree is a straight line.

For the proof, we recall a result by Riedmann and Waschbusch that symmetric algebras of finite type are Morita equivalent to one of

(i) the trivial extension algebras of tilted algebras of Dynkin type,

(ii) Brauer tree algebras,

(iii) modified Brauer tree algebras.

Recall that we assume that the characteristic of the base field is odd. Thus, cellularity is preserved under Morita equivalence.

Rickard's star theorem and the end of the proof

As blocks of Hecke algebras are symmetric cellular algebras, we may apply Ohmatsu's theorem. Therefore, the blocks of finite type are Brauer tree algebras, and they are derived equivalent to Brauer tree algebras without exceptional vertex. Now we appeal to the Rickard star theorem.

Theorem 2.22 (Rickard, 1989)

Any Brauer tree algebra is derived equivalent to a Brauer tree algebra of star shape such that only the center of the star can be an exceptional vertex. The number of edges and the multiplicity of the exceptional vertex determine the derived equivalence classes of Brauer tree algebras.

Thus, if a block of $H_n(q, -q^s)$ is of finite type then it is a Brauer tree algebra whose Brauer tree is a straight line without exceptional vertex. Then, blocks of $H_W(q)$ for $W = W(D_n)$ are of the same kind by the Clifford theory. As we know the result for type A, the proof is complete. Note that derived equivalence becomes Morita equivalence here.

Thank you for your attention.