de Boor's conjecture, past and present

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Summary

- Key-definitions and de Boor's conjecture
- Initial points of interest (and different approaches)

  Construction of orthonormal bases in C^k (Ciesielski)

  Spline interpolation problems (de Boor)

  Galerkin approximations (Douglas-Dupont-Wahlbin)
- Key-ideas of the general proof
- New results

  Exact upper and lower bounds for the max-norms (Foucart)

  Bi-infinite and periodic cases (de Boor)

  Unconditional convergence of spline interpolants in W_p^r (Volkov)

  Linear splines on triangulations (Oswald)

  A short proof (v. Golitschek)

  Convergence almost everywhere (Passenbrunner, S)

Orthogonal spline projector

Smooth splines of order k:

$\mathcal{S} := \mathcal{S}_k(\Delta_N) :=$ piecewise polynomials of degree $< k$

with the knot sequence $\Delta_N = (a = t_0 < t_1 < \cdots < t_N = b)$

in C^{k-1}

Consider $P_{\mathcal{S}}$, the orthogonal spline projector onto $\mathcal{S}$, i.e.

$$(f, \sigma) = (P_{\mathcal{S}} f, \sigma), \qquad \forall \sigma \in \mathcal{S}.$$

We are interested in its norm as an operator from L_∞ to L_∞,

$$\|P_{\mathcal{S}}\|_\infty := \sup_f \frac{\|P_{\mathcal{S}}(f)\|_\infty}{\|f\|_\infty}\,.$$

de Boor's conjecture

Conjecture [de Boor] (1972). For any k, the L_∞-norm of the L_2-projector $P_{\mathcal{S}}$ onto the spline space $\mathcal{S}_k(\Delta_N)$ is bounded independently of Δ_N, i.e.,

$$\sup_{\Delta} \|P_{\mathcal{S}_k(\Delta)}\|_\infty \le c_k.$$

"I offer the modest sum of $(m - 1972) \times 10\$$ to the first person who communicates to me a proof or a counterexample (but not both) of his or her making of this conjecture (known to be true for $k = 2$ or $k = 3$). Here m is the year (A.D.) of such communication."

Initial points of interest (and their developments)

1) Ciesielski [1963]. Construction of orthonormal bases in $C[0,1]$

general Franklin systems, unconditonal bases in L_p, in Hardy spaces $H^p[0,1]$,

bases in $C^1[0,1]^2$ and with higher dimensions and smoothness, etc.

2) de Boor [1968]. Error of spline interpolation

B-spline basis condition number, solvability of cardinal interpolation problem,

optimal knots of interpolation, multivariate interpolation, etc.

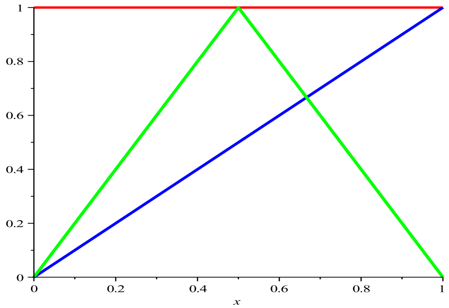
3) Douglas-Dupont-Wahlbin [1976]. Galerkin approximations

exponential decay of finite element approximation to finitely supported functions,

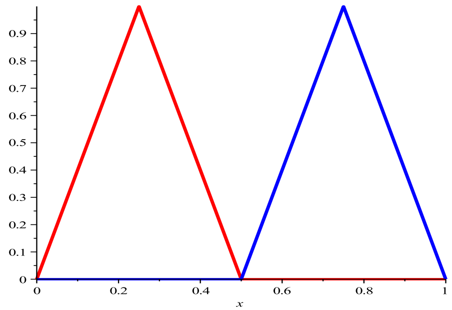
exponential decay of fundamental splines, inverses of the band matrices, etc.

Faber-Shauder and Franklin systems

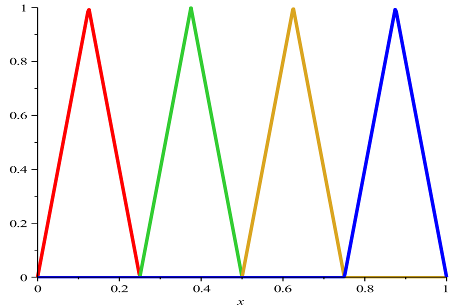
The Faber-Shauder system $(\psi_i)_{i=0}^{\infty}$ is a basis in $C[0,1]$ (but not in $L_p[0,1]$ for $p < \infty$):



$\psi_0,\ \psi_1,\ \psi_2$



$\psi_3,\ \psi_4$

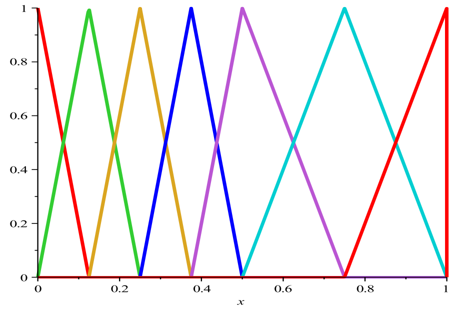


$\psi_5,\ \psi_6,\ \psi_7,\ \psi_8$

The Franklin system (ϕ_i) is obtained from the Faber-Shauder one by orthogonalization. It follows that

$$f \sim \sum_{i=0}^{n} a_i \phi_i = P_{\mathcal{S}}(f),$$

$$\mathcal{S} = \mathrm{span}\,(\phi_i)_{i=0}^{n} = \mathrm{span}\,(N_i)_{i=0}^{n} =$$



Normal equations

If (u_j) is a basis for $\mathcal{S}_n$, and if we write $P_{\mathcal{S}}(f) = s^* = \sum a_j u_j$, then running σ in

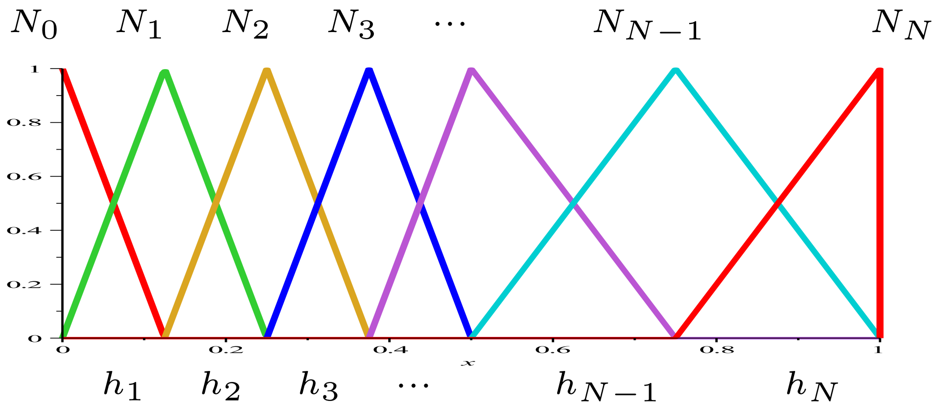
$$(f, \sigma) = (P_{\mathcal{S}} f, \sigma), \qquad \forall \sigma \in \mathcal{S}\,,$$

through the basis functions (u_i) we obtain $(s^*, u_i) = (f, u_i)$, i.e., a linear system of equations for determining the coefficients a,

$$Ga = b, \qquad G = [(u_i, u_j)]_{i,j=1}^n, \qquad b = [(f, u_i)]_{i=1}^n.$$

These equations are called the normal equations. The matrix G is called the Gram matrix.

Linear B-spline basis:



Estimating orthogonal projector in terms of a Gram matrix

With the B-spline expansion of $P_{\mathcal{S}}(f)$,

$$P_{\mathcal{S}}(f) = \sum_{j=0}^{n} a_j N_j \,,$$

it is more convenient to consider a renormalized Gram matrix which is obtained from the normal equations $(P_{\mathcal{S}} f, M_i) = (f, M_i)$:

$$Ga = b, \qquad G = [(M_i, N_j)]_{i,j=0}^{n}, \qquad b = [(f, M_i)]_{i=0}^{n} \,,$$

where $M_i = \frac{k}{t_{i+k} - t_i} N_i \ge 0$. Then

$$\sum N_j \equiv 1, \qquad \int M_i = 1 \,,$$

and respectively

$$\|P_{\mathcal{S}}(f)\|_\infty \le \|a\|_\infty, \qquad \|b\|_\infty \le \|f\|_\infty \,.$$

Since $\|a\|_\infty \le \|G^{-1}\|_\infty \|b\|_\infty$, it follows that $\|P_{\mathcal{S}}(f)\|_\infty \le \|G^{-1}\|_\infty \|f\|_\infty$, hence

$$\|P_{\mathcal{S}}\|_\infty \le \|G^{-1}\|_\infty \,.$$

Ciesielski result

For linear B-splines, the entries of G are easily computed:

$$G = \frac{1}{3}\begin{bmatrix} 2 & 1 & & & \\ \frac{h_1}{h_1+h_2} & 2 & \frac{h_2}{h_1+h_2} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \frac{h_i}{h_i+h_{i+1}} & 2 & \frac{h_{i+1}}{h_i+h_{i+1}} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \frac{h_{N-1}}{h_{N-1}+h_N} & 2 & \frac{h_N}{h_{N-1}+h_N} \\ & & & 1 & 2 \end{bmatrix}.$$

It is readily seen that the matrix G is row-wise diagonally dominant, with the dominance

$$\gamma := \min_i \Big(|g_{ii}| - \sum_{j\neq i} |g_{ij}|\Big) = \frac{1}{3}\,,$$

hence

$$\|G^{-1}\| \le 1/\gamma \le 3 \;\;\Rightarrow\;\; \|P_{\mathcal{S}_2(\Delta)}\|_\infty \le 3\,.$$

Spline interpolation

If $s := s_{2k,\Delta}$ is the spline of degree $2k-1$ with a knot sequence Δ that interpolates f at knots, then

$$P_{\mathcal{S}_k(\Delta)}(f^{(k)}) = s^{(k)}_{2k,\Delta}\,.$$

This is a consequence of the fact that B-splines are Peano kernels for divided differences:

$$\frac{1}{k!}(f^{(k)}, M_i) = [t_i, \ldots, t_{i+k}]f = [t_i, \ldots, t_{i+k}]s = \frac{1}{k!}(s^{(k)}, M_i)\,.$$

Therefore,

$$\|f^{(k)} - s^{(k)}\| = \|f^{(k)} - P_{\mathcal{S}}(f^{(k)})\| \le (1 + \|P_{\mathcal{S}}\|)E_{k,\Delta}(f^{(k)})$$

Since $f - s$ vanishes at every $t_i \in \Delta$, we have $\|f - s\| \le c_k \overline{h}^k \|f^{(k)} - s^{(k)}\|$, hence

$$\|f - s\| \le \begin{cases} c_k(P)\overline{h}^{2k}\|f^{(2k)}\|\,, \\ c_k(P)\overline{h}^k \omega_k(f^{(k)}, \overline{h})\,. \end{cases}$$

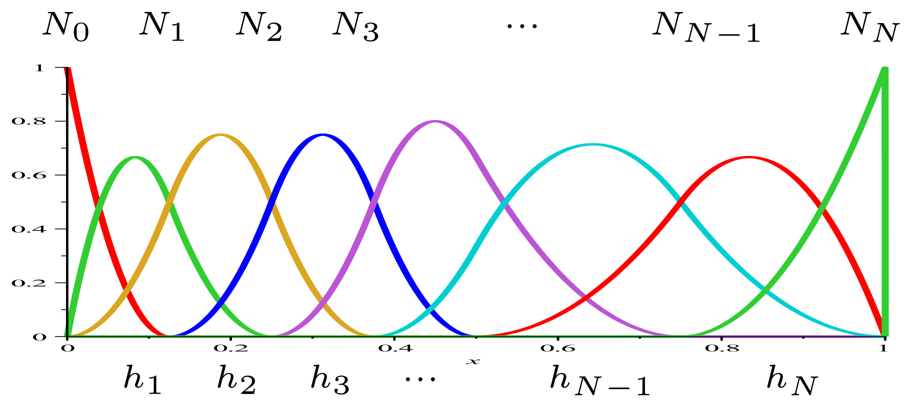
Orthogonal projector on $\mathcal{S}_k(\Delta)$ for any k

Similarly to what we have done with linear splines, for the orthoprojector onto the splines of degree $k-1$, we may consider its expansion

$$P_{\mathcal{S}}(f) = \sum a_j N_j$$

with respect to the B-spline basis of degree $k-1$, and form the normal equations $Ga = b$.

B-spline basis of degree $k-1$:



Then again we have the estimate

$$\|P_{\mathcal{S}_2(\Delta)}\|_\infty \le \|G^{-1}\|_\infty \,,$$

however, generally, for $k \ge 3$, the Gram matrix $G = [(M_i, N_j)_{i,j=0}^n]$ is not diagonally dominant anymore, so some other tools for estimating its inverse should be found.

Total positivity (TP)

Theorem [Karlin]. Let G be the Gram-matrix $[(M_i, N_j)]_{i,j=0}^n$. Then G is totally positive meaning that all its minors are non-negative.

Corollary [de Boor]. The matrix G^{-1} is checkerboard, i.e.

$$\operatorname{sgn} G^{-1} = \left[\begin{array}{c|c|c|c|c|c} + & - & + & \dots & + & - \\ \hline - & + & - & \dots & - & + \\ \hline \multicolumn{6}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ \hline + & - & + & \dots & + & - \\ \hline - & + & - & \dots & - & + \end{array}\right]$$

Proof. Let G_{ji} be the algebraic adjoint to g_{ij}. By Cramer's rule

$$g_{ij}^{(-1)} = (-1)^{i+j} \frac{\det G_{ji}}{\det G} = (-1)^{i+j} |g_{ij}^{(-1)}| \,,$$

the latter equality because both determinants are non-negative by TP.

Inverse of a totally positive matrix

Lemma [de Boor]. Let H^{-1} be a checkerboard matrix, and let $a, b \in \mathbb{R}^N$ be two vectors, such that

$$
\begin{aligned}
&(a_0) \qquad Ha = b; \\
&(a_1) \qquad (-1)^i \operatorname{sgn} b_i = \mathrm{const} \quad \forall i; \\
&(a_2) \qquad |b_i| \ge c_{\min} \quad \forall i; \\
&(a_3) \qquad \|a\|_\infty \le c_{\max}.
\end{aligned}
$$

Then

$$\|H^{-1}\|_\infty \le \frac{c_{\max}}{c_{\min}}.$$

Proof. Let a, b satisfy (a_0)-(a_3), and let $H^{-1} := (h_{ij}^{(-1)})$, where $|h_{ij}^{(-1)}| = (-1)^{i+j} h_{ij}^{(-1)}$. Then

$$|a_i| = |(H^{-1}b)_i| := |\sum_j h_{ij}^{(-1)} b_j| = \sum_j |h_{ij}^{(-1)} b_j| \ge \min_j |b_j| \cdot \sum_j |h_{ij}^{(-1)}|,$$

hence

$$\|a\|_\infty := \max_i |a_i| \ge \min_j |b_j| \cdot \max_i \sum_j |h_{ij}^{(-1)}| = \min_j |b_j| \cdot \|H^{-1}\|_\infty .$$

Inverse of a totally positive matrix (contd)

Example. For $k = 2$ (linear splines), we can take $a = [(-1)^i]$, and obtain

$$\underbrace{\frac{1}{3}\begin{bmatrix} 2 & 1 & & & \\ \frac{h_1}{h_1+h_2} & 2 & \frac{h_2}{h_1+h_2} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \frac{h_i}{h_i+h_{i+1}} & 2 & \frac{h_{i+1}}{h_i+h_{i+1}} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \frac{h_{N-1}}{h_{N-1}+h_N} & 2 & \frac{h_N}{h_{N-1}+h_N} \\ & & & 1 & 2 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ \cdots \\ -1 \\ 1 \\ -1 \end{bmatrix}}_{a} = \frac{1}{3}\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ \cdots \\ -1 \\ 1 \\ -1 \end{bmatrix}}_{b}$$

so that, by the last theorem,

$$\|G^{-1}\|_\infty \le \frac{\max |a_j|}{\min |b_i|} = 3\,,$$

and that proves again Ciesielski bound.

de Boor's result for qudratic and cubic splines

Based on his lemma, de Boor suggested the following choices of the vector a:

$$k=3, \quad (-1)^i a_i = 1 + \frac{(t_{i+2}-t_{i+2})^2}{(t_{i+2}-t_i)(t_{i+3}-t_{i+1})}, \qquad \operatorname{supp} M_i = [t_i, t_{i+3}]$$

$$k=4, \quad (-1)^i a_i = 3 + 4\frac{(t_{i+3}-t_{i+2})^2}{(t_{i+3}-t_i)(t_{i+4}-t_{i+1})}, \quad \operatorname{supp} M_i = [t_i, t_{i+4}]\,.$$

and that resulted in the upper bounds:

$$k=3, \qquad \|P_{\mathcal{S}_3(\Delta)}\| \le \|G^{-1}\| \le 30,$$

$$k=4, \qquad \|P_{\mathcal{S}_4(\Delta)}\| \le \|G^{-1}\| \le 81\tfrac{2}{3}\,.$$

However, already for quadratic B-splines ($k=3$), even to compute the entries of $G=[(M_i,N_j)]$ turned out to be a challenge. No wonder that de Boor, in his 1968 paper devoted to this case, made arithmetic mistakes, so that he had to repeat his calculations in 1980, both versions taken several pages. The calculations for the case $k=4$ had never appeared.

Galerkin approximations to solutions of two-point boundary problems

Consider the two-point boundary value problem:

$$-(a(x)y')' + b(x)y' + d(x)y = f(x), \qquad y(0) = y(1) = 0,$$

or, in a weak form,

$$(ay', v') + (by', v) + (dy, v) = (f, v), \qquad \forall v \in \overset{\circ}{H}_1 \ .$$

The approximate solution Y is sought in the space $\mathcal{S}_k(\Delta)$ according to the rule:

$$(aY', V') + (bY', V) + (dY, V) = (f, V), \qquad \forall V \in \mathcal{S}_k(\Delta)\,.$$

Theorem [Douglas-Dupont-Wahlbin] (1975). If $\max\limits_{i,j} \frac{h_i}{h_j} < M$, then there exists a constant $c = c(a, b, d, k, M)$ such that

$$\|y - Y\|_\infty \le c\overline{h}^k \|y\|_{W_\infty^k} \,.$$

The main tool was the following

Lemma. If $\max\limits_{i,j} \frac{h_i}{h_j} < M$, then

$$\|P_{\mathcal{S}_k(\Delta)}\|_\infty \le c_k(M)\,.$$

Quasi-uniform partitions

The arguments used by Douglas-Dupont-Wahlbin revealed that the boundedness of $\|P_{\mathcal{S}}\|_\infty$ for the quasi-uniform meshes has nothing to do with the particular spline nature as piecewise polynomial functions. The essential structural requirements on a subspace $\mathcal{S}$ for these proofs are that

$\mathcal{S}$ has a good-conditioned basis (ϕ_i) of finitely supported functions.

A bit later, Demko eliminated even the bases and reduced the statement to the pure fact about band matrices.

Theorem [Demko]. Let A be a band matrix, which is bounded and has a bounded inverse, i.e.,

$$1) \quad a_{ij} = 0, \quad \text{if} \quad |i-j| > r,$$
$$2) \quad \|A\|_2 \le c_1, \quad \|A^{-1}\|_2 \le c_2.$$

Then the elements of $A^{-1} = (b_{ij})$ decay exponentially away from the diagonal. i.e.,

$$|b_{ij}| \le c\gamma^{|i-j|}, \qquad \gamma < 1.$$

Boundedness of $\|P_{\mathcal{S}}\|$ for the quasi-uniform meshes

Proof. Let $(\widehat{N}_i)$ be the L_2-normalized B-spline basis, i.e.,

$$\widehat{N}_i := \Big(\frac{k}{t_{i+k}-t_i}\Big)^{1/2} N_i = \Big(\frac{t_{i+k}-t_i}{k}\Big)^{1/2} M_i .$$

Then

$$\kappa_k^{-1}\|a\|_{\ell_2} \le \|\sum a_j \widehat{N}_j\|_{L_2} \le \|a\|_{\ell_2}$$

If $\widehat{G} = (\widehat{N}_i, \widehat{N}_j)$ is the corresponding Gram matrix, then $\widehat{G}$ is banded and

$$\kappa_k^{-2}\|a\|^2 \le (Ga, a) \le \|a\|_2^2 \quad \Rightarrow \quad \|\widehat{G}\|_2 \le 1, \quad \|\widehat{G}^{-1}\|_2 \le \kappa_k^2 .$$

Hence, by the Demko theorem

$$|\widehat{g}_{ij}^{(-1)}| \le c_k \gamma_k^{|i-j|} .$$

We need, however, to estimate the max-norm of the inverse of G not of $\widehat{G}$. But

$$G = (M_i, N_j) = D^{-1/2}\widehat{G}D^{1/2}, \qquad D = \operatorname{diag} \frac{t_{i+k}-t_i}{k} \,,$$

and it follows that

$$|g_{ij}^{(-1)}| = (d_i/d_j)^{1/2}\, |\widehat{g}_{ij}^{(-1)}| \le c_k M^{1/2} \gamma_k^{|i-j|} .$$

and

$$\|G^{-1}\| < c_k'(M) \,.$$

General proof

It is very easy to prove de Boor's conjecture.

I know it from my own experience because I did it many many times.

General proof: analytic version of de Boor's lemma on the inverses of TP matrices

Lemma. Let ϕ be any spline such that

$$
\begin{aligned}
&(A_0) \qquad \phi \in \mathcal{S}_k(\Delta); \\
&(A_1) \qquad (-1)^i \operatorname{sgn}\,(\phi, M_i) = \text{const} \quad \forall i; \\
&(A_2) \qquad |(\phi, M_i)| \ge c_{\min}(k) \quad \forall i; \\
&(A_3) \qquad \|\phi\|_\infty \le c_{\max}(k).
\end{aligned}
$$

Then

$$
\|P_{\mathcal{S}_k(\Delta)}\|_\infty \le \kappa_k \frac{c_{\max}}{c_{\min}}.
$$

Proof. Let $\phi = \sum a_j N_j$ and $G = [(M_i, N_j)]$. Then

$$
Ga = b \qquad \Leftrightarrow \qquad (\phi, M_i) = b_i, \quad \|a\|_\infty \le \kappa_k \|\phi\|_\infty \,,
$$

and by de Boor's lemma

$$
\|G^{-1}\| \le \frac{\|a\|_\infty}{\min |b_i|} \le \kappa_k \frac{c_{\max}(k)}{c_{\min}(k)} \,.
$$

General proof: choice of ϕ

Let $\sigma \in \mathcal{S}_{2k-1}(\Delta)$ be the spline of even degree $2k-2$ on Δ that satisfies the following conditions:

$$\sigma \in \mathcal{S}_{2k-1}(\Delta), \qquad \begin{cases} \sigma(t_i) &= 0, \quad i = 0, \ldots, N; \\ \sigma^{(m)}\Big|_{t_0, t_N} &= 0, \quad m = 1, \ldots, k-2; \\ \sigma^{(k-1)}(t_0) &= 1\,, \end{cases}$$

i.e., σ is a null-spline with zero boundary conditions normalized at the left end-point of Δ.

Theorem. Spline $\phi := \sigma^{(k-1)}$ satisfies conditions (A_0)-(A_3) above.

$$\mathrm{sgn}\,(\phi, M_i) = \mathrm{sgn}\,(\sigma^{(k-1)}, M_i) = \mathrm{sgn}\,(\sigma, M_i^{(k-1)}) = (-1)^i$$

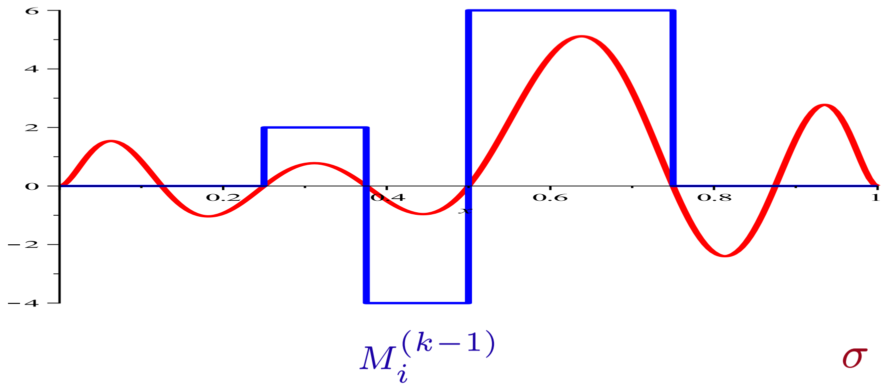


Illustration. Condition (A_1) $\mathrm{sgn}\,(\phi, M_i) = (-1)^i$ follows from the sign patterns of σ and $M_i^{(k-1)}$.

General proof: further remarks

Remark 1. The main technicalities of the proof were concerned with the upper estimate (A_3):

$$\|\phi_\Delta\|_\infty \le c_k \quad (= c_k|\phi_\Delta(t_0)|)\,.$$

Remark 2. Spline $\phi \in \mathcal{S}_k(\Delta)$ has a very distinctive property over all other splines in $\mathcal{S}_k(\Delta)$: it is the unique spline of degree $k-1$ which is orthogonal to all splines of degree $k-2$

$$\phi_\Delta \in \mathcal{S}_k(\Delta), \qquad \phi_\Delta \perp \mathcal{S}_{k-1}(\Delta).$$

Conjecture. If $\phi_\Delta \perp \mathcal{S}_{k-1}(\Delta)$, then

$$\|\phi_\Delta\|_\infty = |\phi_\Delta(t_0)| \quad (= |\phi_\Delta(t_N)|)\,.$$

Example 1. For $k=2$, σ is a quadratic null-spline, and its first derivative $\phi = \sigma'$ is the broken line that alternates detween ± 1 at knots, i.e.,

$$\phi_\Delta = \sum (-1)^i N_i$$

Example 2. For $N=1$ (no interior knots) on $[-1,1]$, $\sigma(x) = c(1-x^2)^{k-1}$, and $\phi = \sigma^{(k-1)}$ is the Legendre polynomial L_{k-1} of degree $k-1$, and $\|L_{k-1}\| = L_{k-1}(1)$.

New results I: exact values of the Lebesgue constants $\|P_{\mathcal{S}_k(\Delta)}\|_\infty$

There are two constants in de Boor's problem:

(a) the norm of the orthoprojector $\|P_{\mathcal{S}_k(\Delta)}\|_\infty$,

(b) the norm of the inverse of the B-spline Gramian $\|G_\Delta^{-1}\|_\infty$,

and we proved that

$$(\|P_{\mathcal{S}_k(\Delta)}\|_\infty \le) \quad \|G_\Delta^{-1}\|_\infty \le c_k .$$

For $k = 2$, both values coincide

$$k = 2, \qquad \sup_\Delta \|P_{\mathcal{S}_k(\Delta)}\| = \|G_\Delta^{-1}\| = 3 ,$$

but for $k > 2$ the situation is unknown and there are indications that those two values have different order in k. To be sure,

$$\kappa_{k,\Delta,\infty}^{-1} \kappa_{k,\Delta,1}^{-1} \|G_\Delta^{-1}\|_\infty \le \|P_{\mathcal{S}_k(\Delta)}\|_\infty \le \|G_\Delta^{-1}\|_\infty ,$$

but $\kappa_{k,p}$, the condition numbers of the B-spline basis, grow like 2^k in the worst case.

Example. If $N = 1$ (no interior knots), then $P_\mathcal{S}$ is simply the orthoprojector onto the space $\mathcal{P}_k$ of polynomials, and in this case

$$\|P\| = \mathcal{O}(\sqrt{k}), \qquad \|G^{-1}\| = \mathcal{O}(k^{-1/2} 4^k) .$$

New results I: exact values of the Lebesgue constants (contd)

In my paper, I proved that

$$\sup_{\Delta} \|P_{\mathcal{S}_k(\Delta)}\| \ge 2k-1, \qquad \sup_{\Delta} \|G_{\Delta}^{-1}\| \ge ck^{-1/2}4^k\,,$$

and I conjectured that both values are actually the upper bounds for suprema..

Foucart studied orthogonal projectors onto the spline spaces $\mathcal{S}_{k,m}(\Delta)$ with low smoothness m:

$$\mathcal{S}_{k,m}(\Delta) := \text{piecewise polynomials of degree} < k, \text{ on } \Delta, \text{ in } C^{m-1}$$

He proved that $\mathcal{O}(\sqrt{k})$-behaviour of the norm for $m = 0$ (from the previous example) is not radically changed if we increase the smoothness to $m = 1$ or $m = 2$.

Theorem [Foucart] (2006). We have

$$\sup_{\Delta} \|P_{\mathcal{S}_{k,m}(\Delta)}\| = \mathcal{O}(\sqrt{k}), \qquad m = 1, 2.$$

He also established the lower bounds which clarify the nature of the contant $2k-1$ for $m = k-2$.

Theorem [Foucart] (2006). We have

$$\sup_{\Delta} \|P_{\mathcal{S}_{k,m}(\Delta)}\| \ge \frac{k}{k-m}\rho_{k,m} \quad \left(\asymp \frac{k}{\sqrt{k-m}}\right)$$

where $\rho_{k,m}$ is the norm of the orthoprojector onto the space of incomplete polynomials.

New results II: bi-infinite and periodic cases

In my paper, I considered also the case of bi-infinite knot-sequences Δ_∞ and deduced that

$$\sup_{\Delta_\infty} \|G_{\Delta_\infty}^{-1}\|_\infty < c_k ,$$

where I used the observation that (i) all principle submatrices G_{Δ_N} of G_{Δ_∞} are uniformly boundedly invertible,

(ii) hence, so must G_{Δ_∞} be and with the same bound.

This would also imply the periodic case, although I did not mention that in my paper.

However, in 2011, I discovered some gaps: (i) my proof is given for the finite knot-sequences which are k-complete, whereas finite sections of Δ_∞ are not necessarily such

(ii) I could not find the reference for that pass (it should have been de Boor).

Clean-up [de Boor] (2011). (i) If $\Delta_N \subset \Delta_M$, then $\|G_{\Delta_N}^{-1}\|_\infty < \|G_{\Delta_M}^{-1}\|_\infty$

(ii) C. de Boor, Trans. AMS 274 (1982) (not explicitly stated) and, moreover, simpler alternative proof is given.

Theorem [de Boor] (2011). For the splines on bi-infinite knot-sequences, we have

$$\sup_{\Delta_\infty} \|G_{\Delta_\infty}^{-1}\|_\infty < c_k ,$$

and the same is true for periodic splines of any order k.

New results III: Unconditional convergence of spline interpolants

For $1 \le p \le \infty$, and f from the Sobolev space W_p^r, let $s := s_{2k,\Delta}$ be the spline of degree $2k-1$ which interpolates f on Δ:

$$s \in \mathcal{S}_{2k}(\Delta), \qquad s\big|_\Delta = f\big|_\Delta \,.$$

Problem. Find, whether the following value is finite:

$$L^*(k,r,p) := \sup_\Delta \|f - s_{2k,\Delta}\|_{W_p^r} \,.$$

It has been known for a while that a necessary condition for that was

$$L^*(k,r,p) < \infty \quad \Rightarrow \quad W_p^r \subset \{W_\infty^{k-1}, W_p^k, W_1^{k+1}\}\,,$$

and the question was whether it is also a sufficient condition. For $r = k$, since

$$P_{\mathcal{S}_k(\Delta)}(f^{(k)}) = s_{2k,\Delta}^{(k)},$$

affirmative answer to de Boor's conjecture implied

$$W_p^r = W_p^k \quad \Rightarrow \quad L^*(k,r,p) < \infty \,.$$

Theorem [Volkov] (2005, 2012). For periodic case, and for non-periodic case as well, we also have

$$W_p^r = W_\infty^{k-1} \quad \Rightarrow \quad L^*(k,r,p) < \infty \,.$$

New results IV: multivariate case

The following two results can be useful in the context of obtaining L_p error estimates in the finite element method.

Theorem. Let $\mathcal{S}_k(\Delta)$ be a tensor product spline space in d dimensions. Then

$$\sup_{\Delta} \|P_{\mathcal{S}_k(\Delta)}\|_\infty < c_{k,d}$$

Proof. Kronecker product of matrices.

On the other hand, for $d = 2$, and for the spaces $V(\mathcal{T})$ of linear splines on triangulations $\mathcal{T}$, the analogue of de Boor's problem has a negative solution.

Theorem [Oswald] (2009, 2013). For any m, there is a triangulation $\mathcal{T}_m$ of the square into $8m + 4$ triangles such that the orthogonal projector $P_{V(\mathcal{T}_m)}$ satisfies

$$\|P_{V(\mathcal{T}_m)}\|_\infty \ge m\,.$$

Thus, for spatial dimension $d = 2$ we have

$$\sup_{\mathcal{T}} \|P_{V(\mathcal{T})}\|_\infty = \infty\,.$$

New results V: a short proof by v. Golitschek

In 2014, v. Golitschek found a short and simple proof of de Boor's conjecture.

Theorem [v. Golitchek (2014)]. For any k, the L_∞-norm of the L_2-projector $P_{\mathcal{S}}$ onto the spline space $\mathcal{S}_k(\Delta_N)$ is bounded independently of Δ_N, i.e.,

$$\sup_{\Delta} \|P_{\mathcal{S}_k(\Delta)}\|_\infty \le c_k \,.$$

Proof. This proof consists of three steps

$$31) \quad P_{\mathcal{S}}(t_0) := \sup_{\|f\|_\infty \le 1} |P_{\mathcal{S}}(f, t_0)| = K_0 < c_k \,,$$

$$2) \quad \|P_{\mathcal{S}}\|_{[t_0,t_1]} := \sup_{\|f\|_\infty \le 1} \|P_{\mathcal{S}}(f)\|_{[t_0,t_1]}) = K_1 < c_k K_0 \,,$$

$$3) \quad \|P_{\mathcal{S}}\|_{[t_i,t_{i+1}]} := \sup_{\|f\|_\infty \le 1} \|P_{\mathcal{S}}(f)\|_{[t_i,t_{i+1}]} = K_2 < c_k K_1 \,.$$

So, in Step 1 the value of the projector is estimated at the end-points, in Step 2 it is shown then that ist norm is bounded on the first (and the last) intervals, and finally in Step 3, it is shown that the global projector restricted to any interior interval $[t_i, t_{i+1}]$ can be represented as a linear combination of two "half-interval" projectors on $[t_0, t_{i+1}]$ and $[t_i, t_N]$, respectively, and the latter are bounded on $[t_i, t_{i+1}]$ by Step 2.

New results V: a short proof by v. Golitschek (contd)

Step 1. We have

$$P_{\mathcal{S}}(t_0) := \sup_{\|f\|_\infty \le 1} |P_{\mathcal{S}}(f, t_0)| = K_0 < c_k$$

The proof of this step is based on the following two statements.

Claim 1 [v. Golitchek (2014)]. There exists a spline $Q_0 = \sum a_i M_i$ from $\mathcal{S}_k(\Delta_N)$ such that

$$1) \quad (Q_0, s) = s(t_0) \quad \forall s \in \mathcal{S}, \qquad 2) \quad \operatorname{sgn} a_i = (-1)^i, \qquad 3) \quad P_{\mathcal{S}}(t_0) = \|Q_0\|_1 \le \sum |a_i|.$$

Another ingredient turned out to be the (A_1)-(A_2) properties of the spline ϕ which I introduced for my proof and those properties had a short and simple proof as well (unlike (A_3)).

Claim 2 [S (2001)]. There exists $\phi \in \mathcal{S}_k(\Delta_N)$ such that

$$4) \quad \phi(t_0) = 1, \qquad 5) \quad (-1)^i (\phi, M_i) > 1/K_0 > 0$$

Proof of Step 1. We have

$$1 \overset{(4)}{=} \phi(t_0) \overset{(1)}{=} (Q_0, \phi) = \sum a_i (M_i, \phi) \overset{(2),(5)}{=} \sum |a_i| \cdot |(M_i, \phi)|$$
$$\overset{(5)}{\ge} 1/K_0 \sum |a_i| \overset{(3)}{\ge} \|Q_0\|_1 / K_0 \overset{(3)}{\ge} P_{\mathcal{S}}(t_0)/K_0 \,.$$

New results V: a short proof by v. Golitschek (an insight)

We have already mentioned that the Douglas-Dupont-Wahlbin result on the boundedness of the orthogonal spline projector for quasi-uniform partitions Δ) remains valid for any subspace $\mathcal{S}$ such that

$\mathcal{S} \in \Phi_{k,d}(\Delta)$: $\mathcal{S}$ has a good d-conditioned basis (ϕ_i) of finitely k-supported functions.

There were several attempts to prove that, for such spaces $\mathcal{S} \in \Phi_{k,d}(\Delta)$, de Boor's conjecture remains valid without assumption of quasi-uniformness of Δ. However, in 1998, I proved that this is not true.

Theorem [S (1998)]. For $k = 2$, and any $d \ge 16$,

$$\sup_{\Delta} \sup_{\mathcal{S}\in\Phi_{k,d}(\Delta)} \|P_{\mathcal{S}}\|_p = \infty, \qquad \left|\frac{1}{p} - \frac{1}{2}\right| > \frac{3}{\sqrt{d}}\,.$$

On the positive side, I proved that for all such spaces we have L_p-boundedness for p in a small neighbourhood of $p = 2$. (I proved the same result, i.e., L_p-boundedness of $P_{\mathcal{S}}$ for splines in 1994, this general result shows that it has nothing to do with the spline nature.)

Theorem [S (1998)]. For any $k \in \mathbb{N}$, $d \in \mathbb{R}$, $d \ge k$, we have

$$\sup_{\Delta} \sup_{\mathcal{S}\in\Phi_{k,d}(\Delta)} \|P_{\mathcal{S}}\|_p \le c_{k,d} \qquad \left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{1}{2kd^2 \ln d}\,.$$

New results V: a short proof by v. Golitschek (insight)

Actually, the proof by v. Golitchek reveals an additional assumption on $\Phi_{k,d}$-spaces which guarantees uniform boundedness of orhogonal projectors. Namely, it shows that the orthogonal projectors onto the $\Phi_{k,d}$-spaces (with finitely supported and well-conditioned bases) are uniformly bounded for all partitions Δ_N if they are uniformly bounded at the end-points t_0 and t_n of Δ_N.

Theorem [S]. Let a $\mathcal{S}$ satisfies the following properties

$$
\begin{aligned}
&1)\quad \mathcal{S} \in \Phi_{k,d}(\Delta_N) \\
&2)\quad P_{\mathcal{S}}(t_0) := \sup_{\|f\|_\infty \le 1} |P_{\mathcal{S}}(f, t_0)| = K_0 \le c_{k,d}\,.
\end{aligned}
$$

Then the max-norm $\|P_{\mathcal{S}}\|$ is uniformly bounded for all partitions Δ_N.

Remark. There have been several attempts to extend de Boor's conjecture to splines formed by Chebyshev systems other than polynomials. So, this theorem shows a way how this could be achieved. On the other hand, Property 2 above is rather delicate, so it does bot seem to be a simple task.

New results VI: convergence almost everywhere

The max-norm boundeness of $P_{\mathcal{S}} = P_{\Delta_n}$ implies that

$$P_{\Delta_n}(f) \to f \quad \text{in the } L_p\text{-norm}, \qquad 1 \le p \le \infty\,.$$

A natural question whether, for $f \in L_p$ we have pointwise convergence almost everywhere (a.e.) was addressed by Ciesielski (1975) for dyadic partitions and, together with Kamont (1997), for linear splines, however the general case has not been resolved.

Remark. From my result, we have of course not just pointwise but the uniform convergence for continuous functions, however for $f \in L_p$ with $p < \infty$ it is not a trivial question at all as one may judge from the results for the Fourier series.

Theorem [Passenbrunner-S (2014)]. For any k and any sequence of partitions (Δ_n) such that $|\Delta_n| \to 0$, we have

$$f \in L_1 \quad \Rightarrow \quad P_{\Delta_n}(f, x) \to f(x) \quad \text{a.e.}$$

Proof. The proof followed the standard approach of proving a weak $(1,1)$-type estimate for the maximal operator, in fact we proved a bit more, namely that

$$|P_{\Delta_n}(x)| \le c_k M(f, x)$$

where $M(f, x)$ is the Hardy-Littlewood maximal function (which is of the weak $(1,1)$-type).

However, although we relied on the max-norm boundedness of $\|P_{\Delta_n}\|_\infty$, the proof required much more sophisticated estimates of the elements of G^{-1} than those which follow from the previous results.

de Boor's conjecture: a final touch

Conjecture [de Boor (1972)]. For any k, the L_∞-norm of the L_2-projector $P_{\mathcal{S}}$ onto the spline space $\mathcal{S}_k(\Delta_N)$ is bounded independently of Δ_N, i.e.,

$$\sup_{\Delta} \|P_{\mathcal{S}_k(\Delta)}\|_\infty \le c_k\,.$$

Theorem [S (1999)]. This is true indeed.

However, the exact wording of the original conjecture was in terms of the max-norm of the Gramian inverse.

Original conjecture [de Boor (1972)]. For given k and Δ_N, let G be the $N \times N$ matrix whose entries are given by the scalar products (M_i, N_j). Then

$$\sup_{\Delta} \|G^{-1}\| \le \infty.$$

The point is that this was conjectured for any knot-sequence, not just for the one that appears for orthogonal spline-projectors and which assumes that, in terms of the knots for the underlying B-splines, we have

$$t_0 = t_1 = \cdots = t_{k-1}, \qquad t_N = t_{N+1} = \cdots = t_{N+k-1},$$

i.e. with k-multiple end-knots (those knot-sequences are called *complete*). It may seem a tiny difference, but it was not obvious at all whether results for incomplete knot-sequences could be easily derived.

Remark. This mismatch was noticed by de Boor in 2011 when he and I were discussing the issues related to the gap in my arguments for bi-infinite knot-sequences. He was kind enough to reassure me "I DO NOT INTEND TO MAKE THIS AN ISSUE!"

de Boor's conjecture: a final touch

At the end, de Boor himself found how to fill the gap.

Lemma [de Boor, Jia, Pinkus (1982)]. If $B \in \mathbb{R}^{n \times n}$ is invertible and total positive, then, for any integer interval $\boldsymbol{m} \subset \{1, 2, ..., n\}$, so is the principal submatrix $C := B(\boldsymbol{m}, \boldsymbol{m}) \in \mathbb{R}^{m \times m}$ of B involving only the rows and columns of B with index $i \in \boldsymbol{m}$, and

$$0 \le (1)^{i+j} C^{-1}(i,j)(1)^{i+j} B^{-1}(i,j), \qquad i, j \in \boldsymbol{m}\,,$$

and as a consequence

$$\|C^{-1}\| \le \|B^{-1}\|\,.$$

Corollary [de Boor (2011)]. For all $k \in \mathbb{N}$, and all finite knot sequences Δ

$$\|G_\Delta^{-1}\| \le c_k < \infty\,.$$

Proof. Any finite knot sequence Δ can be embedded in a k-complete knot sequence $\widehat{\Delta}$ (in many ways), and, for any such choice, $G_\Delta = G_{\widehat{\Delta}}(\boldsymbol{m}, \boldsymbol{m})$ for some integer interval $\boldsymbol{m}$, hence

$$\|G_\Delta^{-1}\| \le \|G_{\widehat{\Delta}}^{-1}\| \le c_k\,,$$

the latter by my result.

Some open problems

Problem 1. Prove (or disprove) that

$$\sup_{\Delta} \|P_{\mathcal{S}_k(\Delta)}\|_\infty = \mathcal{O}(k)\,.$$

Problem 2. For the uniform partition δ_n, find the order of

$$\|P_{\mathcal{S}_k(\delta_n)}\|_\infty, \qquad \delta = \big(\tfrac{i}{n}\big)_{i=0}^{n}\,.$$

Note that in this case

$$\|G_{\delta_n}^{-1}\|_\infty \sim \Big(\frac{\pi}{2}\Big)^{2k},$$

and also that, in the periodic case,

$$\lim_{k\to\infty} \|P_{\mathcal{S}_k(\delta_n)}\|_\infty = \|P_{T_n}\|_\infty = \mathcal{O}(\ln n)\,.$$

Who is the nice guy?

"I offer the modest sum of $(m - 1972) \times \$10$ to the first person who communicates to me a proof or a counterexample (but not both) of his or her making of this conjecture (known to be true for $k = 2$ or $k = 3$). Here m is the year (A.D.) of such communication."

I communicated my proof to de Boor in the year $m = 1999$ (A.D.), and, after three revisions, he accepted it, so according to his formula of financial obligations I expected to receive

$$(m - 1972) \times \$10 = \$270$$

However, the check was written for \$540.

The question:

Why was the sum twice as much as promised? (so that the "modest sum" turned out to be not that modest at all). Is it me who is the nice guy, or is it de Boor? ("but not both")

And **the answer** is (the prize goes to): the nice guy is Rong Quang Jia.

Here is de Boor's explanation:

"...well, about 5-6 years ago, I stated at some occasion that, given inflation and all that, I was doubling the rate. In fact, Jia was kind enough to remind me of that."

Thank you all,

and thank you Carl - I have very much enjoyed and appreciated knowing you for all those years and I am very grateful for everything you have given to me.

Happy Birthday!