The Big Bang Theory of Multivariate Splines†

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Workshop on Spline Approximation

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Multivariate splines…

Appear in many different areas under different incarnations:

- Approximation Theory: Box Spline, Simplicial Splines
- Enumerative Combinatorics: Partition Functions
- Representation Theory: Schur Functions, MacDonald Polynomials
- Symplectic Geometry: Moment Maps
- …

Different Setups → different objects

For example:

- Box Splines – the translation group
- Schur Function – the reflection group

Unifying theory?

What is therefore the common ground in all these application domains?

There is exactly one object that underlies the different constructions:

The Truncated Power!!!

On this talk:

- Goal: understanding truncated powers
- Limitation: Graph Case only
- Setup: G is a connected graph, $n+1$ vertices: $[0{:}n]$
- Fundamental notions:

a) (Maximal) Parking Functions: $S_{max}(G)$

b) The $\mathcal{P}$-polynomials: $\mathrm{soc}\big(\mathcal{P}(G)\big)$

c) The $\mathcal{D}$-polynomials: $\mathrm{soc}\big(\mathcal{D}(G)\big)$

d) The fundamental quantity:

$$\mathrm{Q}(G) \coloneqq \#\mathrm{S}_{max}(G) = \dim \mathrm{soc}\big(\mathcal{P}(G)\big) = \dim \mathrm{soc}(\mathcal{D}(G))$$

Example: G is a complete graph with arbitrary multiplicities:

$$\mathrm{Q}(G) = n!$$

Previous state-of-the-art:

- $\mathcal{P}$-polynomials: explicit, easy to understand, many bases, but no canonical
- Parking Functions: is 'a gimmick', mostly combinatorial
- $\mathcal{D}$-polynomials:

  a) Too complicated for any intrinsic understanding
  b) No direct connection with parking functions
  c) Understood via duality with $\mathcal{P}$-polynomials

- Conclusions:

  1) Matter is doomed: no canonical basis for $\mathcal{D}$-polynomials
  2) Matter is doomed: no any basis for them
  3) Matter is doomed: truncated powers are hopeless
  4) Matter is doomed: one cannot come with a unifying theory

The Big Bang Theory

Can be described in many different ways:

- Ideal Theory: writing the “torsion ideal” (de Boor, DeVore, Höllig) as an intersection of complete intersection ideals
- Jeffrey – Kirwan decompositions (Brion – Vergne): finding a canonical decomposition for the point evaluation
- Convex Geometry: Explicit computations of volume defined by incidence matrices
- Truncated powers: resolving truncated powers based on their restrictions to the positive octant
- Partition Functions: constructing truncated powers from partition functions

Finally we start (with previous state of the art):

G is a connected graph with vertex set $[0\!:n]$:

- $e_0 := 0 \in \mathbb{R}^n$, $e_i,\ i \in [1\!:n]$, is the standard basis.
- vertex $i \quad \longleftrightarrow \quad e_i$
- An edge $x \in G$:

  $x\!:\ i \to j \quad \longleftrightarrow \quad x = e_j - e_i$
- $\mathrm{N} := \#G$
- $\mathbb{B}(G) := \{\text{The spanning trees of } G\}$
- $\Pi := \mathbb{R}[t(1), \ldots, t(n)]$
- $G \ni x \quad \longleftrightarrow \quad p_x(t) := t(j) - t(i)$

  $G \supset Y \quad \longleftrightarrow \quad p_Y(t) := \Pi_{x \in Y}\, p_x$

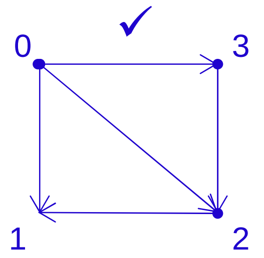
Edges are anti-matter - they only appear as differential operators:

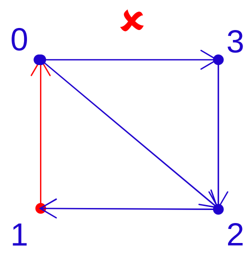
$$p_x(D) = D_x$$

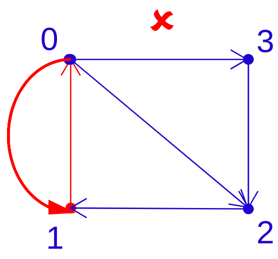
Two important sets: orienting the graph

$\overline{\mathrm{O}}(G) \coloneqq \{\text{All the acyclic orientations of } G\}$.

Example:

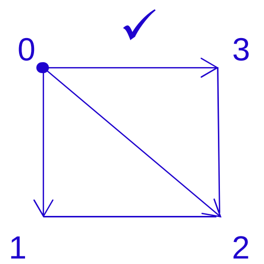


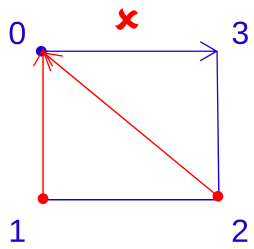


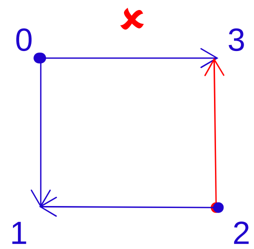


$\mathrm{O}(G) \coloneqq \{\vec{G} \in \overline{\mathrm{O}}(G) : 0 \text{ is the only source of } \vec{G}\}$.

Example:







Parking Functions: $S_{\max}(G)$

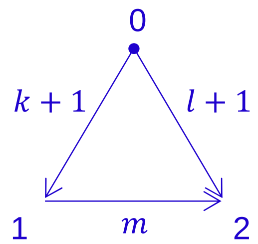
Many equivalent definitions. One here follows [Benson, Chakraparty, Tetali, 2010]

- Example: complete graph, $n = 2$.

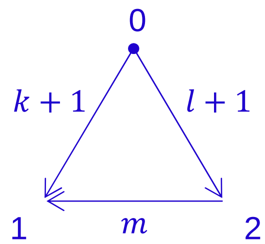
$$\mathcal{O}(G) \ni \vec{G} \mapsto s \coloneqq s(\vec{G}) \in S_{\max}(G)$$

$$s\colon [1{:}n] \to \mathbb{Z}_+$$

$$s(i) \coloneqq \#\{\text{inflow edges at } i\} - 1$$



$$s = (k, l+m)$$
$$m_s\colon t \mapsto t^{(k,l+m)}$$



$$s = (k+m, l)$$
$$m_s\colon t \mapsto t^{(k+m,l)}$$

The $\mathcal{P}$-polynomials

- What is the role of the $\mathcal{P}$-polynomials?
  - Take any $p \in soc\big(\mathcal{P}(G)\big)$: TP is truncated power.
  - Then $p(D)\mathrm{TP}$ is piecewise-constant.

Truncated Powers

- Bijection:

$$\overline{\mathrm{O}}(G) \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{TP}(G)$$

$$\vec{G} \mapsto \mathrm{TP}_{\vec{G}} : \mathbb{R}^n \to \mathbb{R}_+$$

- Properties of truncated powers:

  1) Supported on the positive hull of $\vec{G}$

  2) Piecewise-polynomials: each polynomial is homogeneous of degree $N - n$

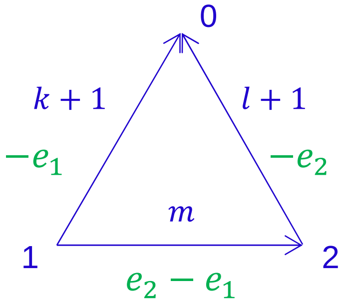
- Definition of $\mathcal{D}$-polynomials:

  - $\mathrm{soc}\big(\mathcal{D}(G)\big) \coloneqq$ span$\{\text{of the polynomials in the local structure of } \mathrm{TP}_{\vec{G}}\}$

  - Comment: definition depends on G only

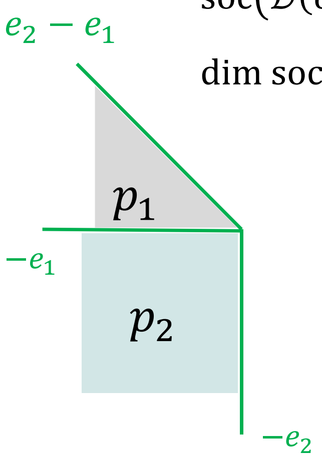
Truncated Powers - Examples

- Example 1: $n = 2$.

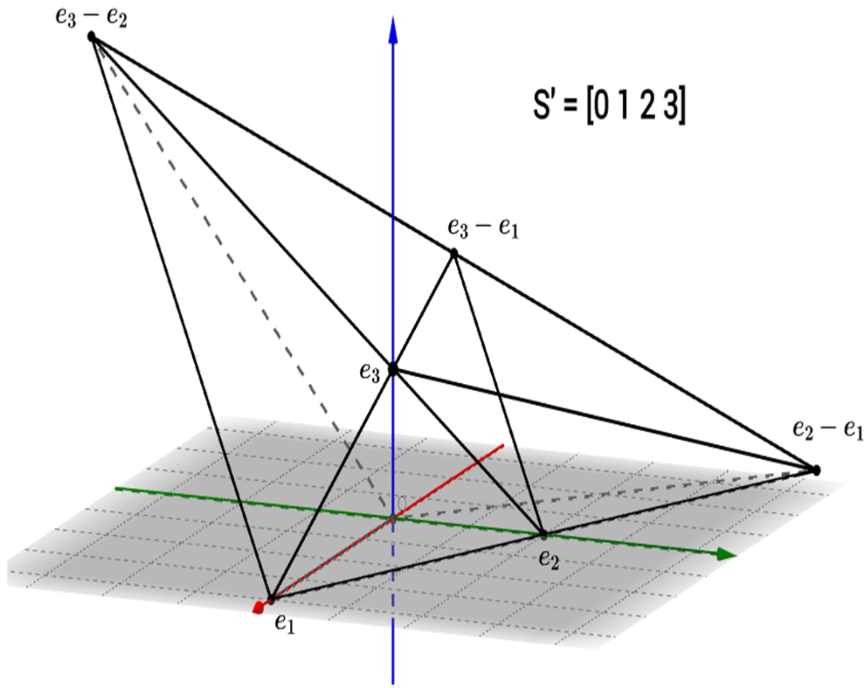


$\mathrm{soc}\big(\mathcal{D}(G)\big) = \text{span}\{\, p_1, p_2\}$

$\dim \mathrm{soc}\big(\mathcal{D}(G)\big) = 2$



- Example 2: complete graph, $n = 3$.



Truncated Powers – cont’d

End of previous state-of-the-art!!

The Big Bang Theory: the main achievement

- $\mathcal{D}$-polynomials are *simple,* simpler than $\mathcal{P}$-polynomials:
  - There is a simple, canonical basis for the $\mathcal{D}$-polynomials.
  - The basis has a natural bijection with $\mathrm{O}(G)$ hence with $\mathrm{S}_{\max}(G)$.
  - Each basis polynomial $\mathrm{M}_s,\ s \in \mathrm{S}_{\max}(G)$ is nicknamed flow polynomial.

Seven major characteristics of flow polynomials

1) Evolve from the parking functions
2) Dual to the parking functions
3) Involve only positive integer coefficients
4) Lead to complete intersection decomposition of the torsion ideal
5) Recorded by truncated power in the positive octant
6) Represented via partition function of reduced graphs
7) Satisfy heredity analogous to truncated powers

FP characteristic: Only positive integer coefficients

Each flow polynomial:

$$\mathrm{M}_s(t) = \sum_{|\alpha|=N-n} c(G,s,\alpha)\ [t^\alpha], \qquad s \in \ \mathrm{S}_{max}(G)$$

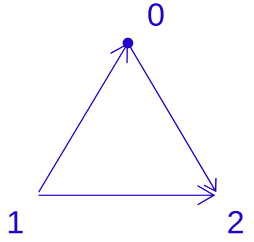
$$[t^\alpha] \ \coloneqq \ \frac{t^\alpha}{\alpha!} \ = \ \frac{t(1)^{\alpha(1)} \cdots t(n)^{\alpha(n)}}{\alpha(1)! \cdots \alpha(n)!}$$

Then **each** $c(G,s,\alpha)$ is a non-negative integer.

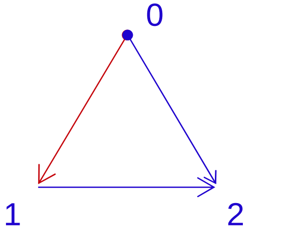
FP characteristic: truncated power in the positive octant

$\vec{G} \in \overline{\mathrm{O}}(G)$:

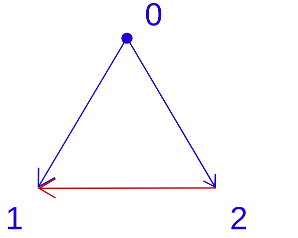
- If $\vec{G} \notin \mathrm{O}(G)$, then $\mathrm{TP}_{\vec{G}}|_{\mathbb{R}^n_+} = 0$.
- If $\vec{G} \in \mathrm{O}(G)$, $s = s(\vec{G})$, then $\mathrm{TP}_{\vec{G}}|_{\mathbb{R}^n_+} = \mathrm{M}_s$.
- Examples:



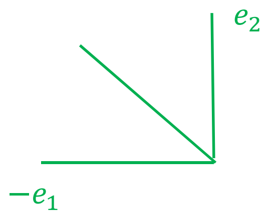
orientation: 1 – 0 – 2

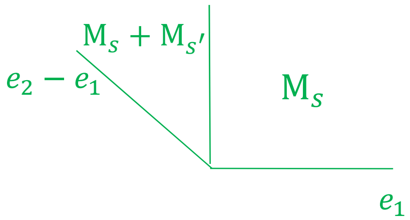


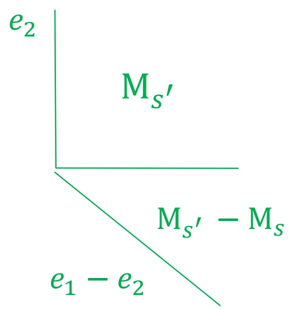
orientation: 0 – 1 – 2



orientation: 0 – 2 – 1







FP characteristic: Dual to the parking functions

Duality with the parking function:

$$s, s' \in \mathrm{S}_{max}(G),$$

$$\mathrm{D}^{s'}\,\mathrm{M}_s = \begin{cases} 1, & s = s', \\ 0, & s \neq s'. \end{cases}$$

FP characteristic: heredity analogous to truncated powers

$x \in \vec{G}, \vec{G} \in \mathrm{O}(G), s = s(\vec{G})$:

- If $\vec{G}\backslash x$ has a source $i \in [1{:}n]$, then

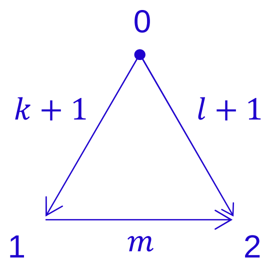
$$D_x \, \mathrm{M}_s = 0.$$

- Otherwise,

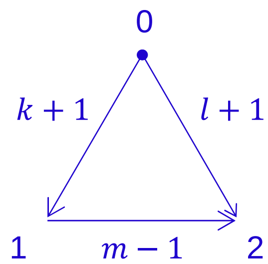
$$D_x \, \mathrm{M}_s = \mathrm{M}_{s'},$$

with s' the parking function of $\vec{G}\backslash x$.

- Example:



$s = (k, l + m)$



$s' = (k, l + m - 1)$

$$D_{e_2 - e_1} \, \mathrm{M}_s = \mathrm{M}_{s'}$$

FP characteristic: Complete intersection decompositions

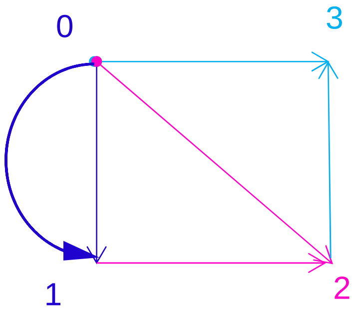
$\vec{G} \in \mathrm{O}(G)$, s the parking function,

$$X_i \coloneqq \{\text{all the edges of } \vec{G} \text{ that flow into } i\}, \qquad i \in [1:n]$$

Then, up to normalization, M_s is the only polynomial s.t.

- It is homogeneous of degree $N - n$.
- $p_{X_i}(\mathrm{D})\ \mathrm{M}_s = 0, \qquad i \in [1:n]$.

Example:



$$X_1 = (e_1, e_1)$$

$$X_2 = (e_2, e_2 - e_1)$$

$$X_3 = (e_3, e_3 - e_2)$$

FP characteristic: Partition functions of reduced graphs

How to compute the flow polynomials?

$$\mathrm{M}_s(t) = \sum_{|\alpha|=N-n} c(G,s,\alpha)\ \ [t^{\alpha}],\ \ s = s(\vec{G}), \vec{G} \in \mathrm{O}(G)$$

$$c(G,s,\alpha) = ?$$

- Step I: remove from $\vec{G}$ all the edges connected to 0. Get $\mathrm{rd}(\vec{G})$.
- Step II: $\mathrm{tp}_{\mathrm{rd}(\vec{G})}$ is the discrete truncated power associated with $\mathrm{rd}(\vec{G})$.
  - If Y is the incidence matrix of $\mathrm{rd}(\vec{G})$, then

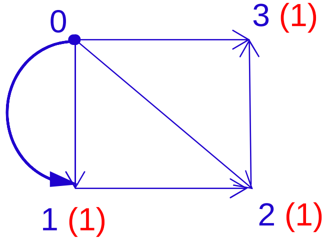
$$tp_{rd(\vec{G})}(\alpha) \coloneqq \#\{\beta \in \mathbb{Z}_+^k : Y\beta = \alpha\}.$$

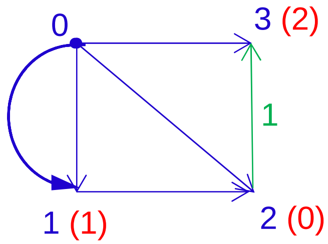
  - Then,

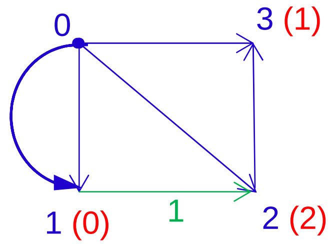
$$c(G,s,\alpha) = \mathrm{tp}_{\mathrm{rd}(\vec{G})}(\alpha - s).$$

FP characteristic: Evolution from parking functions

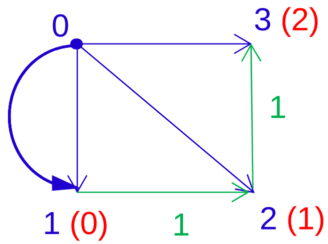
Example:

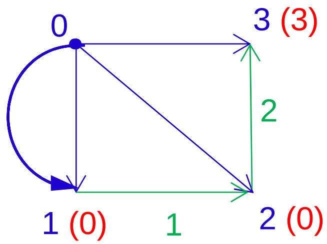






$$[t^{(1,1,1)}] \quad + \quad [t^{(1,0,2)}] \quad + \quad [t^{(0,2,1)}]$$





$$+ \quad [t^{(0,1,2)}] \quad + \quad [t^{(0,0,3)}] \quad = \quad \mathrm{M}_s(t)$$

The end of the beginning

- All good things eventually come to an end.

- This is the end of the beginning of the Big Bang Theory.

Thank you!