From B-splines to Box Splines — the Insight and Influence of Carl de Boor's Work

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Polynomial Splines

Let k be a positive integer. The linear space of all polynomials of order $\leq k$ is denoted by $\mathbb{P}_k$.

A real sequence $\mathbf{t} := (t_i)_{i\in\mathbb{Z}}$ is called a *knot sequence* if $t_i \leq t_{i+1}$ for all $i \in \mathbb{Z}$. If $t_{i-1} < t_i = \cdots = t_{i+m-1} < t_{i+m}$ and $\tau = t_i$, then we say that the *multiplicity* of τ in $\mathbf{t}$ is m.

Let $\mathbf{t}$ be a knot sequence with $t_i < t_{i+k}$ for all $i \in \mathbb{Z}$. A function f from $\mathbb{R}$ to $\mathbb{R}$ is called a *spline* of order k with knot sequence $\mathbf{t}$ if it satisfies the following two conditions:

1. $f|_{(t_j,t_{j+1})} \in \mathbb{P}_k|_{(t_j,t_{j+1})}$ for all j with $t_j < t_{j+1}$;
2. If τ is a knot in $\mathbf{t}$ with multiplicity m, then $D^r f(\tau-) = D^r f(\tau+)$ for $r = 0, 1, \ldots, k-m-1$, where $D^r f$ denotes the rth derivative of f.

The collection of all splines of order k with knot sequence $\mathbf{t}$ is denoted by $\mathbb{S}_{k,\mathbf{t}}$.

B-splines

B-splines were introduced by Curry and Schoenberg in 1947.

Let $\mathbf{t}$ be a given knot sequence with $t_i < t_{i+k}$ for all $i \in \mathbb{Z}$. The *B-splines* of order k for $\mathbf{t}$ are given by

$$B_i(x) := B_{i,k,\mathbf{t}}(x) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - x)_+^{k-1}, \quad x \in \mathbb{R},$$

where $[t_i, \ldots, t_{i+k}]$ denotes the kth order divided difference at the points $t_i, \ldots, t_{i+k}$, and $u_+ := \max\{u, 0\}$.

The kth *divided difference* of a function g at the points $\tau_i, \ldots, \tau_{i+k}$ is the leading coefficient of the polynomial of order $k+1$ which agrees with g at the points $\tau_i, \ldots, \tau_{i+k}$. It is denoted by

$$[\tau_i, \ldots, \tau_{i+k}]g.$$

See Carl de Boor, A Practical Guide to Splines, Springer-Verlag, 1978.

Properties of B-splines

It is easily seen that each B_i lies in $\mathbb{S}_{k,\mathbf{t}}$. Moreover, by using the properties of divided difference we have

$$B_i(x) = 0 \quad \text{for} \quad x \notin [t_i, t_{i+k}]$$

and

$$\sum_{i \in \mathbb{Z}} B_i(x) = 1 \quad \forall\, x \in \mathbb{R}.$$

Theorem (Curry and Schoenberg, 1966)

Any function $f \in \mathbb{S}_{k,\mathbf{t}}$ can be uniquely represented as

$$f = \sum_{i \in \mathbb{Z}} a_i B_{i,k,\mathbf{t}},$$

where $a_i \in \mathbb{R}$ for $i \in \mathbb{Z}$.

Quasi-interpolants

The above theorem can be proved by using quasi-interpolants.

Theorem (de Boor and Fix, 1973)

Let λ_i *be the linear functional given by the rule*

$$\lambda_i f = \sum_{r=1}^{k} (-D)^{r-1} \psi_i(\xi) D^{k-r} f(\xi),$$

with $\psi_i(t) := (t_{i+1} - t) \cdots (t_{i+k-1} - t)/(k-1)!$*, and* ξ *some arbitrary point in the open interval* (t_i, t_{i+k})*. Then*

$$\lambda_i B_j = \delta_{ij} \quad \forall\, i, j \in \mathbb{Z},$$

where $\delta_{ij} = 1$ *for* $i = j$*, and* $\delta_{ij} = 0$ *for* $i \neq j$*.*

The quasi-interpolation scheme made a profound impact on many areas of analysis beyond spline theory.

Discrete B-splines

Suppose $\mathbf{s} = (s_i)_{i\in\mathbb{Z}}$ is a subsequence of $\mathbf{t}$. Then any B-spline $B_{j,k,\mathbf{s}}$ is a linear combination of the B-splines $B_{i,k,\mathbf{t}}$:

$$B_{j,k,\mathbf{s}} = \sum_{i\in\mathbb{Z}} \beta_{j,k,\mathbf{s},\mathbf{t}}(i) B_{i,k,\mathbf{t}}.$$

For each $j \in \mathbb{Z}$, the function that maps $i \in \mathbb{Z}$ to $\beta_{j,k,\mathbf{s},\mathbf{t}}(i)$ is called a *discrete B-spline*. Clearly, $\beta_j(i) = \delta_{ij}$ if $\mathbf{s} = \mathbf{t}$.

Suppose that the knot sequence $\mathbf{s}$ is formed by dropping an entry t_z from $\mathbf{t}$, that is, $s_i = t_i$ for $i < z$ and $s_i = t_{i+1}$ for $i \geq z$. Then we have $\beta_j(i) = \delta_{ij}$ for $j + k \leq z - 1$ and $\beta_j(i) = \delta_{i-1,j}$ for $j \geq z$. In the case $j < z \leq j + k$ we have $\beta_j(i) = 0$ for $j < i - 1$ or $j > i$. Moreover,

$$\beta_{i-1}(i) = \frac{t_{i+k} - t_z}{t_{i+k} - t_i} \quad \text{and} \quad \beta_i(i) = \frac{t_z - t_i}{t_{i+k} - t_i}.$$

Total Positivity

Theorem (Jia 1983)

Let k be a positive integer, let $\mathbf{t} = (t_i)_{i\in\mathbb{Z}}$ be a knot sequence with $t_i < t_{i+k}$ for all $i \in \mathbb{Z}$, and let $\mathbf{s}$ be a subsequence of $\mathbf{t}$. Suppose $I = (i_1, \ldots, i_m) \in \mathbb{Z}^m$ and $J = (j_1, \ldots, j_m) \in \mathbb{Z}^m$ are two sequences such that $i_1 < \ldots < i_m$ and $j_1 < \ldots < j_m$. Then

$$\det Q(I, J) \geq 0,$$

where $Q(I, J)$ denotes the matrix $(\beta_{j,k,\mathbf{s},\mathbf{t}}(i))_{i\in I, j\in J}$.

We abbreviate $\beta_{j,k,\mathbf{s},\mathbf{t}}(i)$ to $\beta_j(i)$. If $\mathbf{s} = \mathbf{t}$, or if $\mathbf{s}$ is obtained from dropping one knot from $\mathbf{t}$, then

$$Q(I, J) = \beta_{j_1}(i_1) \cdots \beta_{j_m}(i_m) \geq 0.$$

For the general case we use induction on the number of new knots in $\mathbf{t}$ but not in $\mathbf{s}$ to finish the proof.

Spline Collocation Matrix

The above theorem can be used to give an easy proof for total positivity of the spline collocation matrix. Write $B_j = B_{j,k,\mathbf{t}}$ for $j \in \mathbb{Z}$. Suppose $x_1 < \cdots < x_n$. If $I = (1, \ldots, n)$ and $J = (j_1, \ldots, j_n)$ with $j_1 < \ldots < j_n$, then

$$\det\big(B_j(x_i)\big)_{i \in I, j \in J} \geq 0.$$

This result was originally established by Karlin in 1968.

Let $\mathbf{s}$ be a knot refinement of $\mathbf{t}$ such that each x_i ($i = 1, \ldots, n$) appears exactly k times in $\mathbf{s}$. Then

$$B_{j,k,\mathbf{t}} = \sum_{i \in \mathbb{Z}} \beta_{j,k,\mathbf{t},\mathbf{s}}(r) B_{r,k,\mathbf{s}}.$$

Suppose $s_{r_i - 1} < x_i = s_{r_i} = \cdots = s_{r_i+k-1} < s_{r_i+k}$. Then $B_{j,k,\mathbf{t}}(x_i) = \beta_{j,k,\mathbf{t},\mathbf{s}}(r_i)$. So the desired result can be derived from the corresponding theorem on discrete B-splines.

Sign Changes

Given a vector $a = (a_1, \ldots, a_n)$ of real numbers, we use $S^-(a)$ to denote the sigh changes in the sequence $a_1, \ldots, a_n$.

Theorem (Lane and Riesenfeld 1983)

If $f = \sum_i a_i B_{i,k,\mathbf{t}} = \sum_i b_i B_{i,k,\mathbf{s}}$, where $\mathbf{s}$ is a refinement of $\mathbf{t}$, then $S^-(b) \leq S^-(a)$.

The following proof was given by de Boor and DeVore in 1985. It suffices to show that the statement is true for the insertion of a single additional knot. Then $B_{i,k,\mathbf{t}} = \alpha_i B_{i,k,\mathbf{s}} + \beta_i B_{i+1,k,\mathbf{s}}$ with $\alpha_i \geq 0$ and $\beta_i \geq 0$ for each i. It follows that

$$b_i = \beta_{i-1} a_{i-1} + \alpha_i a_i.$$

Thus b_i will have the same sign as either a_{i-1} or a_i. Therefore,

$$S^-(a) = S^-(\ldots, a_{i-1}, b_i, a_i, b_{i+1}, a_{i+1}, \ldots) \geq S^-(b).$$

Variation Diminishing Property

Given a function f, we use $S^-(f)$ to denote the number of sign changes of f.

The original result on variation diminishing was stated by Schoenberg in 1967 and proved by Karlin in 1968. It says that

$$S^-\left(\sum_i a_i B_{i,k,\mathbf{t}}\right) \leq S^-(a)$$

for any finite sequence a.
Indeed, let $f := \sum_i a_i B_{i,k,\mathbf{t}}$ and suppose $x_1 < \cdots < x_{r+1}$ are so chosen that $S^-(f) = S^-(f(x_1), \ldots, f(x_{r+1}))$. Let $\mathbf{s}$ be a knot refinement of $\mathbf{t}$ such that each x_i $(i = 1, \ldots, r+1)$ appears exactly k times in $\mathbf{s}$. Then f can be written as $\sum_j b_j B_{j,k,\mathbf{s}}$. If $s_{j_i - 1} < x_i = s_{j_i} = \cdots = s_{j_i+k-1} < s_{j_i+k}$, then $f(x_i) = b_{j_i}$. Hence $S^-(f) \leq S^-(b)$. But $S^-(b) \leq S^-(a)$. This shows

$$S^-(f) \leq S^-(a).$$

Zeros of Splines

A point y is a *zero* of f with *multiplicity* m if $f^{(j)}(y) = 0$ for $j = 0, \ldots, m-1$, $f^{(m)}$ is continuous at y and $f^{(m)}(y) \neq 0$.

Theorem (Goodman1994)

Let $f = \sum_{j=1}^{n} a_j B_{j,k,\mathbf{t}}$. Suppose that for every $x \in (t_1, t_{n+k})$ there exists some j with $a_j \neq 0$ and $t_j < x < t_{j+k}$. Then $Z(f) \leq S^{-}(a)$, where $Z(f)$ denotes the total number of zeros of f in (t_1, t_{n+k}), counting multiplicities.

Let x be a zero of f in (t_1, t_{n+k}) of multiplicity m. Then the multiplicity of x in the knot sequence $\mathbf{t}$ does not exceed $k - m - 1$. Insert x repeatedly into the knot sequence until x has multiplicity $k - m - 1$ in the resulting knot sequence. Also insert knots in $x - h$ and $x + h$, each with multiplicity k, where $h > 0$ is sufficiently small. Recall that knot insertion does not increase the number of sign changes. We can use the quasi-interpolants of de Boor and Fix to finish the proof.

B-spline Collocation Matrix

Let k and n be positive integers, $\mathbf{t} := (t_i)_{i\in\mathbb{Z}}$ a knot sequence, and $\mathbf{z} = (z_i)_{1\le i\le n}$ a nondecreasing sequence of real numbers. It is required that any point appear at most k times totally in $\mathbf{z}$ and $\mathbf{t}$. For $1 \le i \le n$ we use μ_i to denote the number of $j < i$ for which $z_j = z_i$.

Theorem (de Boor 1976)

Let $I := (1, \dots, n)$, $J = (j_1, \dots, j_n)$ a sequence of integers with $j_1 < \cdots < j_n$, and

$$A(I, J) := \big(D^{\mu_i} B_j(z_i)\big)_{i\in I, j\in J}.$$

Then $\det A(I, J) \ge 0$ with strict inequality if and only if $t_{j_i} < z_i < t_{j_i+k}$ for all $i = 1, \dots, n$.

The last statement of this theorem can be proved by considering zeros of splines.

Spline Interpolation

Let $\mathbf{t} = (t_i)_{1 \le i \le n}$ and $\mathbf{z} = (z_i)_{1 \le i \le n+k}$ be nondecreasing sequences of real numbers. It is assumed that any point appear at most k times totally in $\mathbf{z}$ and $\mathbf{t}$. The following result was first obtained by Schoenberg and Whitney in 1953 for the case where $\mathbf{z}$ is a strictly increasing sequence. For $1 \le i \le n$ we use μ_i to denote the number of $j < i$ for which $z_j = z_i$.

Theorem (de Boor 1976)

The interpolations problem

$$\begin{cases} f \in \mathbb{S}_{k,\mathbf{t}} \\ D^{\mu_i} f(z_i) = y_i, i = 1, \dots, n+k, \end{cases}$$

has a unique solution for arbitrary $(y_i)_{1 \le i \le n+k}$ *if and only if* $z_i < t_i < z_{i+k}$ *for all* $i = 1, \dots, n$.

Bounds for Least-squares Approximation by Splines

Let $\mathbf{t} = (t_i)_{1\le i\le n+k}$ be a finite knot sequence. Given $f \in L_\infty[t_1, t_{n+k}]$, we use $L_{\mathbf{t}} f$ to denote the least-squares approximation to f from $\mathbb{S}_{k,\mathbf{t}}$. A basic question is whether the linear projector $L := L_{\mathbf{t}}$ is bounded on $L_\infty[t_1, t_{n+k}]$.

This problem can be reformulated as follows. For $f, g \in L_2(a, b)$ with $a := t_1$ and $b := t_{n+k}$, we use $\langle f, g\rangle$ to denote the integral $\int_a^b f(x)g(x)\,dx$. Let

$$M_i := kB_i/(t_{i+k} - t_i), \quad i = 1, \ldots, n.$$

Then $\int_a^b M_i(x)\,dx = 1$. We use $G_{\mathbf{t}}$ to denote the matrix $(\langle M_i, B_j\rangle)_{1\le i,j\le n}$.

In 1973 de Boor raised the conjecture

$$\sup_{\mathbf{t}} \|(G_{\mathbf{t}})^{-1}\|_\infty < \infty.$$

This is equivalent to saying that $L_{\mathbf{t}}$ is bounded on $L_\infty[t_1, t_{n+k}]$.

Confirmation of de Boor's Conjecture

Theorem (Shadrin 2001)

For all $k \in \mathbb{N}$,

$$s_k := \sup_{\mathbf{t}} \|G_{\mathbf{t}}^{-1}\|_\infty < \infty,$$

with the supremum taken over all finite knot sequence $\mathbf{t}$ that are k-complete, meaning that the first and the last knot appear with maximal multiplicity k.

Shadrin's proof is rather long and complicated. A short and simple proof was given by Golitschek recently in 2014.

Shadrin's Theorem is proved only for k-complete finite sequences hence says nothing about principal submatrices of $G_{\mathbf{t}}$, since there is no reason for any finite section of $\mathbf{t}$ to be k-complete. Also, it is not clear how the result can be extended to (bi)infinite knot sequences.

These problems were settled by de Boor himself in 2012.

Totally Positive Matrices

Shadrin's Theorem together with the following lemma shows $\sup_{\mathbf{t}} \|(G_{\mathbf{t}})^{-1}\|_\infty < \infty$.

Lemma (de Boor, Jia, and Pinkus 1982)

If $B \in \mathbb{R}^{n\times n}$ is totally positive and invertible, then for any integer interval $\mathbf{m} \subseteq \{1, 2, \ldots, m\}$, so is the principal submatrix $C := B(\mathbf{m}, \mathbf{m}) \in \mathbb{R}^{m\times m}$, and

$$0 \leq (-1)^{i+j} C^{-1}(i,j) \leq (-1)^{i+j} B^{-1}(i,j), \quad i, j \in \mathbf{m}.$$

The (bi)infinite case is settled in the following theorem.

Theorem (de Boor 2012)

Let $A \in \mathbb{R}^{I\times I}$ with I equals $\mathbb{N}$ or $\mathbb{Z}$, and assume that A is totally positive and banded. If for some positive number s and all finite integer intervals $\mathbf{m} \subset I$, the principal submatrix $A_{\mathbf{m}} := A(\mathbf{m}, \mathbf{m})$ is invertible and $\|(A_{\mathbf{m}})^{-1}\| \leq s$, then A is invertible as a linear map on $\ell_\infty(I)$, and $\|A^{-1}\|_\infty \leq s$.

Multivariate Splines

In his talk "The way things were in multivariate splines: A personal view" presented in 2009, de Boor gave a personal account of his encounters with multivariate splines during their early history.

In 1960, as a research assistant for G. Birkhoff, de Boor was aware that Birkhoff and H. Garabedian had developed a scheme for interpolation to data on a rectangular grid. Then de Boor pointed out that the same scheme could be achieved by using the tensor-product of univariate cubic spline interpolation. The resulting surface will be C^2 rather than just C^1. This is now known as bicubic spline interpolation, and has become a mainstay in the construction of smooth interpolants to gridded data.

Multivariate B-splines

In his talk given at the second Texas conference in 1976, on the central role played by B-splines in the univariate spline theory, de Boor finished with a brief discussion of Schoneberg's multivariate B-splines.

In 1965, in a letter to Phil Davis, I. J. Schoenberg sketched a bivariate quadratic B-spline. This geometric construction preceded the later development of polyhedral splines.

In 1978, C. A. Micchelli constructed simplex splines and obtained a recurrence formula for simplex splines. In the spirit of Micchelli's view of Schoenberg's multivariate B-spline, for a convex body B in $\mathbb{R}^n$ and a linear map P from $\mathbb{R}^n$ to $\mathbb{R}^s$, the corresponding B-spline M_B is defined as the distribution on $\mathbb{R}^s$ given by

$$M_B(\phi) := \int_B \phi \circ P, \quad \text{all test functions } \phi.$$

Polyhedral Splines

Under the assumption that B is a convex polytope, it can be shown that M_B is indeed a piecewise polynomial. Carl de Boor and Klaus Höllig called M_B a polyhedral spline.

Thus Schoenberg's B-spline became a simplex spline. In early 1980's W. Dahmen and C. A. Micchelli were engaged in an extensive study of simplex splines. In particular, Dahmen introduced multivariate truncated powers, which can be regarded as cone splines.

In 1980 de Boor and DeVore had a discussion on the approximation order achieved by a space of piecewise polynomials on a partition. They suddenly realized that Courant's hat function is the 2-dimensional (skewed) shadow of a 3-cube. Some other finite elements could also be obtained as shadows of higher-dimensional cubes. This motivated them to introduce box splines (cube splines).

Box Splines

Given an $s \times n$ real matrix X, the box spline M_X associated with X is the distribution given by the rule

$$\phi \mapsto \int_{[0,1)^n} \phi(Xt)\, dt \quad \text{for } \phi \in C(\mathbb{R}^s).$$

Suppose $X = [x^1, \ldots, x^n]$. We assume that X is of full rank and $x^j \in \mathbb{Z}^s \setminus \{0\}$ for $j = 1, \ldots, n$. The box spline M_X is nonnegative on $\mathbb{R}^s$ and its support is the zonotope

$$[X] := \Big\{ \sum_{j=1}^{n} x^j t_j : 0 \leq t_j \leq 1 \Big\}.$$

Moreover, it is a piecewise polynomial.

C. de Boor, K. Hölliog, and S. Riemenschneider, Box Splines, Springer-Verlag, 1993.

The Influence of Box Splines

The study of box splines made a profound impact on many areas of mathematics.

Box splines are piecewise polynomials, so they provide a powerful tool for approximation and interpolation. Their study led to comprehensive research on approximation by scaled shift-invariant spaces.

Box splines are refinable. Thus they are suitable for subdivision schemes and wavelet construction. Subdivision schemes play a vital role in computer aided geometric design. The wavelets induced by box splines are useful for numerical analysis.

From their construction, box splines, or more generally, polyhedral splines, inherit certain nice algebraic and geometric properties. Indeed, box splines have found some interesting applications in algebra, geometry, and combinatorics.

The Linear Algebra of Box Spline Spaces

The multi-integer shifts of a box spline form a partition of unity:

$$\sum_{j \in \mathbb{Z}^s} M_X(\cdot - j) = 1.$$

The sequence $(M_X(\cdot - j))_{j \in \mathbb{Z}^s}$ is said to be (globally) *linearly independent* if $\sum_{j \in \mathbb{Z}^s} c(j) M_X(\cdot - j) = 0$ implies $c(j) = 0$ for all $j \in \mathbb{Z}^s$. The sequence is *locally linearly independent* if, for any bounded open subset G of $\mathbb{R}^s$, all shifts of M_X having some support in G are linearly independent there.

Theorem (Dahmen–Micchelli 1983-5, Jia 1984-5)

The following statements are equivalent:

1. *The shifts of M_X are linearly independent.*
2. *The shifts of M_X are locally linearly independent.*
3. *All bases in X have determinant ± 1.*

Polynomials in Box Spline Spaces

For a vector y in $\mathbb{R}^s$, the directional derivative operator D_y is defined as $D_y f := \lim_{t \to 0}[f(\cdot + ty) - f]/t$. For a multiset Y of vectors in $\mathbb{R}^s$, we define $D_Y := \prod_{y \in Y} D_y$. Let $\ker(D_Y)$ be the space of all smooth functions f on $\mathbb{R}^s$ such that $D_Y f = 0$.

We view X as a multiset of vectors in $\mathbb{R}^s$. Let $\mathcal{A}(X)$ be the collection of all smallest subsets Y of X for which $X \setminus Y$ does not span $\mathbb{R}^s$. Define

$$D(X) := \cap_{Y \in \mathcal{A}(X)} \ker(D_Y).$$

Then the box spline M_X is piecewise in $D(X)$. Further, $D(X)$ is the space of all polynomials generated by shifts of M_X.

Theorem (Dahmen and Micchelli 1985)

The dimension of $D(X)$ is equal to the number of bases of $\mathbb{R}^s$ in X. It is also equal to the volume of the zonotope $Z[X]$.

Approximation Power of Box Splines

Given a compactly supported function $\phi \in L_p(\mathbb{R}^s)$, $1 \le p \le \infty$, we use $\mathbb{S}(\phi)$ to denote the space of functions of the form $\sum_{j\in\mathbb{Z}^s} b(j)\phi(\cdot - j)$. For $h > 0$, let σ_h be the scaling operator given by $\sigma_h f := f(\cdot/h)$.

Theorem

If $\sum_{j\in\mathbb{Z}^s} q(j)\phi(\cdot - j) = q$ for all polynomials q of degree at most $k-1$, then $\mathbb{S}(\phi)$ provides L_p-approximation order k, that is, for every sufficiently smooth function f in $L_p(\mathbb{R}^s)$,

$$\inf_{g\in\sigma_h(\mathbb{S}(\phi))} \{\|f-g\|_{L_p}\} \le C_f h^k \quad \forall\, h > 0,$$

where C_f is a constant independent of h.

For a box spline M_X, the approximation order of $\mathbb{S}(M_X)$ is $\min\{\#Z : Z \in \mathcal{A}(X)\}$. A quasi-interpolation scheme can be used to achieve the approximation order.

Splines on a Three Direction Mesh

Let $e_1 := (1, 0)$ and $e_2 := (0, 1)$ be the two unit coordinate vectors in $\mathbb{R}^2$, and let $e_3 := e_1 + e_2$. Further, let Δ be the partition of $\mathbb{R}^2$ into triangles obtained from three families of meshlines $\{v + te_i : v \in \mathbb{Z}^2, t \in \mathbb{R}\}$, $i = 1, 2, 3$. We use $\mathbb{S}^\rho_{k,\Delta}$ to denote the space of all C^ρ-functions which are piecewise polynomials of degree at most k on the partition Δ. Let Π_m denote the space of all polynomials of degree at most m.

Theorem (de Boor and Höllig 1983)

A necessary condition for $\mathbb{S}^\rho_{k,\Delta}$ to provide approximation order m is $\mathbb{S}^\rho_{k,\Delta} \supset \Pi_{m-1}$. But the optimal approximation order of $\mathbb{S}^1_{3,\Delta}$ is 3, not 4, even $\mathbb{S}^1_{3,\Delta} \supset \Pi_3$.

Let ϕ_1 be the box spline M_{X_1} with $X_1 := \{e_1, e_1, e_2, e_3\}$, and let ϕ_2 be the box spline M_{X_2} with $X_2 := \{e_1, e_2, e_2, e_3\}$. Then $\phi_1, \phi_2 \in \mathbb{S}^1_{3,\Delta}$. A quasi-interpolation scheme with ϕ_1 and ϕ_2 can be used to achieve approximation order 3.

Subdivision Schemes

The box spline M_X associated with $X = (x^1, \ldots, x^n)$ satisfies the refinement equation

$$M_X(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) M_X(2x - \alpha), \quad x \in \mathbb{R}^s,$$

where the refinement mask a is given by

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha \cdot \xi} = \prod_{j=1}^{n} \frac{1 + e^{-ix^j \cdot \xi}}{2}, \quad \xi \in \mathbb{R}^s.$$

Given a mask $a : \mathbb{Z}^s \to \mathbb{R}$, the subdivision operator S_a is the operator on sequences on $\mathbb{Z}^s$ given by

$$S_a \lambda(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - 2\beta) \lambda(\beta), \quad \alpha \in \mathbb{Z}^s.$$

A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, Stationary Subdivision, AMS Memoirs, No. 453, 1991.

Spline Wavelets

During 1988–1990 malltiresolution analysis was introduced by Y. Meyer and S. Mallat, and compactly supported orthogonal wavelets were constructed by I. Daubechies.

In early 1990's, C. K. Chui and J. Z. Wang used cardinal B-splines to construct compactly supported prewavelets.

Box splines are refinable, so they are suitable for constructing wavelets. This was first done by S. Riemenschneider and Z. W. Shen for dimensions 2 and 3 in their 1991 paper.

Extensions of mutliresolution analysis as well as constructions of wavelets and prewavelets in higher dimensions using box splines were investigated by R. Q. Jia and C. A. Micchelli in their 1991 paper, and by C. de Boor, R. Devore, and A. Ron in their 1993 paper.

The book "Box Splines" also contains some interesting results on the construction of wavelets in $\mathbb{R}^s$ for $s \leq 3$.

Dimension of Kernels of Linear Operators

The algebraic theory of box splines was extended to the study of kernels of certain linear operators. Around 1990 de Boor and Ron applied polynomial ideals to multivariate splines. Meanwhile Dahmen and Micchelli connected piecewise polynomial spaces with solutions of systems of partial differential equations. Soon after Z. W. Shen solved a difficult conjecture of Dahmen and Micchelli. This motivated the joint work of Jia, Riemenschneider, and Shen on dimesnion of kernals of linear operators, using matroid theory and algebraic geometry.

In this direction the following two important papers appeared in Advances in Applied Mathematics in 1996: C. de Boor, A. Ron, and Z. W. Shen, On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators. W. Dahmen, A. Dress, and C. A. Micchelli, On multivariate splines, matroids, and the Ext-functor. Both papers were selected by Math Reviews to have featured reviews.

Perturbation of Polynomial Ideals

My paper "Perturbation of polynomial ideals" appeared in the same issue. It was influenced by the previous work of de Boor and Ron on polynomial ideals. Let K be a field. We denote by $K[Z_1, \ldots, Z_s]$ (resp. $K[[Z_1, \ldots, Z_s]]$) the ring of polynomials (resp. the ring of formal power series) in s indeterminates over K. Let I be an ideal of $K[Z_1, \ldots, Z_s]$. The codimension of I is the dimension of the quotient space $K[Z_1, \ldots, Z_s]/I$ over K. The kernel of I is the set

$$I_\perp := \{f \in K[[Z_1, \ldots, Z_s]] : p(D)f = 0 \quad \forall\, p \in I\}.$$

Theorem (de Boor and Ron 1991, Jia 1996)

Let K be an algebraically closed field of characteristic zero. If I is an ideal of $K[Z_1, \ldots, Z_s]$ with finite codimension, then

$$codim(I) = dim(I_\perp).$$

Polynomial ideals played a vital role in the study of de Boor on multivariate interpolation.

Magic Squares

An $m \times m$ matrix with nonnegative integer entries is called a magic r-square of order m if every row and column sums to r. Let $H_m(r)$ denote the number of all magic r-squares of order m. It was proved by R. Stanley in 1973 that $H_m(r)$ is a polynomial of degree $(m-1)^2$.

A study of magic squares led to linear diophantine equations. It was Dahmen and Micchelli who first revealed the close relationship between linear diophantine equations and the so-called discrete truncated powers. Thus the theory of multivariate splines could be applied to certain combinatorial and algebraic problems. Using this approach, they succeeded in re-proving and extending certain results of Stanley on magic squares. Their insight into this problem opened a new way of attacking the more difficult problem of Stanley's conjecture about symmetric magic squares, which had remained unsolved for a long time by using commutative algebra.

Symmetric Magic Squares

Theorem (Stanley 1976)

Let $m \geq 1$, and let $S_m(r)$ be the number of $m \times m$ symmetric magic r-squares. Then

1. $S_m(r) = P_m(r) + (-1)^r Q_m(r)$ for all $r \in \mathbb{N}$, where $P_m(r)$ and $Q_m(r)$ are polynomials in r.
2. $\deg P_m = \binom{m}{2}$.
3. $\deg Q_m \leq \binom{m-1}{2} - 1$ if m is odd; $\deg Q_m \leq \binom{m-2}{2} - 1$ if m is even.

He conjectured that equality holds for all m in part 3 of the above theorem.

In 1992, using multivariate discrete splines, I gave a proof of this long-outstanding conjecture. My two papers "Multivariate discrete splines and linear diophatine equations" and "Symmetric magic squares and multivariate splines" were cited in the book R. Stanley, Enumerative Combinatorics: Volume 1, Second Edition, Cambridge University Press, 2012.

Recent Developments

Written by two Lie algebraists, Corrado De Concini and Claudio Procesi, the book "Topics in Hyperplane Arrangements, Polytopes and Box-Splines" (Springer 2010) brought together many areas of research that focus on methods to compute the number of integral points in suitable families or variable polytopes. Multivariate splines, in particular box splines, play a central role in the book.

In their 2011 paper, Olga Holtz and Amos Ron introduced zonotopal algebra associated with a given zonotope and further enhanced the algebraic theory originated from the study of box splines.

In their 2010 paper Bernd Sturmfels and Zhiqiang Xu used multivariate splines to solve some interesting problems in commutative algebra related to combinatorics. In his 2011 paper Xu showed that box splines could be employed to recast many important results in discrete geometry.