

# From Bernstein approximation to Zauner's conjecture

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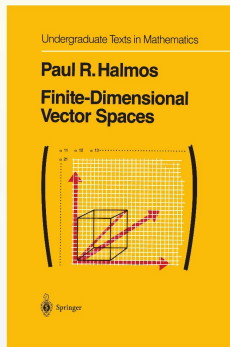
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# Outline

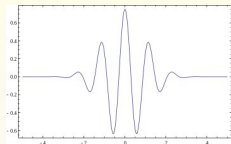
- ▶ A brief (personal) history of tight frames.
- ▶ Examples of finite tight frames
  - ▶ Tight frames for multivariate orthogonal polynomials.
  - ▶ Group frames
  - ▶ Equiangular lines
- ▶ SICs and Zauner's conjecture.

# Finite dimensional linear algebra



- ▶ Vectors written (uniquely) in terms of bases - for “efficient calculation”, e.g., **B-splines**
- ▶ Nice (natural) bases important for vector spaces with additional structure, e.g., spaces of polynomials
- ▶ Assumes that when there is a natural spanning set, then there is a natural basis
- ▶ Orthonormal bases are considered to be the best possible

# Infinite dimensional analysis



- ▶ Wavelet tight frame expansions, i.e., with  $\psi_{m,n}(t) = 2^{-\frac{m}{2}} \psi(2^{-m}t - n)$ ,

$$f = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle f, \psi_{m,n} \rangle \psi_{m,n}.$$

- ▶ The time–frequency shifts applied to the mother wavelet  $\psi$  do not form group (multiresolution analysis).
- ▶ There is a parallel theory where they do, i.e., Weyl–Heisenberg systems (the group is generated by a shifts and modulations)

# Tight frames - generalised orthonormal bases

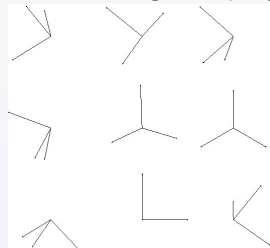
A sequence of vectors  $(v_j)$  is a (normalised) tight frame for  $\mathcal{H}$  if

$$f = \sum_j \langle f, v_j \rangle v_j, \quad \forall f \in \mathcal{H}.$$

Equivalently,  $(v_j)$  is the orthogonal projection of an orthonormal basis, or the Gramian

$$P = V^* V = [\langle v_k, v_j \rangle]_{j,k=1}^n, \quad V = [v_1, \dots, v_n]$$

is an orthogonal projection matrix of rank  $\dim(\mathcal{H})$ .



# The variational characterisation

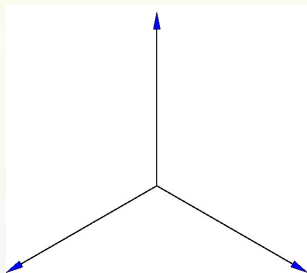
## Theorem

Let  $v_1, \dots, v_n$  be vectors in  $\mathcal{H}$ , not all zero, and  $d = \dim(\mathcal{H})$ . Then

$$\sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k \rangle|^2 \geq \frac{1}{d} \left( \sum_{j=1}^n \|v_j\|^2 \right)^2, \quad (1)$$

with equality if and only if  $(v_j)_{j=1}^n$  is a tight frame for  $\mathcal{H}$ .

# The Mercedes-Benz frame for $\mathbb{R}^2$



# An example where there is no natural basis

Let  $T = \text{conv}(V)$  be a simplex in  $\mathbb{R}^d$  with  $d + 1$  vertices  $V$ , with corresponding barycentric coordinates  $\xi = (\xi_\nu)_{\nu \in V}$ , and define the **Jacobi inner product**

$$\langle f, g \rangle_\nu := \int_T fg \xi^{\nu-1}, \quad \nu = (\nu_\nu)_{\nu \in V} > 0.$$

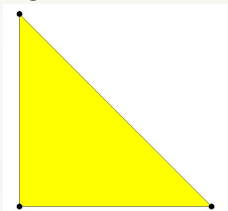


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e.g., for  $d = 2$ ,  $T = \text{conv}\{e_1, e_2, 0\}$ ,  $\nu - 1 = (\alpha, \beta, \gamma)$



- ▶  $\xi_0(x, y) = 1 - x - y$
- ▶  $\xi_{e_2}(x, y) = y$
- ▶  $\xi_{e_1}(x, y) = x$

$$\langle f, g \rangle_\nu = \int_0^1 \int_0^{1-x} f(x, y) g(x, y) x^\alpha y^\beta (1 - x - y)^\gamma dy dx$$

# Multivariate Jacobi polynomials

The **Jacobi polynomials** of degree  $k$  are

$$\mathcal{P}_k^\nu := \{f \in \Pi_k : \langle f, p \rangle_\nu = 0, \forall p \in \Pi_{k-1}\}.$$

This space has

$$\dim(\mathcal{P}_k^\nu) = \binom{k+d-1}{d-1}.$$

Each polynomial in  $\mathcal{P}_k^\nu$  is uniquely determined by its leading term, e.g., for  $\xi_0^2$  + lower order terms, the leading term is

$$\{(1-x-y)^2\}_\downarrow = x^2 - 2xy + y^2.$$

The limits of the eigenfunctions of the **Bernstein operator** on a simplex are related to these polynomials for the “singular weight”  $\nu = (-1, \dots, -1)$ .

# Orthogonal and biorthogonal systems

We describe the known representations for  $\mathcal{P}_k^\nu$  in terms of the leading terms (for the case  $d = 2$ ,  $k = 2$ ).

**Biorthogonal system** (Appell 1920's): partial symmetries

$$x^2, \quad xy, \quad y^2.$$

**Orthogonal system** (Prorial 1957, et al): no symmetries

$$x^2 + y^2 + 2xy, \quad x^2 - y^2, \quad x^2 - y^2 - 4xy.$$

For the *three* dimensional space of all quadratic Jacobi polynomials on the triangle, we want an orthonormal basis with leading terms determined by the *six* polynomials

$$x^2, \quad xy, \quad y^2, \quad x(1 - x - y), \quad y(1 - x - y), \quad (1 - x - y)^2.$$

# Scaling to obtain a tight frame

## Theorem

(W, Peng) Let  $\mathcal{H}$  be a Hilbert space of dimension  $d$ , and

$$n = \begin{cases} \frac{1}{2}d(d+1), & \mathcal{H} \text{ real} \\ d^2, & \mathcal{H} \text{ complex} \end{cases}$$

Then for a generic sequence of vectors  $v_1, \dots, v_n$  there are unique positive scalars  $c_j$  so that  $(c_j v_j)$  is a normalised tight frame for  $\mathcal{H}$ .

- ▶ You will recognise this  $n$  as the maximal number of equiangular lines in  $\mathcal{H} = \mathbb{R}^d, \mathbb{C}^d$ .
- ▶ For our six quadratic orthogonal polynomials in a space of dimension three, there should be a unique scaling to a tight frame

# A miracle

This works, even when it shouldn't!

Let  $\phi_\alpha^\nu$  be the orthogonal projection of  $\xi^\alpha/(\nu)_\alpha$ ,  $|\alpha| = n$  onto  $\mathcal{P}_n^\nu$ , which is given by

$$\phi_\alpha^\nu = \frac{(-1)^n}{(n + |\nu| - 1)_n} \sum_{\beta \leq \alpha} \frac{(n + |\nu| - 1)_{|\beta|} (-\alpha)_\beta}{(\nu)_\beta} \frac{\xi^\beta}{\beta!}.$$

## Theorem

(W, Xu, Rosengren) *The Jacobi polynomials on a simplex have the tight frame representation*

$$f = (|\nu|)_{2n} \sum_{|\alpha|=n} \frac{(\nu)_\alpha}{\alpha!} \langle f, \phi_\alpha^\nu \rangle_\nu \phi_\alpha^\nu, \quad \forall f \in \mathcal{P}_n^\nu,$$

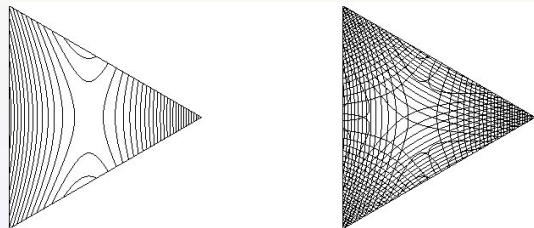
where the normalisation is  $\langle 1, 1 \rangle_\nu = 1$ .

## A nice example

The group of symmetries of the triangle (the dihedral group  $G = D_3 \approx S_3$ ) induces a representation on the quadratic Legendre polynomials  $\mathcal{P}_2$  on the triangle, and so (by algebra) we can construct

$$f = (2\sqrt{5} - 5\sqrt{2})\left(\xi_v^2 + \xi_w^2 + \xi_u^2 - \frac{1}{2}\right) + 15\sqrt{2}\left(\xi_v^2 - \frac{4}{5}\xi_v + \frac{1}{10}\right) \in \mathcal{P}_2$$

a single polynomial whose orbit under  $G$  consists of *three* polynomials which form an orthonormal basis for  $\mathcal{P}_2$ .



Contour plots of  $f$  and those of its orbit showing the triangular symmetry.

# Group frames

Let  $\mathcal{H}$  be a finite dimensional Hilbert space ( $\mathbb{R}^d$  or  $\mathbb{C}^d$ ), and  $G$  be a finite abstract group with a unitary action on  $\mathcal{H}$ . Then

$$(gv)_{g \in G}$$

is called a  **$G$ -frame** (or group frame) for its span.

Since the action is unitary

- ▶ The frame operator  $S$  commutes with (the action of)  $G$ .
- ▶ The dual and canonical tight frames of a  $G$ -frame are  $G$ -frames.
- ▶ The complement of a tight  $G$ -frame is a tight  $G$ -frame.

# The Gramian of a $G$ -frame is a $G$ -matrix

The Gramian of a  $G$ -frame is a  $G$ -**matrix**, i.e.,

$$\langle gv, hv \rangle = \langle h^* gv, v \rangle = \langle g^{-1} hv, v \rangle = \mu(g^{-1} h).$$

## Example

For  $G$  a cyclic group, a  $G$ -matrix is a *circulant matrix*. As with circulant matrices,  $G$ -matrices can be diagonalised using the characters of the group as eigenfunctions.

## Theorem

Let  $G$  be a finite group. Then  $\Phi = (\phi_g)_{g \in G}$  is a  $G$ -frame (for its span  $\mathcal{H}$ ) if and only if its Gramian is a  $G$ -matrix.

Further,  $G$ -frames can also be identified with elements of the group algebra  $\mathbb{C}G$ .

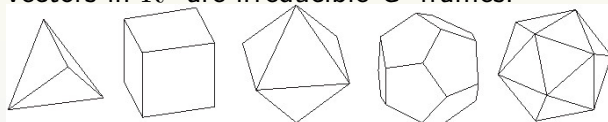


# Irreducible $G$ -frames

If the (unitary) action of  $G$  is **irreducible**, i.e., every nonzero orbit spans  $\mathcal{H}$ , then  $(gv)_{g \in G}$  is a tight  $G$ -frame for every  $v \neq 0$ .

## Example

The vertices of a simplex, Platonic solid, or the  $n$  equally spaced unit vectors in  $\mathbb{R}^2$  are irreducible  $G$ -frames.



These can be constructed from their corresponding symmetry groups (as abstract groups).

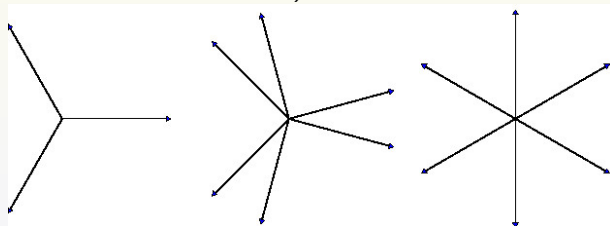
The vertices of the platonic solids are distinguished from other orbits of the symmetry group (which have more vectors), but the fact a vertex is stabilised by a nontrivial subgroup. This idea leads to **highly symmetric tight frames**.

# The dihedral group

Consider the irreducible action of the **dihedral group**

$$G = D_3 = \langle a, b : a^3 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle,$$

on  $\mathbb{R}^2$  as symmetries of three equally spaced unit vectors (the Mercedes-Benz frame).



- ▶ There are *uncountably* many unitarily inequivalent  $D_3$ -frames for  $\mathbb{R}^2$ . This is always the case for  $G$  *nonabelian*.
- ▶ For the abelian subgroup  $C_3 = \langle a \rangle$  of rotations all  $C_3$ -frames are unitarily equivalent.

# Harmonic frames

For  $G$  *abelian* there are finitely many tight  $G$ -frames, the **harmonic frames**. These can be obtained by selecting rows of the character table, e.g., for  $C_3$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \omega := e^{\frac{2\pi i}{3}}$$

gives two unitarily inequivalent harmonic frame for  $C^2$

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \omega \end{bmatrix} \right\} \text{ (real)} \quad \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix} \right\} \text{ (complex)}$$

- Equivalently (taking columns) one can restrict the characters to a subset  $J$  of  $G$

$$\Phi_J = (\xi|_J)_{\xi \in \hat{G}}$$

Here  $\hat{G}$  is the character group of  $G$  ( $\hat{\hat{G}} \approx G$ ).

# Difference sets and equiangular lines

We say that a sequence of equal-norm vectors  $(v_j)$  (or the lines that they determine) is **equiangular** if

$$|\langle v_j, v_k \rangle| = C, \quad j \neq k,$$

for some constant  $C$ . These vectors/lines can be thought of as being well spread out in  $F^d$ , and are desired in many applications.

## Theorem

Let  $G$  be a finite group of order  $n$ , and  $\Phi = (\xi|_J)_{\xi \in \hat{G}}$  be the harmonic frame of  $n$  vectors for  $C^d$  given by  $J \subset G$ ,  $|J| = d$ . Then

1.  $\Phi$  has distinct vectors if and only if  $J$  generates  $G$ .
2.  $\Phi$  is a real frame if and only if  $J$  is closed under taking inverses.
3.  $\Phi$  is equiangular if and only if  $J$  is an  $(n, d, \lambda)$ -difference set for  $G$ , i.e., each nonidentity element of  $G$  can be written as a difference  $j_1 j_2^{-1}$  of two elements  $j_1, j_2 \in J$  in exactly  $\lambda$  ways.

# Real equiangular lines

If there are  $n$  real equiangular lines in  $\mathbb{R}^d$ , then

$$n \leq \frac{1}{2}d(d+1).$$

Since a tight frame is determined by its Gram matrix (up to unitary equivalence), systems of  $n$  real equiangular lines correspond to graphs (the inner product between vectors is  $\pm C$ ). The graphs that they correspond to are **strongly regular graphs** on  $n - 1$  vertices, with certain parameters.

# Complex equiangular lines

If there are  $n$  complex equiangular lines in  $\mathbb{R}^d$ , then

$$n \leq d^2.$$

A set of  $d^2$  equiangular lines in  $\mathbb{C}^2$  is called a **SIC** (Symmetric informationally complete positive valued operator measure).

- ▶ SICs exist numerically for every dimension  $d$ .
- ▶ Zauner's conjecture (or the SIC problem) is to show that a SIC exists (analytically) for every dimension  $d$ .
- ▶ Zauner's conjecture has been proved (by explicit constructions) for

$$d = 2, \dots, 16, 19, 24, 28, 35, 48 \quad [\text{SG10}],$$

$$d = 17, 18, 20, 21, 30, 31, 37, 39, 43 \quad [\text{ACFW16}].$$

# The Heisenberg group

Let  $\omega$  and  $\mu$  be the primitive  $d$ -th and  $2d$ -th root of unity, and  $S$  and  $\Omega$  be the cyclic shift and modulation operators on  $\mathbb{C}^d$ . For  $d = 2$ , these are the Pauli matrices  $\sigma_1 = \sigma_x$  and  $\sigma_3 = \sigma_z$ , and for  $d = 3$  they are

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}.$$

The nonabelian group generated by  $S$  and  $\Omega$  is the **Heisenberg group**. With one exception (the Hoggar lines in  $\mathbb{C}^8$ ) all SICs are the orbit of a *fiducial* vector  $v$ .

# The Clifford group

The normaliser of the Heisenberg group in the unitary matrices is the Clifford group, which is generated by the Fourier matrix  $F$  and the diagonal matrix

$$R = \text{diag}(\omega^{j^2})_{j \in \mathbb{Z}_d}.$$

It is conjectured that

- ▶ There is a SIC fiducial  $v$  in the eigenspace of the Zauner matrix (of order 3)

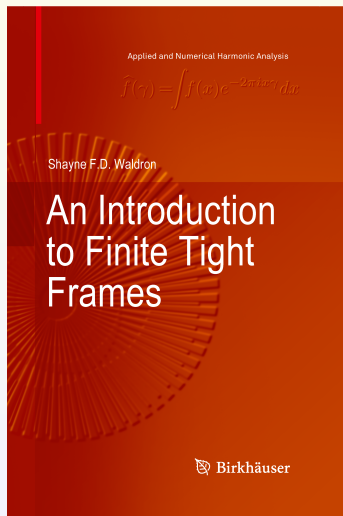
$$Z := e^{\frac{2\pi i}{24}(d-1)} RF.$$

- ▶ The (triple products) of the SIC lie in the ray class field over  $\mathbb{Q}(\sqrt{(d-3)(d+1)})$  (with conductor  $d'$  and ramification allowed at both infinite places).



# Thank you for your attention

If you want to learn more about finite tight frames, then ...



# Tight frames and Approximation, 20-23 February 2018

at Taipa, Doubtless Bay, New Zealand



# Unitarily equivalent harmonic frames

We say  $J, K \subset G$  are

- ▶ **translates** if  $K = J - b$ ,  $b \in G$
- ▶ **multiplicatively equivalent** if  $K = \sigma J$ ,  $\sigma : G \rightarrow G$  an automorphism

## Theorem

(W, Chien) Let  $J, K$  be subsets of a finite abelian group  $G$ . Then

1. If  $J$  and  $K$  are translates, then  $\Phi_J$  and  $\Phi_K$  are projectively unitarily equivalent.
2. If  $J$  and  $K$  are multiplicatively equivalent, then  $\Phi_J$  and  $\Phi_K$  are unitarily equivalent after reordering (by an automorphism)

This essentially allows one to classify all harmonic frames up to (projective) unitary equivalence.

# $G$ nonabelian (or abelian)

## Theorem

(Characterisation). Let there be a unitary action of a finite group  $G$  on  $\mathcal{H} = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ , an orthogonal direct sum of irreducible  $G$ -invariant subspaces (irreducible  $FG$ -modules). Then

$$(gv)_{g \in G}, \quad v = v_1 + \cdots + v_m, \quad v_j \in V_j$$

is a tight  $G$ -frame for  $\mathcal{H}$  if and only if

$$v_j \neq 0, \quad \forall j, \quad \frac{\|v_j\|^2}{\|v_k\|^2} = \frac{\dim(V_j)}{\dim(V_k)}, \quad j \neq k,$$

and when  $V_j \neq V_k$  are  $FG$ -isomorphic,  $(gv_j)_{g \in G}$  and  $(gv_k)_{g \in G}$  are orthogonal, i.e.,

$$\sum_{g \in G} \langle v_j, gv_j \rangle gv_k = 0. \quad (2)$$

Moreover, if  $V_j$  is absolutely irreducible, then (2) can be replaced by

$$\langle \sigma v_j, v_k \rangle = 0, \quad (3)$$

where  $\sigma : V_j \rightarrow V_k$  is any  $\mathbb{F}G$ -isomorphism.

- ▶ Continuing this line of reasoning, allows one to determine the minimal number of generators for a  $G$ -invariant tight frame (or spanning set).

# Projective unitary equivalence

We say frames  $\Phi = (v_j)$  and  $\Psi = (w_j)$  are **projectively unitarily equivalent** if

$$w_j = c_j U v_j, \quad \forall j$$

where  $U$  is unitary and  $c_j$  are unit modulus scalars. In other words we now view the vectors as (weighted) lines. The term *fusion frame* is also used.

Unitary equivalence is characterised by the Gramian (inner products). These are *not* projective unitary invariants

$$\langle c_j U v_j c_k U v_k \rangle = c_j \overline{c_k} \langle v_j, v_k \rangle.$$

# The $m$ -products

The norm and inner product squared are projective unitary invariants, e.g.,

$$\begin{aligned} |\langle c_j Uv_j, c_k Uv_k \rangle|^2 &= \langle c_j Uv_j, c_k Uv_k \rangle \langle c_k Uv_k, c_j Uv_j \rangle \\ &= c_j \overline{c_k} \langle Uv_j, Uv_k \rangle c_k \overline{c_j} \langle Uv_k, Uv_j \rangle \\ &= |\langle v_j, v_k \rangle|^2. \end{aligned}$$

In the same way, the  $m$ -**products**

$$\Delta(v_{j_1}, v_{j_2}, \dots, v_{j_m}) := \langle v_{j_1}, v_{j_2} \rangle \langle v_{j_2}, v_{j_3} \rangle \cdots \langle v_{j_m}, v_{j_1} \rangle$$

are projective unitary invariants.

These (finitely many)  $m$ -products determine  $\Phi = (v_j)$  up to projective unitary equivalence.

# The frame graph

The **frame graph** of a sequence of vectors  $(v_j)$  is the graph with vertices  $\{v_j\}$  (or the indices  $j$  themselves) and

an edge between  $v_j$  and  $v_k$ ,  $j \neq k \iff \langle v_j, v_k \rangle \neq 0$ .

Cycles (with a direction) correspond to nonzero  $m$ -products.

## Theorem

*A finite frame  $\Phi$ , with frame graph  $\Gamma$ , is determined up to projective unitary equivalence by a determining set for the  $m$ -products, e.g.,*

- 1. The 2-products.*
- 2. The  $m$ -products,  $3 \leq m \leq n$ , corresponding to a fundamental cycle basis (for the cycle space of  $\Gamma$ ) formed from a spanning tree (forest)  $\mathcal{T}$  for  $\Gamma$ .*

*In particular, if  $M$  is the number of edges of  $\Gamma \setminus \mathcal{T}$ , then it is sufficient to know all of the 2-products, and  $M$  of the  $m$ -products,  $3 \leq m \leq n$ .*



# Example SICs

The frame graph of a SIC, or any set of equiangular lines is the complete graph.

By taking the star graph with internal vertex say  $v_1$ , it follows that a set of equiangular lines is determined by the 2-products and the 3-products (triple products) involving the point  $v_1$ . For  $\mathcal{H} = \mathbb{R}^2$  this can be interpreted in terms of *two graphs*.

## Example MUBs

Let  $\Phi = (v_j)$  be the two MUBs for  $C^2$  given by

$$\Phi = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right), \quad \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}.$$

The frame graph  $\Gamma$  of  $\Phi$  is the 4-cycle  $(v_1, v_3, v_2, v_4)$ . Thus this MUB is determined up to projective unitary equivalence by the 2-products and the 4-product given by this cycle.

This case is an anomaly.

### Corollary

*A frame consisting of three or more MUBs is determined by to projective unitary equivalence by its 2-products and 3-products.*