Efficient numerical methods for L0-norm related minimizations

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ABSTRACT

The splines and its induced systems are widely used in the sparsity based image processing problems. Usually, it is formulated as solving a L0-norm related minimization problems. In this talk, I will propose numerical methods for solving these L0-norm minimizations. Moreover, the convergence analysis of the proposed method is discussed and the results in image restoration and recognition are also present.

Parametric PDEs: Adaptive High-Dimensional Polynomial Approximation

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ABSTRACT

Design or optimization tasks are often based on families of partial differential equations (PDEs) involving a large (or even infinite) number of parameters. These parameters typically range over a compact set and determine the possible states of the modeled process. Each state is the solution of the PDE for a certain parameter specification and hence a function of spatial and possibly infinitely many parametric variables. The entity of states forms the so called solution manifold. In this talk we discuss some recent results from [1, 2] on the adaptive approximation of parametric solutions for a class of uniformly elliptic parametric PDEs. We briefly review first essential approximability properties (see also [4, 3]) of the solution manifold which then serves as benchmark for numerical algorithms. We then discuss a fully adaptive algorithm which generates approximations in terms of tensor products of spatial wavelets and Legendre polynomials in the parametric variables. The algorithm can be shown to exhibit optimal performance for these benchmark classes in the sense of best N-term approximation. That is, it achieves a given target accuracy by keeping the number of adaptively generated degrees of freedom near-minimal at linear computatiuonal cost. We conclude with some comments on alternative approximation formats using low-rank or hierarchical tensor mesthods.

References

[1] Bachmayer, M., Dahmen, W. *Adaptive Near-Optimal Rank Tensor Approximation for High-Dimensional Operator Equations*, Journal of Foundations of Computational Mathematics, 15, pp. 839–898, 2015. DOI: 10.1007/s10208-013-9187-3,

[2] Bachmayr , M., Cohen, A., Dahmen, W. *Parametric PDEs: Sparse or low-rank approximations?*, to appear in IMA Journal of Numerical Analysis, arxiv.org/abs/1607.04444 [math.NA]

[3] Bachmayr, M., Cohen, A., Dũng, D., Schwab, C. *Fully discrete approximation of parametric and stochastic elliptic PDEs*, arXiv:1702.03671, 2017,

[4] Cohen, A. Devore, R. Schwab, C. *Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's*, Anal. Appl. (Singap.), 9 (1), pp. 11–47, 2011, DOI = 10.1142/S0219530511001728

Estimating a Quantity of Interest from Given Data

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ABSTRACT

A common scientific problem is that we are given some data about a function f and we wish to use this information to either (i) approximate f or (ii) answer some question about f called a quantity of interest. In approximation circles this problem is referred to as optimal recovery. Meaningful results require extra information about f known as model class assumptions. We discuss recent results on optimal recovery which determine optimal algorithms for the two scenarios above under the assumption that f is in a model class described by approximation.

Wavelet Frame Transforms and Differential Operators: Bridging Discrete and Continuum for Image Restoration

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ABSTRACT

In image restoration, wavelet frame based approach and PDE based approach, which includes variational models and PDE models, are two of the most successful approaches and are widely adopted in both academia and industry. Their development followed rather different paths and were generally considered as different approaches. This talk is based on a series of our recent work [1, 2, 3, 4], where we established rigorous and generic connections between wavelet frame based and PDE based approach. One of the key observations is that the B-spline tight wavelet frame transforms are standard approximations to differential operators. In particular to the general results, connection of wavelet frame based approach to total variation model was established in [1], to the Mumford-Shah model was established in [2], and to the total generalized variational model was established in [4]. On the other hand, connection of wavelet frame shrinkage to a general form of nonlinear evolution PDEs was established in [3], where the Perona-Malik equation, Osher-Rudin's shock filters and Navier-Stokes image inpainting equation are special cases. Other than connections to existing variational and PDE models, brand new models were also discovered, which combine merits from both wavelet frame and PDE based approach, and thus outperform existing models in various applications in image restoration.

References

[1] J. Cai, B. Dong, S. Osher, and Z. Shen, "Image restorations: total variation, wavelet frames and beyond," *Journal of American Mathematical Society*, vol. 25(4), pp. 1033–1089, 2012.

[2] J. Cai, B. Dong, and Z. Shen, "Image restorations: a wavelet frame based model for piecewise smooth functions and beyond," *Applied and Computational Harmonic Analysis*, vol. 41(1), pp. 94–138, 2016.

[3] B. Dong, Q. Jiang, and Z. Shen, “Image restoration: wavelet frame shrinkage, nonlinear evolution PDEs, and beyond,” *Multiscale Modeling & Simulation: A SIAM Interdisciplinary Journal*, vol. 15, no. 1, pp. 606–660, 2017.

[4] B. Dong, Z. Shen, and P. Xie, “Image restoration: a general wavelet frame based model and its asymptotic analysis,” *SIAM Journal on Mathematical Analysis*, vol. 49, no. 1, pp. 421–445, 2017.

B-spline generated frames and their extensions to locally compact abelian groups

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ABSTRACT

It is well known that using the refinable uniform B-splines, compactly supported tight wavelet frames on the real line can be obtained via the unitary extension principle. It is also known that based on the translation partition of unity property of uniform B-splines, pairs of dual Gabor frames can be constructed explicitly. Corresponding to the latter for wavelet frames, by considering an integral transform, we obtain B-splines with geometric knots that satisfy the scaling partition of unity property. Such B-splines with geometric knots lead to explicit pairs of band-limited dual wavelet frames on the real line. To obtain wavelet frames and Gabor frames on other domains of interest, we extend the ideas of B-spline generated frames to locally compact abelian (LCA) groups. Our general results give different types of frames on LCA groups. These include tight wavelet frames with B-splines as refinable functions and pairs of dual frames from partition of unity properties of B-splines. This is joint work with Ole Christensen.

Solving systems of spline equations: A linear programming-based approach

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ABSTRACT

For the past 25 years, the method of choice for solving tensor product spline systems of equations has been the projected polyhedron method of Sherbrooke and Patrikalakis [SP93]. Although the method can sometimes bog down and perform badly on certain problems, it has the virtue of being very robust, and its use has enabled many other robust algorithms for geometry processing to be developed on top of it, aided by a number of refinements published over the years (see [H14] and [MPR03]). One of the potential refinements discussed in [H14] is to use linear programming as a means of accelerating the performance of the algorithm. The author ultimately discards the idea as being too computationally expensive to be practical. This talk will explore this idea in more detail than was done in [H14] and suggest that, once one takes advantage of special problem structure arising in large numbers of applications, the method may prove to be computationally competitive after all.

References

[H14] Hanniel, I. "Solving multivariate polynomial systems using hyperplane arithmetic and linear programming," *Computer-Aided Design* **46**, pp. 101–109.

[MPR03] Mourrain, B., V. Y. Pan, and O. Ruatta. "Accelerated solution of multivariate polynomial systems of equations," *SIAM Journal of Computing* **32**, pp. 435–454.

[SP93] Sherbrooke, E. C. and N. M. Patrikalakis. "Computation of the solutions of nonlinear polynomial systems," *Computer Aided Geometric Design* **10**, pp. 379–405.

Multivariate wavelet frames through constant matrix completion via the duality principle

Zhitao Fan[a], Andreas Heinecke[a,b], Hui Ji[a], and Zuowei Shen[a]

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ABSTRACT

The duality principle, ultimately a statement about adjoint operators, is a universal principle in frame theory. We take a broad perspective on the duality principle and discuss how the mixed unitary extension principle for MRA-wavelet frames can be viewed as the duality principle in a sequence space. This leads to a construction scheme for dual MRA-wavelet frames which is strikingly simple in the sense that it only requires the completion of an invertible constant matrix. Under minimal conditions on the multiresolution analysis our construction guarantees the existence and easy constructability of multivariate non-separable dual MRA-wavelet frames of compactly supported wavelets. As an example we discuss box spline constructions of tight wavelet frames for any dimension.

References

[1] Fan Z., Heinecke A., Shen Z., *Duality for frames*, Journal of Fourier Analysis and Applications, 2016.

[2] Fan Z., Ji H., Shen Z., *Dual Gramian analysis: duality principle and unitary extension principle*, Mathematics of Computation, 2016.

B-spline Wavelet Based Blind Image Recovery

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ABSTRACT

B-spline wavelet tight frame has been an important tool in various image recovery tasks. The sparse representation of image under B-spline wavele tight frame allows the design of powerful regularization for solving challenging inverse problems in image processing, e.g, blind Image de-convolution. Blind image deconvolution aims at recovering the clear image from its blurred observation without knowing how it is blurred. Even worse, in many cases, the blurring process is non-stationary in the sense that different image regions are blurred by different kernels, which makes it even more difficult. In this talk, I will present several mathematical techniques toward solving blind image deblurling, in which B-spline wavelet frames plays an important role.

From B-splines to Box Splines – the Insight and Influence of Carl de Boor's Work

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ABSTRACT

Carl de Boor's pioneering work on splines made a profound impact on many areas of mathematics as well as applications in engineering. In this talk we will give a brief review of the development he made from B-splines to box splines. First, we will survey the central role played by B-splines in the univariate spline theory and the solution of his famous conjecture on boundedness in the uniform norm for least-square approximation by splines. Next, we will recall how de Boor developed multivariate splines from B-splines and established the beautiful theory of box splines. Finally, we will discuss applications of box splines to approximation theory, wavelet anlysis and, in particular, applications to algebra and combinatorics.

Design and Analysis on Manifolds with Irregular Layout

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ABSTRACT

Irregularities in parameterizations of manifolds are places where shift-invariance does not hold. For tensor-product arrangements, depending on the dimension, irregularities can be neighborhoods of points, edges, etc. For tensor-product surfaces, they are neighborhoods of points where three or more than four quadrilateral polynomial pieces meet. Irregularities play an important role both in the design of manifolds and in their analysis.

This talk reviews and categorizes techniques for creating surfaces with irregularities and surveys the state of the art of analysis on surfaces with irregular layout. Time permitting, a glimpse and one solution to the trivariate challenge will be presented.

The ‘Big Bang Theory’ of multivariate splines

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ABSTRACT

At the beginnings, the vibrant universe of multivariate splines did not exist. There was just one **graph**, a finite graph, whose edges continuously flipped their directions in a random way. From time to time, like a good slot machine in Las Vegas, all the orientations of the edges clicked into a harmonious configuration, and small nuclei-like shape known as parking function bursted in the chaotic vacuum and began orbiting the graph. It was William Tutte who was able to count the number of parking functions emitted by a graph: a complicated number in general, but sometimes simple: $n!$ for a complete graph with $n+1$ vertices, for example.

According to the **Big Bang Theory** of multivariate splines, one day the big bang occurred and the parking functions, each, exploded.

On the negative side, anti-matter was created. Its destructive force was culminated by the thunder power of the **torsion ideal**. The torsion ideal was discovered in the 80’s by a group of scientists led by Carl de Boor, Ron DeVore, Klaus Höllig and others. It received its name from its role in Jeffrey-Kirwan (JK) decompositions a la Brion-Vergne. One thunder ball from the torsion ideal destroys all matter. Below the torsion ideal rests the evil empire of the $\mathcal{P}$-polynomials. Those were discovered a few years later by Dahmen and Micchelli, and by Dyn and Ron. They destroy matter partially, sending some matter to dust, and reducing other to more rudimentary form. In JK decompositions they are known as “free”, but make no mistake about these guys and their destructive power. The evolution of the evil empire from the parking functions is well understood, is not too complicated, and goes under the umbrella of “monomization of power ideals”. It is well documented in the work of Postnikov, Shapiro and others in the 00’s.

On the positive side there is matter. The small building blocks of matter are the $\mathcal{D}$-polynomials. These were thought to be complex objects beyond imagination, with magic powers of combining themselves, in a smooth piecewise-manner, to create wonderful complicated constructs known as *truncated powers* and *partition functions*.

These truncated powers and partitions functions then spread themselves over the science universe, helping the mortals in their daily tasks of computing volumes, counting number of solutions, representing Schur functions and moment maps, building the even more complex guys known as box splines, and much, much more. Matter was introduced by the same people who introduced the torsion ideal: at the end, the only power of the torsion ideal is to destroy mat-ter, it can destroy nothing else! The torsion ideal kills the truncated power all the way down to hyperplane dust, which occupies no volume. The evil empire is not so powerful, as the $\mathcal{P}$-polynomials destroy the truncated power into flat constants. Matter can be classified by duality according to the action of the $\mathcal{P}$-polynomials. But the intrinsic meaning of that classification has never been understood, and the duality does not help in understanding the shape and form of matter.

We complete in this talk the Big Bang Theory of multivariate splines, by showing that matter also evolved from the same parking functions, and describe exactly how that evolution worked. So, each parking function led to the creation of one atom of matter, crowned as **flow polynomial** and one dual evil-atom of anti-matter, creating a universe of basis of atoms and dual basis of evil-atoms. It turns out that the evolution of matter from the parking functions all the way to the flow polynomials is simple and concrete, making matter simpler to understand than anti-matter!. The destructive nature of the evil-atoms on matter, became a simple task of monitoring orientations in directed graphs, a task that all mortals of the universe (even those known as undergraduate students) can easily perform. And finally, the assembly of truncated powers from their matter, i.e., from flow polynomials, becomes a simple task: it is just a simple interaction between matter and anti-matter that makes it happen.

The reported work is a joint endeavor with Shengnan (Sarah) Wang.

de Boor's conjecture, past and present

Alexei Shadrin

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ABSTRACT

The famous conjecture of de Boor that the max-norm of the orthogonal spline-projector of any order k is bounded independently of the knot-sequence was a challenge for spline theorists for almost 30 years. The interest was heightened by its connection to spline interpolation problems, construction of bases in function spaces, finite element methods, and not least by de Boor's promise to pay $10 for every year passed since 1972 when it was formulated.

I proved this conjecture in 1999. The story did not end though, as several important issues which I emphasized in my paper remained unanswered, in particular the exact order of the Lebesgue constant, unconditional convergence of spline interpolants, generalizations to multivariate splines and, intriguingly, existence of a simpler proof as my proof ran on 78 pages with quite sophisticated arguments. Many of these issues were resolved in subsequent studies including a short proof found by von Golitschek in 2014.

In my talk, I will give an overview of several old and recent results related to de Boor's conjecture. It is worth mentioning that, as it was noticed only few years ago, I did not actually prove de Boor's conjecture in full in its exact wording, but then, aptly, de Boor himself found how to fill the gap.

Spline Tight Frame for Image Restoration and Beyond

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ABSTRACT

We are living in the era of big data. The discovery, interpretation and usage of the information, knowledge and resources hidden in all sorts of data to benefit human beings and to improve everyone?s day to day life is a challenge to all of us. The huge amount of data we collect nowadays is so complicated, and yet what we expect from it is so much. This provides many challenges and opportunities to many fields. As images are one of the most useful and commonly used types of data, in this talk, we start from reviewing the development of the spline wavelet frame (or more general redundant system) based approach for image restoration. We will observe that a good system for any data, including images, should be capable of effectively capturing both global patterns and local features. One of the examples of such system is the spline wavelet frame. We will then show how models and algorithms of spline wavelet frame based image restoration are developed via the generic knowledge of images. Then, the specific information of a given image can be used to further improve the models and algorithms. Through this process, we shall reveal some insights and understandings of the wavelet frame based approach for image restoration and its connections to other approaches, e.g. the partial differential equation based methods. Finally, we will also show, by many examples, that ideas given here can go beyond image restoration and can be used to many other applications in data science.

From analog to digital: On the unifying role of splines in Science and Engineering 4.0

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ABSTRACT

We like to view splines as a comprehensive and elegant framework that enables the representation of continuous-domain functions in a discrete form adapted to a computer; i.e., the mathematical version of analog-to-discrete conversion. Splines are used extensively in computer aided design and are already playing a prominent role in the move of our society towards digitalization (Industry/Engineering 4.0). In this talk, we shall argue that this is only a beginning and that their influence is spreading to other areas of science and engineering. Our main point is that the framework is not only applicable to functions and signals, but also to whole domains of applications and theories. In particular, we shall discuss the unifying role of splines in

- sampling theory
- signal/image processing
- linear system theory [1]
- stochastic processes (including fractals) and estimation theory
- inverse problems and regularisation theory
- imaging, compressed sensing [2]
- machine learning

The topics will be illustrated with concrete examples of applications.

References

[1] M. Unser, *Cardinal Exponential Splines: Part II—Think Analog, Act Digital,* IEEE Trans. Signal Processing, vol. 53, no. 4, pp. 1439-1449, 2005.

[2] M. Unser, J. Fageot, J.P. Ward, *Splines Are Universal Solutions of Linear Inverse Problems with Generalized-TV Regularization,* SIAM Review, vol. 59, No. 4, pp. 769-793, 2017.

From Bernstein approximation to Zauner's conjecture

Shayne Waldron

The University of Auckland, New Zealand

ABSTRACT

I will talk about my recent interest in *finite tight frames* with symmetries: from multivariate orthogonal polynomials to systems of equiangular lines. I will give some fascinating examples including maximal sets of complex equiangular lines constructed in ray class fields as the orbit of the Weyl–Heisenberg group.

References

[1] Waldron, S. *An introduction to finite tight frames*, Birkhauser, 2017

On B-spline framelets derived from the unitary extension principle

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ABSTRACT

Spline wavelet tight frames of have been used widely in frame based image analysis and restorations. However, except for the tight frame property and the approximation order of the truncated series, there are few other properties of this family of spline wavelet tight frames to be known. In this talk, we introduce a few new properties of this family that will provide further understanding of it and, hopefully, give some indications why it is efficient in image analysis and restorations. In particular, we present a recurrence formula of computing generators of higher order spline wavelet tight frames from the lower order ones. We also represent each generator of spline wavelet tight frames as certain order of derivative of some univariate box spline. With this, we further show that each generator of sufficiently high order spline wavelet tight frames is close to a right order of derivative of a properly scaled Gaussian function. This leads to the result that the wavelet system generated by a finitely many consecutive derivatives of a properly scaled Gaussian function forms a frame whose frame bounds can be almost tight.

Wavelet frame based scattered data reconstruction

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ABSTRACT

In real world applications many signals contain singularities, like edges in images. Recent wavelet frame based approaches were successfully applied to reconstruct scattered data from such functions while preserving these features. In this talk we present a recent approach which determines the approximant from shift invariant subspaces by minimizing an L_1-regularized least squares problem which makes additional use of the wavelet frame transform in order to preserve sharp edges. We give a detailed analysis of this approach, i.e., how the approximation error behaves dependent on data density and noise level. Moreover, a link to wavelet frame based image restoration models is established and the convergence of these models is analyzed. We present some numerical examples, for instance how to apply this approach to handle coarse-grained models in molecular dynamics. This is a joint work with Prof. Zuowei Shen and Dominik Stahl in Department of Mathematics, NUS; and with Prof. Lanyuan Lu, Guanhua Zhu and Dudu Tong in School of Biological Sciences, NTU.

MRA-based wavelet frames and digital Gabor filters

Hui Ji, Zuowei Shen, and Yufei Zhao

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ABSTRACT

With the benefits of orientation selectivity and localized time-frequency analysis, digital Gabor filters are indispensable tools in signal and image processing. However, owing to the lack of multi-scale structure, discrete Gabor frames generated by Gabor filters are less effective than multi-resolution analysis based wavelet frames for modeling local structures of signals with varying sizes. In this talk, I will present our work about constructing MRA-based wavelet frames induced by the digital Gabor filters. It is found that the digital Gabor filters can generate MRA-based wavelet tight frame via the Unitary Extension Principle, only if the Gabor filters are local discrete Fourier basis. And the digital Gabor filters can generate MRA-based wavelet bi-frames, when only a mild condition is imposed on the window sequences. The examples of window sequences satisfying such conditions include the discretization of B-spline functions and the refinable masks of B-spline functions. Several image restoration experiments illustrate the efficiency of constructed systems in sparse image representation.

References

[1] Ji Hui, Shen Zuowei and Zhao Yufei, *Digital Gabor filters do generate MRA-based wavelet tight frames*, Appl. Comput. Harmon. Anal., In Press (2017).

[2] Ji Hui, Shen Zuowei and Zhao Yufei, *Digital Gabor filters with MRA structure*, SIAM J. Multiscale Model. Simul. In Press (2017).