B-spline wavelet frames

Zhiqiang Xu (许志强)

Joint work with Zuowei Shen

LSEC, Inst. Comp. Math., Academy of Mathematics and System Science,
Chinese Academy of Sciences, Beijing, China

Workshop on Spline Approximation and its Applications on
Carl de Boor's 80th Birthday, Dec. 2017

Frames

Frames

A family $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a *frame* with bounds A and B if

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq B\|f\|^2$$

holds for all $f \in \mathcal{H}$. If $A = B$, then $\{f_j\}_{j\in J}$ is called a *tight frame*.

Frames

Frames

A family $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a *frame* with bounds A and B if

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le B\|f\|^2$$

holds for all $f \in \mathcal{H}$. If $A = B$, then $\{f_j\}_{j\in J}$ is called a *tight frame*.

Hilbert space frames were introduced by Duffin and Schaeffer in 1952.
R.J. Duffin and A.C. Schaeffer, A class of nonharmonic Fourier series. Trans. AMS 72 (1952) 341-366.

Gabor frames

Suppose that $g \in L_2(\mathbb{R})$. The frame is generated by

$$\{\exp(2\pi inax)g(x - mb)\}_{n,m\in\mathbb{Z}}$$

is called Gabor frame, where $a, b \in \mathbb{R}$.

Gabor frames and wavelet frames

Gabor frames

Suppose that $g \in L_2(\mathbb{R})$. The frame is generated by

$$\{\exp(2\pi inax)g(x - mb)\}_{n,m\in\mathbb{Z}}$$

is called Gabor frame, where $a, b \in \mathbb{R}$.

Wavelet frames

For given $\Psi := \{\psi_1, \ldots, \psi_r\} \subset L_2(\mathbb{R})$, the *wavelet system* generated by Ψ is defined as

$$X(\Psi) := \{\psi_{\ell,n,k} := 2^{n/2}\psi_\ell(2^n \cdot -k) : 1 \leq \ell \leq r;\ n, k \in \mathbb{Z}\}.$$

If $X(\Psi)$ is a frame of $L^2(\mathbb{R})$, $X(\Psi)$ is called wavelet frames.

Wavelet tight frames

If $X(\Psi)$ is a tight frame with $A = B = 1$, then

$$f = \sum_{g \in X(\Psi)} \langle f, g \rangle g$$

holds for all $f \in L_2(\mathbb{R})$.

Multiresolution analysis (MRA)

MRA: A popular tool for constructing wavelet bases and wavelet frames.

Multiresolution analysis (MRA)

MRA: A popular tool for constructing wavelet bases and wavelet frames.
A multiresolution analysis is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$ that satisfies:

- $V_j \subset V_{j+1}$,
- $\bigcup_j V_j$ is dense in $L_2(\mathbb{R})$,
- $\bigcap_j V_j = \{0\}$.

Multiresolution analysis (MRA)

MRA: A popular tool for constructing wavelet bases and wavelet frames.

A multiresolution analysis is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$ that satisfies:

- $V_j \subset V_{j+1}$,
- $\bigcup_j V_j$ is dense in $L_2(\mathbb{R})$,
- $\bigcap_j V_j = \{0\}$.

V_0 be the closed shift invariant space generated by $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ and $V_j := \{f(2^j\cdot) : f \in V_0\}$, $j \in \mathbb{Z}$.

Multiresolution analysis (MRA)

MRA: A popular tool for constructing wavelet bases and wavelet frames.
A multiresolution analysis is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$ that satisfies:

- $V_j \subset V_{j+1}$,
- $\bigcup_j V_j$ is dense in $L_2(\mathbb{R})$,
- $\bigcap_j V_j = \{0\}$.

V_0 be the closed shift invariant space generated by $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ and $V_j := \{f(2^j\cdot) : f \in V_0\}$, $j \in \mathbb{Z}$.
The function φ satisfies a refinement equation

$$\varphi(x) = 2\sum_{j\in\mathbb{Z}} a_j \varphi(2x - j). \tag{1}$$

B-splines

A special family of refinable functions is B-splines.

B-splines

A special family of refinable functions is B-splines. Let $\varphi^{(m)}$ be the centered B-spline of order m, which is defined in Fourier domain by

$$\hat{\varphi}^{(m)}(\omega) = \operatorname{sinc}(\frac{\omega}{2})^m,$$

where

$$\operatorname{sinc}(x) := \begin{cases} \sin(x)/x, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}. \tag{2}$$

Then $\varphi^{(m)}$ is a refinable function.

B-spline wavelet frame

For a given B-spline $\varphi^{(m)}$ of order m, it was shown by Ron-Shen (by UEP) that the m functions, $\Psi^{(m)} = \{\psi_\ell^{(m)} : \ell = 1, \ldots, m\}$, defined in Fourier domain by

$$\hat{\psi}_\ell^{(m)}(\omega) := i^\ell e^{-\frac{i\omega j m}{2}} \sqrt{\binom{m}{\ell}} \frac{\cos^{m-\ell}(\omega/4) \sin^{m+\ell}(\omega/4)}{(\omega/4)^m},$$

form a tight wavelet frame in $L_2(\mathbb{R})$.
A. Ron and Z. Shen, Affine system in $L_2(\mathbb{R}^d)$: the analyis of the analysis operator, J. Func. Anal., 148: 408-447, 1997.

B-spline framelet

Set $\Psi^{(m)} = \{\psi_\ell^{(m)} : \ell = 1, \ldots, m\}$. We call $\Psi^{(m)}$ as the *B-spline framelet of order m*.

B-spline framelet

Set $\Psi^{(m)} = \{\psi_\ell^{(m)} : \ell = 1, \ldots, m\}$. We call $\Psi^{(m)}$ as the *B-spline framelet of order m*. The B-spline framelets $\Psi^{(m)}$ are used in various applications:

1. image inpainting; image denoising;
2. high and super resolution image reconstruction;
3. deblurring and blind debluring ; and image segmentation.

Z. Shen, Wavelet frames and image restorations, Proceedings of the International congress of Mathematicians, Vol IV, Hyderabad, India, (2010).

Box Splines

The *box spline* $B(\cdot|\Xi)$ associated with a matrix $\Xi \in \mathbb{R}^{s\times n}$ is the distribution given by the rule

$$\int_{\mathbb{R}^s} B(x|\Xi)\varphi(x)dx = \int_{[-\frac{1}{2},\frac{1}{2})^n} \varphi(\Xi u)du, \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^s), \qquad (3)$$

where $\mathcal{D}(\mathbb{R}^s)$ is the test function space.

Box Splines

The *box spline* $B(\cdot|\Xi)$ associated with a matrix $\Xi \in \mathbb{R}^{s\times n}$ is the distribution given by the rule

$$\int_{\mathbb{R}^s} B(x|\Xi)\varphi(x)dx = \int_{[-\frac{1}{2},\frac{1}{2})^n} \varphi(\Xi u)du, \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^s), \qquad (3)$$

where $\mathcal{D}(\mathbb{R}^s)$ is the test function space.If we take $\Xi = (1, 1, \ldots, 1) \in \mathbb{R}^{1\times m}$, then the box spline $B(\cdot|\Xi)$ is reduced to a B-spline of order m.

C. de Boor, K. Höllig and S. Riemenschneider, Box Splines, Springer-Verlag, New York, 1993.

A univariate box spline and B-spline framelet

Theorem

Set

$$\Xi_{m,\ell} := [\underbrace{1, \ldots, 1}_{m-\ell}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{2\ell}].$$

Then

$$\psi_\ell^{(m)}(x) = \sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot \frac{d^\ell}{dx^\ell} B(x|\Xi_{m,\ell}).$$

A univariate box spline and B-spline framelet

Theorem

Set

$$\Xi_{m,\ell} := [\underbrace{1, \ldots, 1}_{m-\ell}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{2\ell}].$$

Then

$$\psi_\ell^{(m)}(x) = \sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot \frac{d^\ell}{dx^\ell} B(x|\Xi_{m,\ell}).$$

The $\psi_\ell^{(m)}$ can be considered as the ℓ order derivative of the box spline $B(\cdot|\Xi_{m,\ell})$ (up to a constant).

Recurrence formula of B-splines:

$$\varphi^{(m+1)}(x) = \frac{2x+m+1}{2m}\varphi^{(m)}\left(x+\frac{1}{2}\right)+\frac{m+1-2x}{2m}\varphi^{(m)}\left(x-\frac{1}{2}\right).$$

Recurrence formula of B-splines framelet

$$\psi_1^{(1)}(x) = \begin{cases} 1, & \text{if } x \in [-1/2, 0)\ , \\ -1, & \text{if } x \in [0, 1/2]\ , \\ 0, & \text{if } |x| > 1/2. \end{cases}$$

If $\ell \leq m-1$

$$\psi_\ell^{(m+1)}(x) = \sqrt{\frac{m+1}{m+1-\ell}} \left(\frac{2x+m+1}{2m} \psi_\ell^{(m)}\left(x+\frac{1}{2}\right) + \frac{m+1-2x}{2m} \psi_\ell^{(m)}\left(x-\frac{1}{2}\right) + \frac{\ell}{m} \psi_\ell^{(m)}(x) \right);$$

Recurrence formula of B-splines framelet

$$\psi_1^{(1)}(x) = \begin{cases} 1, & \text{if } x \in [-1/2, 0)\ , \\ -1, & \text{if } x \in [0, 1/2]\ , \\ 0, & \text{if } |x| > 1/2. \end{cases}$$

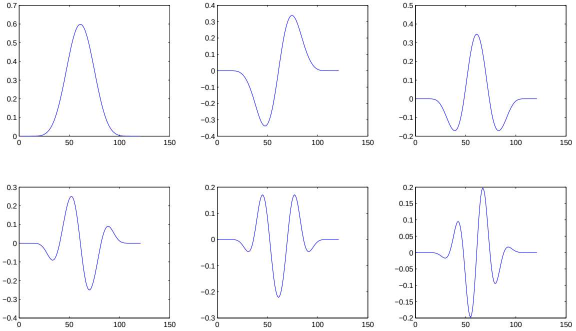
If $\ell \leq m - 1$

$$\psi_\ell^{(m+1)}(x) = \sqrt{\frac{m+1}{m+1-\ell}} \left(\frac{2x+m+1}{2m} \psi_\ell^{(m)} \left(x + \frac{1}{2} \right) + \frac{m+1-2x}{2m} \psi_\ell^{(m)} \left(x - \frac{1}{2} \right) + \frac{\ell}{m} \psi_\ell^{(m)}(x) \right);$$

$$\psi_{m+1}^{(m+1)}(x) = \frac{2x+m+1}{2m} \psi_m^{(m)} \left(x + \frac{1}{2} \right) + \frac{2x-m-1}{2m} \psi_m^{(m)} \left(x - \frac{1}{2} \right) - \frac{2x}{m} \psi_m^{(m)}(x).$$

$$
\begin{array}{ccccccc}
B_1 & & B_2 & & B_3 & & B_4 \\
\psi_1^{(1)} & & \psi_1^{(2)} & & \psi_1^{(3)} & & \psi_1^{(4)} \\
& \searrow & \psi_2^{(2)} & \rightarrow & \psi_2^{(3)} & \rightarrow & \psi_2^{(4)} \\
& & & & \psi_3^{(3)} & & \psi_3^{(4)} \\
& & & & & & \psi_4^{(4)}
\end{array} .
$$

$B_5, \psi_1^{(5)}, \ldots, \psi_5^{(5)}$



The asymptotic convergence of B-splines

Let $\varphi^{(m)}$ be B-spline of order m. Then

$$\lim_{m\to\infty} \sqrt{m} B_m(\sqrt{m}x) = \sqrt{\frac{6}{\pi}} \exp\left(-6x^2\right).$$

M. Unser, A. Aldroubi, and M. Eden, On the asymptotic convergence of B-splines wavelets to Gabor functions, IEEE Trans. Inf. Th., 38(1992), pp. 864-872.

The asymptotic convergence of univariate box splines

Theorem

For each $k \in \mathbb{N}$, *let*

$$\Xi_k := [a_1^{(k)}, \ldots, a_k^{(k)}] \in \mathbb{R}^{1\times k},$$

where $a_j^{(k)} > 0, j = 1, \ldots, k$. *Let* $B(\cdot|\Xi_k)$ *be the box spline associate with* Ξ_k. *Assume that*

$$\|\Xi_k\|_2^2 = \sigma^2 + \epsilon_k,$$

with $\sigma \in \mathbb{R}$ *is a fixed constant and* $\lim_{k\to\infty} \epsilon_k = 0$, *and assume that*

$$c_1 \leq \frac{\max_{1\leq j\leq k} a_j^{(k)}}{\min_{1\leq j\leq k} a_j^{(k)}} \leq c_2$$

where c_1 *and* c_2 *are fixed positive constants independent of* k. *Then,*

$$\max_x \left|\sqrt{\frac{6}{\pi\sigma^2}} \exp\left(-\frac{6x^2}{\sigma^2}\right) - B(x|\Xi_k)\right| \lesssim_{c_1,c_2} \frac{(\ln k)^3}{k} + |\epsilon_k| \cdot |\ln(|\epsilon_k|)| \cdot \ln(k).$$

Observations

1. The $\psi_\ell^{(m)}$ can be considered as the ℓ order derivative of a univariate box spline (up to a constant).

Observations

1. The $\psi_\ell^{(m)}$ can be considered as the ℓ order derivative of a univariate box spline (up to a constant).
2. The univariate box spline tend to a Gaussian function (under some mild conditions).

Observations

1. The $\psi_\ell^{(m)}$ can be considered as the ℓ order derivative of a univariate box spline (up to a constant).
2. The univariate box spline tend to a Gaussian function (under some mild conditions).

Question: Can we construct frames using the derivative of some Gaussian functions?

Observations

1. The $\psi_\ell^{(m)}$ can be considered as the ℓ order derivative of a univariate box spline (up to a constant).
2. The univariate box spline tend to a Gaussian function (under some mild conditions).

Question: Can we construct frames using the derivative of some Gaussian functions?
It is raised by Zuowei Shen (2011).

The asymptotic convergence of B-spline framelet

$$
\begin{aligned}
G(x) &= \sqrt{\frac{6}{\pi}} \frac{\sqrt{\binom{m}{\ell}}}{\sqrt{m-\ell/2}\cdot 4^{\ell}} \cdot \exp\left(-\frac{12\cdot x^2}{2m-\ell}\right) \\
G_\ell^{(m)}(x) &= \frac{d^\ell}{dx^\ell} G(x), \qquad \ell = 1,\dots,m, \\
G^{(m)} &= \{G_1^{(m)},\dots,G_m^{(m)}\}.
\end{aligned}
$$

The asymptotic convergence of B-spline framelet

Theorem

Let $m \in \mathbb{N}$ be given and $1 \le \ell \le m$, and the framelet $\psi_\ell^{(m)}$ be derived from B-spline of order m. Then,

$$\max_{1 \le \ell \le m} \max_{x \in \mathbb{R}} |\psi_\ell^{(m)}(x) - G_\ell^{(m)}(x)| \lesssim \frac{(\ln m)^{5/2}}{m^{3/2}}.$$

Theorem

Let $X(G^{(m)})$ be the wavelet system generated by functions $G^{(m)}$. Then $X(G^{(m)})$ is a frame system with frame bounds A_m and B_m for sufficiently large m. Furthermore, the frame is close to be tight as m is sufficiently large. In fact, asymptotically, we have

$$\lim_{m\to\infty} A_m = \lim_{m\to\infty} B_m = 1.$$

Theorem

Let $\{f_j\}_{j\in J}$ be a frame of $L_2(\mathbb{R})$ with bounds A and B. Assume that $\{g_j\}_{j\in J} \subset L_2(\mathbb{R})$ is such that $\{f_j - g_j\}_{j\in J}$ is a Bessel sequence with a bound $R < A$. Then $\{g_j\}_{j\in J}$ is a frame with bounds $A\left(1-\sqrt{\frac{R}{A}}\right)^2$ and $B\left(1+\sqrt{\frac{R}{B}}\right)^2$.

Oel Christensen,Christopher Heil, Perturbations of Banach Frames and Atomic Decompositions, Mathematische Nachrichten,1997.

Table: The numerical results of frame bounds of $X(G^{(m)})$

m	2	3	4	5	6	7	8
A	0.3855	0.5266	0.5898	0.6407	0.6803	0.7095	0.7274
B	1.9020	1.6239	1.5179	1.4390	1.3811	1.3403	1.3159

Thank you!