A duality formula + Particle Gibbs for continuous time Feynman-Kac measures

M. Arnaudon, P. Del Moral

IMB & INRIA Bordeaux Sud Ouest

IMS - August-30th Singapore 2018

- `this talk` $\rightsquigarrow$ `Arxiv (18)`
- *Discrete time duality + Taylor expansions [+ Kohn-Patras]:* `Arxiv (14), CRAS (15), IHP (16)`
- *Sharp 1st order analysis [+ Jasra]:* `SPA I` + `SPA-II (18)`
- *Influenced by the pioneering article*
  PMCMC - Andrieu, Doucet, Holenstein JRSS-B-10

Basic matrix/operator notation

Integration (Lebesgue or Riemann)

$$\eta(f) = \int \eta(dx) f(x)$$

Basic matrix/operator notation

Integration (Lebesgue or Riemann)

$$\eta(f) = \int \eta(dx) f(x)$$

(Markov) Integral transport formulae

$$(\eta K)(dy) = \int \eta(dx) K(x, dy) \quad \text{and} \quad K(f)(x) = \int K(x, dy)\ f(y)$$

Basic matrix/operator notation

Integration (Lebesgue or Riemann)

$$\eta(f) = \int \eta(dx) f(x)$$

(Markov) Integral transport formulae

$$(\eta K)(dy) = \int \eta(dx) K(x, dy) \quad \text{and} \quad K(f)(x) = \int K(x, dy)\ f(y)$$

Iterated transitions

$$K^n(x, dz) = \int K(x, dy)\ K^{n-1}(y, dz)$$

Basic matrix/operator notation

Integration (Lebesgue or Riemann)

$$\eta(f) = \int \eta(dx) f(x)$$

(Markov) Integral transport formulae

$$(\eta K)(dy) = \int \eta(dx) K(x, dy) \quad \text{and} \quad K(f)(x) = \int K(x, dy)\ f(y)$$

Iterated transitions

$$K^n(x, dz) = \int K(x, dy)\ K^{n-1}(y, dz)$$

In this talk: Measure estimates w.r.t. total variation distance

$$\mu^\epsilon = \mu + \epsilon \quad \Longleftrightarrow \quad \|\mu^\epsilon - \mu\|_{tv} \leq c\ \epsilon \quad \text{for some constant } c \perp \epsilon$$

X_t Markov $\in$ metric space S and potential $V(X_t) \geq 0$

Historical processes

$$\widehat{X}_t := (X_s)_{s \leq t} \in \widehat{S} := \cup_{t \geq 0} S_t \quad \text{and} \quad \widehat{V}_t(\widehat{X}_t) = V_t(X_t)$$

with the path space

$$S_t := \text{càdlàg paths from } [0, t] \text{ to } S$$

X_t Markov $\in$ metric space S and potential $V(X_t) \geq 0$

Historical processes

$$\widehat{X}_t := (X_s)_{s \leq t} \in \widehat{S} := \cup_{t \geq 0} S_t \quad \text{and} \quad \widehat{V}_t(\widehat{X}_t) = V_t(X_t)$$

with the path space

$$S_t := \text{càdlàg paths from } [0,t] \text{ to } S$$

- *Discrete time :* $\widehat{X}_n = (X_0, \ldots, X_n) \in S_n = S^{n+1}$
  $\rightsquigarrow$ Obvious stochastic calculus (Genealogies [AAP-01], FK-04)

X_t Markov $\in$ metric space S and potential $V(X_t) \geq 0$

Historical processes

$$\widehat{X}_t := (X_s)_{s \leq t} \in \widehat{S} := \cup_{t \geq 0} S_t \quad \text{and} \quad \widehat{V}_t(\widehat{X}_t) = V_t(X_t)$$

with the path space

$$S_t := \text{càdlàg paths from } [0, t] \text{ to } S$$

- *Discrete time :* $\widehat{X}_n = (X_0, \ldots, X_n) \in S_n = S^{n+1}$

  $\rightsquigarrow$ Obvious stochastic calculus (Genealogies [AAP-01], FK-04)

- *Continuous time :*

  $\rightsquigarrow$ Discrete time + density argument on cylindrical functions

X_t Markov $\in$ metric space S and potential $V(X_t) \geq 0$

Historical processes

$$\widehat{X}_t := (X_s)_{s \leq t} \in \widehat{S} := \cup_{t \geq 0} S_t \quad \text{and} \quad \widehat{V}_t(\widehat{X}_t) = V_t(X_t)$$

with the path space

$$S_t := \text{càdlàg paths from } [0,t] \text{ to } S$$

- *Discrete time :* $\widehat{X}_n = (X_0, \ldots, X_n) \in S_n = S^{n+1}$

  $\rightsquigarrow$ Obvious stochastic calculus (Genealogies [AAP-01], FK-04)

- *Continuous time :*

  $\rightsquigarrow$ Discrete time + density argument on cylindrical functions

  $\rightsquigarrow$ Infinitesimal generator processes/FK on measurable spaces [+ Miclo SPA-00, Séminaire Probab.-00, SAA-07]

X_t Markov $\in$ metric space S and potential $V(X_t) \geq 0$

Historical processes

$$\widehat{X}_t := (X_s)_{s\leq t} \in \widehat{S} := \cup_{t\geq 0} S_t \quad \text{and} \quad \widehat{V}_t(\widehat{X}_t) = V_t(X_t)$$

with the path space

$$S_t := \text{càdlàg paths from } [0,t] \text{ to } S$$

- *Discrete time :* $\widehat{X}_n = (X_0, \ldots, X_n) \in S_n = S^{n+1}$

  $\rightsquigarrow$ Obvious stochastic calculus (Genealogies [AAP-01], FK-04)

- *Continuous time :*

  $\rightsquigarrow$ Discrete time + density argument on cylindrical functions

  $\rightsquigarrow$ Infinitesimal generator processes/FK on measurable spaces [+ Miclo SPA-00, Séminaire Probab.-00, SAA-07]

  $\rightsquigarrow$ Ito-functional calculus $\in$ [Dupire-09,Cont-Fournié-13,Jazaerli-Saporito-17, Saporito-14]

FK-measures on path space := $FK(\widehat{X}, \widehat{V})$

$$d\mathbb{Q}_t = \frac{1}{Z_t}\ \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\ d\mathbb{P}_t$$

with the normalizing constant

$$Z_t = \mathbb{E}\left(\exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right) \quad \text{and} \quad \mathbb{P}_t = \mathrm{Law}(\widehat{X}_t)$$

FK-measures on path space := $FK(\widehat{X}, \widehat{V})$

$$d\mathbb{Q}_t = \frac{1}{Z_t} \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right] d\mathbb{P}_t$$

with the normalizing constant

$$Z_t = \mathbb{E}\left(\exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right) \quad \text{and} \quad \mathbb{P}_t = \mathrm{Law}(\widehat{X}_t)$$

$$\Updownarrow$$

Weak formulation on test functions F on S_t:

$$\mathbb{Q}_t(F) = \int F(x)\, \mathbb{Q}_t(dx) \propto \mathbb{E}\left(F(\widehat{X}_t)\, \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right)$$

Time marginal FK-measures $= FK(X, V)$

Path-space:

$$\mathbb{Q}_t(F) \;\propto\; \mathbb{E}\left(F(\widehat{X}_t)\;\exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right)$$

Time marginal FK-measures $= FK(X, V)$

Path-space:

$$\mathbb{Q}_t(F) \;\propto\; \mathbb{E}\left(F(\widehat{X}_t)\;\exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right)$$

$$F(\widehat{X}_t) = f(X_t) \quad \Downarrow$$

Time marginal:

$$\eta_t(f) \;\propto\; \mathbb{E}\left(f(X_t)\;\exp\left[-\int_0^t V_s(X_s)ds\right]\right)$$

Time marginal FK-measures $= FK(X, V)$

Path-space:

$$\mathbb{Q}_t(F) \;\propto\; \mathbb{E}\left(F(\widehat{X}_t)\;\exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right)$$

$$F(\widehat{X}_t) = f(X_t) \quad \Downarrow$$

Time marginal:

$$\eta_t(f) \;\propto\; \mathbb{E}\left(f(X_t)\;\exp\left[-\int_0^t V_s(X_s)ds\right]\right)$$

$X_t = \widehat{Y}_t := (Y_s)_{0\leq s\leq t} \rightsquigarrow$ *Path-space again (even more general):*

$$\eta_t(f) \;\propto\; \mathbb{E}\left(F(\widehat{Y}_t)\;\exp\left[-\int_0^t V_s(\widehat{Y}_s)ds\right]\right)$$

$FK(\widehat{X}, \widehat{V})$ - particle sampler = N-path particles

$$\xi_t := (\xi_t^i)_{1 \leq i \leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1 \leq s \leq t}$$

$FK(\widehat{X}, \widehat{V})$ - particle sampler $=$ N-path particles

$$\xi_t := (\xi_t^i)_{1 \leq i \leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1 \leq s \leq t}$$

Interacting jump process:

- i.i.d. initial conditions.

$FK(\widehat{X}, \widehat{V})$ - particle sampler = N-path particles

$$\xi_t := (\xi_t^i)_{1 \leq i \leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1 \leq s \leq t}$$

Interacting jump process:

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.

$FK(\widehat{X}, \widehat{V})$ - particle sampler = N-path particles

$$\xi_t := (\xi_t^i)_{1\leq i\leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1\leq s\leq t}$$

Interacting jump process:

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $\widehat{V}$.

$FK(\widehat{X}, \widehat{V})$ - particle sampler $=$ N-path particles

$$\xi_t := (\xi_t^i)_{1 \leq i \leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1 \leq s \leq t}$$

Interacting jump process:

- i.i.d. initial conditions.
- Between jumps : independent $+$ same evolutions as $\widehat{X}_t$.
- Jump/killing rate $\widehat{V}$.
- Jump onto/offspring birth - uniformly on the path-pool.

$FK(\widehat{X}, \widehat{V})$ - particle sampler = N-path particles

$$\xi_t := (\xi_t^i)_{1 \leq i \leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1 \leq s \leq t}$$

Interacting jump process:

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $\widehat{V}$.
- Jump onto/offspring birth - uniformly on the path-pool.

Historical process: $\widehat{\xi}_t := (\xi_s)_{0 \leq s \leq t}$

$FK(\widehat{X},\widehat{V})$ - particle sampler = N-path particles

$$\xi_t := (\xi_t^i)_{1\leq i\leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1\leq s\leq t}$$

Interacting jump process:

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $\widehat{V}$.
- Jump onto/offspring birth - uniformly on the path-pool.

Historical process: $\widehat{\xi}_t := (\xi_s)_{0\leq s\leq t}$

Ancestral line & Occupation measure of the ancestral tree

$$\mathbb{X}_t \sim m(\xi_t) := \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_t^i}$$

$FK(\widehat{X}, \widehat{V})$ - particle sampler $=$ N-path particles

$$\xi_t := (\xi_t^i)_{1\leq i\leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1\leq s\leq t}$$

Interacting jump process:

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $\widehat{V}$.
- Jump onto/offspring birth - uniformly on the path-pool.

Historical process: $\widehat{\xi}_t := (\xi_s)_{0\leq s\leq t}$

Ancestral line & Occupation measure of the ancestral tree

$$\mathbb{X}_t \sim m(\xi_t) := \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_t^i}$$

In the talk : All particle N-product spaces up to index permutations (= empirical measures)

Equivalent interpretations

Genetic algorithm [Turing - 1950], Spatial branching processes [Galton-Watson - 1873, Yaglom - 1947, Harris - 1951], Moran and/or Nanbu type particle system, Mean field particle interpretations of nonlinear Markov processes [Fermi - 1948, McKean - 1966],...

Equivalent interpretations

Genetic algorithm [Turing - 1950], Spatial branching processes [Galton-Watson - 1873, Yaglom - 1947, Harris - 1951], Moran and/or Nanbu type particle system, Mean field particle interpretations of nonlinear Markov processes [Fermi - 1948, McKean - 1966],...

Many other lively buzzwords :

Reconfiguration, resampling, bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, go with the winner, look-ahead, weighted dynamics, quantum teleportation,...

Equivalent interpretations
Genetic algorithm [Turing - 1950], Spatial branching processes [Galton-Watson - 1873, Yaglom - 1947, Harris - 1951], Moran and/or Nanbu type particle system, Mean field particle interpretations of nonlinear Markov processes [Fermi - 1948, McKean - 1966],...

Many other lively buzzwords :
Reconfiguration, resampling, bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, go with the winner, look-ahead, weighted dynamics, quantum teleportation,...

Looks similar for non initiated but differs from Fleming-Viot & Dawson-Watanabe processes:

- Different jump scales ($\propto$ N)

Equivalent interpretations

Genetic algorithm [Turing - 1950], Spatial branching processes [Galton-Watson - 1873, Yaglom - 1947, Harris - 1951], Moran and/or Nanbu type particle system, Mean field particle interpretations of nonlinear Markov processes [Fermi - 1948, McKean - 1966],...

Many other lively buzzwords :

Reconfiguration, resampling, bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, go with the winner, look-ahead, weighted dynamics, quantum teleportation,...

Looks similar for non initiated but differs from Fleming-Viot & Dawson-Watanabe processes:

- Different jump scales ($\propto$ N)
- $m(\xi_t) \longrightarrow_{N\to\infty}$ *random* measures (a.k.a. genetic noise)

Equivalent interpretations
Genetic algorithm [`Turing - 1950`], Spatial branching processes [Galton-Watson - 1873, Yaglom - 1947, Harris - 1951], Moran and/or Nanbu type particle system, Mean field particle interpretations of nonlinear Markov processes [`Fermi - 1948`, McKean - 1966],...

Many other lively buzzwords :
Reconfiguration, resampling, bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, go with the winner, look-ahead, weighted dynamics, quantum teleportation,...

Looks similar for non initiated but differs from Fleming-Viot & Dawson-Watanabe processes:

- Different jump scales ($\propto$ N)
- $m(\xi_t) \longrightarrow_{N\to\infty}$ *random* measures (a.k.a. genetic noise)
- Requires *symmetric/neutral* branching/selection functions/rates

Some key results (when $FK(X,V)$ stable)

Fluctuation/clt + expo. concentration + ldp :

$$m(\xi_t) = \mathbb{Q}_t + \frac{1}{\sqrt{N}}\,\mathbb{V}_t \qquad \text{with} \quad \mathbb{V}_t \simeq \text{Gaussian field}$$

Some key results (when $FK(X,V)$ stable)

Fluctuation/clt + expo. concentration + ldp :

$$m(\xi_t) = \mathbb{Q}_t + \frac{1}{\sqrt{N}}\,\mathbb{V}_t \qquad \text{with} \quad \mathbb{V}_t \simeq \text{Gaussian field}$$

Bias/propagation of chaos:

$$\text{Law}(\mathbb{X}_t) = \mathbb{Q}_t + \frac{t}{N} \quad \text{and} \quad \text{Law}(\mathbb{X}_{s,t}) = \mathbb{Q}_t + \frac{1+(t-s)}{N}$$

Some key results (when $FK(X, V)$ stable)

Fluctuation/clt + expo. concentration + ldp :

$$m(\xi_t) = \mathbb{Q}_t + \frac{1}{\sqrt{N}}\ \mathbb{V}_t \qquad \text{with} \quad \mathbb{V}_t \simeq \text{Gaussian field}$$

Bias/propagation of chaos:

$$\text{Law}(\mathbb{X}_t) = \mathbb{Q}_t + \frac{t}{N} \quad \text{and} \quad \text{Law}(\mathbb{X}_{s,t}) = \mathbb{Q}_t + \frac{1+(t-s)}{N}$$

Unbiasedness property :

$$\mathbb{E}\left(F(\mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)\ \propto\ \mathbb{Q}_t(F)$$

Some key results (when $FK(X, V)$ stable)

Fluctuation/clt + expo. concentration + ldp :

$$m(\xi_t) = \mathbb{Q}_t + \frac{1}{\sqrt{N}}\ \mathbb{V}_t \qquad \text{with} \quad \mathbb{V}_t \simeq \text{Gaussian field}$$

Bias/propagation of chaos:

$$\text{Law}(\mathbb{X}_t) = \mathbb{Q}_t + \frac{t}{N} \quad \text{and} \quad \text{Law}(\mathbb{X}_{s,t}) = \mathbb{Q}_t + \frac{1+(t-s)}{N}$$

Unbiasedness property :

$$\mathbb{E}\left(F(\mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)\ \propto\ \mathbb{Q}_t(F)$$

↳ *not so new:* ⇝ [discrete time MPRF-96, continuous time SPA-00]

Unbiasedness prop. $\rightsquigarrow$ Many-body FK

Unbiasedness property :

$$\mathbb{E}\left(F(\mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)\ \propto\ \mathbb{Q}_t(F)$$

Unbiasedness prop. $\rightsquigarrow$ Many-body FK

Unbiasedness property :

$$\mathbb{E}\left(F(\mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)\ \propto\ \mathbb{Q}_t(F)$$

↳ **Many-body FK measures on a product space $(\mathbb{S}_1 \times \mathbb{S}_2) \ni (\mathbb{X}_t, \widehat{\xi}_t)$:**

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \int\ F(x_1, x_2)\ \Pi_t(d(x_1, x_2))$$

Unbiasedness prop. $\rightsquigarrow$ Many-body FK

Unbiasedness property :

$$\mathbb{E}\left(F(\mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right) \ \propto\ \mathbb{Q}_t(F)$$

↳ **Many-body FK measures on a product space** $(\mathbb{S}_1 \times \mathbb{S}_2) \ni (\mathbb{X}_t, \widehat{\xi}_t)$**:**

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \int\ F(x_1, x_2)\ \Pi_t(d(x_1, x_2))$$

Nb.:

- Unbiasedness property $\Longleftrightarrow$ *Under* Π_t *we have* $\mathbb{X}_t\ \sim\ \mathbb{Q}_t$

Unbiasedness prop. $\rightsquigarrow$ Many-body FK

Unbiasedness property :

$$\mathbb{E}\left(F(\mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right) \ \propto\ \mathbb{Q}_t(F)$$

$\hookrightarrow$ **Many-body FK measures on a product space** $(\mathbb{S}_1 \times \mathbb{S}_2) \ni (\mathbb{X}_t, \widehat{\xi}_t)$**:**

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \int\ F(x_1, x_2)\ \Pi_t(d(x_1, x_2))$$

Nb.:

- Unbiasedness property $\Longleftrightarrow$ *Under* Π_t *we have* $\mathbb{X}_t\ \sim\ \mathbb{Q}_t$
- *Independent Metropolis-Hastings - obvious*

↳ Many-body FK on a product space $(\mathbb{S}_1 \times \mathbb{S}_2) \ni (\mathbb{X}_t, \widehat{\xi}_t)$

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \int\ F(x_1, x_2)\ \Pi_t(d(x_1, x_2))$$

↳ Many-body FK on a product space $(\mathbb{S}_1 \times \mathbb{S}_2) \ni (\mathbb{X}_t, \widehat{\xi}_t)$

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \int F(x_1, x_2)\ \Pi_t(d(x_1, x_2))$$

$$\Downarrow$$

Gibbs sampler on $(\mathbb{S}_1 \times \mathbb{S}_2)$ with target Π_t

$$\underbrace{((1) \mid (2)) = \left(\mathbb{X}_t \mid \widehat{\xi}_t\right) = m(\xi_t)}_{\textit{easy to sample !}}$$

↳ Many-body FK on a product space $(\mathbb{S}_1 \times \mathbb{S}_2) \ni (\mathbb{X}_t, \widehat{\xi}_t)$

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \int\ F(x_1, x_2)\ \Pi_t(d(x_1, x_2))$$

$$\Downarrow$$

Gibbs sampler on $(\mathbb{S}_1 \times \mathbb{S}_2)$ with target Π_t

$$\underbrace{((1) \mid (2)) = \left(\mathbb{X}_t \mid \widehat{\xi}_t\right) = m(\xi_t)}_{\textit{easy to sample !}} \quad \text{and} \quad ((2) \mid (1)) = \left(\widehat{\xi}_t \mid \mathbb{X}_t\right) =?$$

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$ ↴

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2 \leq i \leq N}$ and $\xi_s^1 = \mathbb{X}_s$

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$ $\neg$

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2\leq i\leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$ $\neg$

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2\leq i\leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$ ↴

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2\leq i\leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $(1 - 1/N)\ \widehat{V}_s$.

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$ $\neg$

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2\leq i\leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $(1 - 1/N)$ $\widehat{V}_s$.
- Jump onto/offspring birth - uniformly on the path-pool.

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2 \leq i \leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $(1 - 1/N)\ \widehat{V}_s$.
- Jump onto/offspring birth - uniformly on the path-pool.
- **Extra jump rate $(2/N)\ \widehat{V}_s$ onto $\mathbb{X}_s$.**

The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2 \leq i \leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $(1 - 1/N)\ \widehat{V}_s$.
- Jump onto/offspring birth - uniformly on the path-pool.
- **Extra jump rate** $(2/N)\ \widehat{V}_s$ **onto** $\mathbb{X}_s$**.**

Nb.: without the blue

the above process coincides with the FK-particle sampler

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2\le i\le N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $(1 - 1/N)\ \widehat{V}_s$.
- Jump onto/offspring birth - uniformly on the path-pool.
- **Extra jump rate $(2/N)\ \widehat{V}_s$ onto $\mathbb{X}_s$.**

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2\leq i\leq N}$ and $\xi_s^1 = \mathbb{X}_s$

- i.i.d. initial conditions.
- Between jumps : independent + same evolutions as $\widehat{X}_t$.
- Jump/killing rate $(1-1/N)\ \widehat{V}_s$.
- Jump onto/offspring birth - uniformly on the path-pool.
- **Extra jump rate $(2/N)\ \widehat{V}_s$ onto $\mathbb{X}_s$.**

Given $\mathbb{X}_t = x \sim \mathbb{Q}_t$

Coincides with the particle sampler $\widehat{\zeta}_t^- = (\zeta_s^i)_{2\leq i\leq N}$ of FK measures:

$$\begin{array}{rcl} \widehat{V}_s & \rightsquigarrow & (1-1/N)\ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \oplus \text{extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } \zeta_s^1 := x_s \end{array}$$

Equivalent formulation - Duality formula

Theo.: [+Arnaudon ⇝ `Arxiv (18)`]

$$\mathbb{E}\left(F(\mathbb{X}_t,\widehat{\xi}_t)\; e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right)=\mathbb{E}\left(F(\widehat{X}_t,\widehat{\zeta}_t)\; e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)$$

Equivalent formulation - Duality formula

Theo.: [+Arnaudon ⇝ Arxiv (18)]

$$\mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t, \widehat{\zeta}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)$$

Nb.:

- = random sample from $\mathbb{Q}_t$

Equivalent formulation - Duality formula

Theo.: [+Arnaudon $\rightsquigarrow$ `Arxiv (18)`]

$$\mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t, \widehat{\zeta}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)$$

Nb.:

- $=$ random sample from $\mathbb{Q}_t$
- $=$ many-body/twisted distribution on $\widehat{\xi}_t$ and

$$(\widehat{\xi}_t \mid \mathbb{X}_t = x) = (\widehat{\zeta}_t \mid \widehat{X}_t = x) \qquad \mathbb{Q}_t - a.e.\ x$$

Equivalent formulation - Duality formula

Theo.: [+Arnaudon $\rightsquigarrow$ Arxiv (18)]

$$\mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{X}_t, \widehat{\zeta}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)$$

Nb.:

- $=$ random sample from $\mathbb{Q}_t$
- $=$ many-body/twisted distribution on $\widehat{\xi}_t$ and

$$(\widehat{\xi}_t \mid \mathbb{X}_t = x) = (\widehat{\zeta}_t \mid \widehat{X}_t = x) \qquad \mathbb{Q}_t - a.e.\ x$$

- *discrete time version*

$\subset$ +Kohn+Patras, Arxiv (14), CRAS (15)+IHP (16)

$\Longrightarrow$ Reversible Gibbs-sampler with target $\mathbb{Q}_t$

$$\mathbb{X} = x \longrightarrow \left[\; \widehat{\zeta}_t^x \;\sim\; \left(\widehat{\zeta}_t \mid \widehat{X}_t = x\right) \;\right] \longrightarrow \overline{\mathbb{X}} = \overline{x} \;\sim\; m(\widehat{\zeta}_t^x)$$

$\Longrightarrow$ Reversible Gibbs-sampler with target $\mathbb{Q}_t$

$$\mathbb{X} = x \longrightarrow \left[\ \widehat{\zeta}_t^x \ \sim \ \left(\widehat{\zeta}_t \mid \widehat{X}_t = x\right)\ \right] \longrightarrow \overline{\mathbb{X}} = \overline{x} \ \sim \ m(\widehat{\zeta}_t^x)$$

$$\Updownarrow$$

$\mathbb{Q}_t$-reversible Markov transition:

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right]$$

$\Longrightarrow$ Reversible Gibbs-sampler with target $\mathbb{Q}_t$

$$\mathbb{X} = x \longrightarrow \left[\ \widehat{\zeta}_t^x \ \sim \ \left(\widehat{\zeta}_t \mid \widehat{X}_t = x\right)\ \right] \longrightarrow \overline{\mathbb{X}} = \overline{x} \ \sim \ m(\widehat{\zeta}_t^x)$$

$$\Updownarrow$$

$\mathbb{Q}_t$-reversible Markov transition:

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right]$$

Key observation for the convergence analysis:

$$m(\widehat{\zeta}_t^x) = \frac{1}{N}\ \delta_x + \left(1 - \frac{1}{N}\right)\ m(\widehat{\zeta}_t^{x,-})$$

with $\widehat{\zeta}_t^{x,-} = (\zeta_s^{x,-})_{0\leq s\leq t}$ = particle sampler of an FK measure:

$$\begin{array}{lll} \widehat{V}_s & \rightsquigarrow & (1 - 1/N)\ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \oplus \text{extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } x_s \end{array}$$

The FK-bias/propagation of chaos

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right] = \mathbb{Q}_t^x(f) + \frac{t}{N}$$

The FK-bias/propagation of chaos

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right] = \mathbb{Q}_t^x(f) + \frac{t}{N}$$

but with FK measure $\mathbb{Q}_t^x$ defined with:

$$\begin{array}{lll} \widehat{V}_s & \rightsquigarrow & (1 - 1/N)\ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \ \oplus \ \text{extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } x_s \end{array}$$

The FK-bias/propagation of chaos

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right] = \mathbb{Q}_t^x(f) + \frac{t}{N}$$

but with FK measure $\mathbb{Q}_t^x$ defined with:

$$\begin{array}{ccl} \widehat{V}_s & \rightsquigarrow & (1 - 1/N)\ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \ \oplus \ \text{extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } x_s \end{array}$$

"A little" perturbation analysis

$$\mathbb{Q}_t^x = \mathbb{Q}_t + \frac{t}{N} \Longrightarrow \delta_x \mathbb{K}_t = \mathbb{Q}_t + \frac{t}{N} \Longrightarrow \delta_x \mathbb{K}_t - \delta_y \mathbb{K}_t = \frac{t}{N}$$

The FK-bias/propagation of chaos

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right] = \mathbb{Q}_t^x(f) + \frac{t}{N}$$

but with FK measure $\mathbb{Q}_t^x$ defined with:

$$\widehat{V}_s \rightsquigarrow (1 - 1/N)\ \widehat{V}_s$$
$$\widehat{X}_s \rightsquigarrow \widehat{X}_s \oplus \text{extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } x_s$$

"A little" perturbation analysis

$$\mathbb{Q}_t^x = \mathbb{Q}_t + \frac{t}{N} \Longrightarrow \delta_x\mathbb{K}_t = \mathbb{Q}_t + \frac{t}{N} \Longrightarrow \delta_x\mathbb{K}_t - \delta_y\mathbb{K}_t = \frac{t}{N}$$

$$\Downarrow$$

For any starting path $x \in S_t$ after n iterations we have

$$\delta_x\mathbb{K}_t^n = \mathbb{Q}_t + \left(c\ \frac{t}{N}\ \right)^n$$

The FK-bias/propagation of chaos

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\widehat{\zeta}_t^x)(f)\right] = \mathbb{Q}_t^x(f) + \frac{t}{N}$$

but with FK measure $\mathbb{Q}_t^x$ defined with:

$$\begin{array}{lcl} \widehat{V}_s & \rightsquigarrow & (1 - 1/N)\ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \oplus \text{ extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } x_s \end{array}$$

"A little" perturbation analysis

$$\mathbb{Q}_t^x = \mathbb{Q}_t + \frac{t}{N} \Longrightarrow \delta_x\mathbb{K}_t = \mathbb{Q}_t + \frac{t}{N} \Longrightarrow \delta_x\mathbb{K}_t - \delta_y\mathbb{K}_t = \frac{t}{N}$$

$$\Downarrow$$

For any starting path $x \in S_t$ after n iterations we have

$$\delta_x\mathbb{K}_t^n = \mathbb{Q}_t + \left(c\ \frac{t}{N}\ \right)^n \qquad \text{more info on "c"?}$$

More refined (discrete time)

Theo. [+Kohn+Patras, Arxiv (14), CRAS (15)+IHP (16)]

$$\mathbb{K}_t = \mathbb{Q}_t + \sum_{1 \leq k < l} \left(\frac{t}{N}\right)^k \partial^{(k)}\mathbb{K}_t + \left(\frac{t}{N}\right)^l \overline{\partial}^l \mathbb{K}_t$$

$(\forall l \geq 1) \oplus$ Differential operators $\sim$ coalescent/colored trees and s.t.

$$\left\|\partial^{(k)}\mathbb{K}_t\right\| \vee \left\|\overline{\partial}^{(k)}\mathbb{K}_t\right\| \leq k^{2k} \quad \text{as soon as } l^2 t/N < 1.$$

More refined (discrete time)

Theo. [+Kohn+Patras, Arxiv (14), CRAS (15)+IHP (16)]

$$\mathbb{K}_t = \mathbb{Q}_t + \sum_{1\leq k<l} \left(\frac{t}{N}\right)^k \partial^{(k)}\mathbb{K}_t + \left(\frac{t}{N}\right)^l \overline{\partial}^l\mathbb{K}_t$$

$(\forall l \geq 1) \oplus$ Differential operators $\sim$ coalescent/colored trees and s.t.

$$\left\|\partial^{(k)}\mathbb{K}_t\right\| \vee \left\|\overline{\partial}^{(k)}\mathbb{K}_t\right\| \leq k^{2k} \quad \text{as soon as } l^2 t/N < 1.$$

$\Downarrow$ [*direct*]

$$\left| \|\mathbb{K}_t^{\mathbf{n}}(f) - \mathbb{Q}_t(f)\|_{\mathbb{L}_p(\mathbb{Q}_t)} - \left(\frac{t}{N}\right)^{\mathbf{n}} \|[\partial\mathbb{K}_t]^{\mathbf{n}}(f)\|_{\mathbb{L}_p(\mathbb{Q}_t)} \right| \leq \left(c\ \frac{t}{N}\right)^{\mathbf{n}+1}$$

More refined (discrete time)

Theo. [+Kohn+Patras, `Arxiv (14)`, `CRAS (15)`+`IHP` (16)]

$$\mathbb{K}_t = \mathbb{Q}_t + \sum_{1 \leq k < l} \left(\frac{t}{N}\right)^k \partial^{(k)}\mathbb{K}_t + \left(\frac{t}{N}\right)^l \overline{\partial}^l \mathbb{K}_t$$

$(\forall l \geq 1) \oplus$ Differential operators $\sim$ coalescent/colored trees and s.t.

$$\left\| \partial^{(k)} \mathbb{K}_t \right\| \vee \left\| \overline{\partial}^{(k)} \mathbb{K}_t \right\| \leq k^{2k} \quad \text{as soon as } l^2 t/N < 1.$$

$\Downarrow$ [*direct*]

$$\left| \|\mathbb{K}_t^{\mathbf{n}}(f) - \mathbb{Q}_t(f)\|_{\mathbb{L}_p(\mathbb{Q}_t)} - \left(\frac{t}{N}\right)^{\mathbf{n}} \|[\partial \mathbb{K}_t]^{\mathbf{n}}(f)\|_{\mathbb{L}_p(\mathbb{Q}_t)} \right| \leq \left(c \, \frac{t}{N}\right)^{\mathbf{n}+1}$$

Continuous time ? $\rightsquigarrow$ *Open problem/question*

"Hint/solution" $\rightsquigarrow$ Taylor/Bias [+ Rubenthaler-Patras] JTP (09)

$\oplus$ backward analysis using prop 2.1 in $\rightsquigarrow$ `Arxiv (18)`