Importance sampling type estimators based on approximate marginal MCMC

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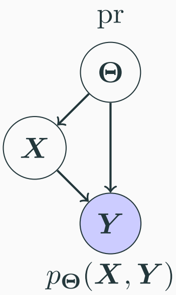
Bayesian latent variable models

Bayesian latent variable models

The following abstraction applies to many relevant statistical models:

- $\boldsymbol{\Theta}$ is a vector of (hyper)parameters
- $\boldsymbol{X}$ is a vector of latent variables
- $\boldsymbol{Y}$ is a vector of observations

Only $\boldsymbol{Y} = \boldsymbol{y}^*$ observed, both $\boldsymbol{\Theta}$ and $\boldsymbol{X}$ are unknown.



The model is defined in terms of the following conditonal laws:

- $\boldsymbol{\Theta} \sim \mathrm{pr}(\,\cdot\,)$.
- $(\boldsymbol{X}, \boldsymbol{Y}) \mid \boldsymbol{\Theta} \sim p_{\boldsymbol{\Theta}}(\,\cdot\,)$
  (Often, $p_{\boldsymbol{\Theta}}(\boldsymbol{X}, \boldsymbol{Y}) = f_{\boldsymbol{\Theta}}(\boldsymbol{X}) g_{\boldsymbol{\Theta}}(\boldsymbol{Y} \mid \boldsymbol{X})$, but this is not relevant here.)

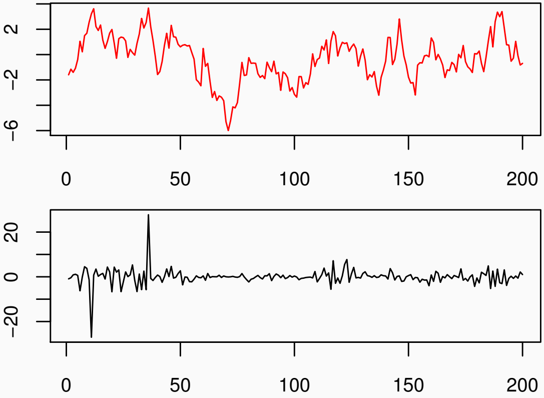
We are interested in the posterior of $(\boldsymbol{\Theta}, \boldsymbol{X})$ after observing $\boldsymbol{Y} = \boldsymbol{y}^*$:

$$\boxed{\pi(\boldsymbol{\theta}, \boldsymbol{x}) = p(\boldsymbol{x}, \boldsymbol{\theta} \mid \boldsymbol{y}^*) \propto p(\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{y}^*) = \mathrm{pr}(\boldsymbol{\theta}) p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*).}$$

Example of $p_{\Theta}(\boldsymbol{X}, \boldsymbol{Y})$: Stochastic volatility model

- $\boldsymbol{\Theta} = (\phi, \sigma_x, \sigma_y)$
- $\boldsymbol{X} = (X^{(1)}, \ldots, X^{(T)})$ stationary Gaussian AR(1) with parameters (ϕ, σ_x^2).
- The observations $\boldsymbol{Y} = (Y^{(1)}, \ldots, Y^{(T)})$ are zero-mean Gaussian with $\mathrm{sd}(Y^{(t)}) = \sigma_y \exp(X^{(t)})$,

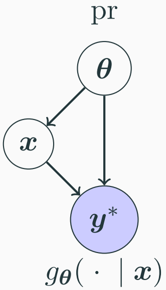
One realisation of $\boldsymbol{X}$ and $\boldsymbol{Y}$ with $\boldsymbol{\theta} = (0.9, 1, 2)$ and $T = 200$.



Challenges for inference

Typical scenario in a latent variable model:

- The hyperparameters $\boldsymbol{\Theta}$ are low-dimensional
    - $\dim(\boldsymbol{\Theta}) = 3$ in the SV-example.
- The latent variables $\boldsymbol{X}$ are high-dimensional
    - Often, $\dim(\boldsymbol{X}) \propto \dim(\boldsymbol{Y})$.
    - $\dim(\boldsymbol{X}) = 200$ in the SV-example.



Standard 'out-of-the-box' inference (e.g. using `BUGS`, `Stan`...):

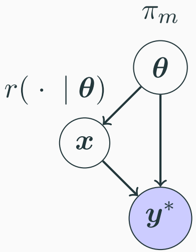
- Simulate MCMC chain $\boldsymbol{Z}_k = (\boldsymbol{\Theta}_k, \boldsymbol{X}_k)$, targetting π.
- High overall dimension & high correlations $\implies$ inefficient

Some popular inference algorithms

Structure of the model & notation for inference

Consider the following factorisation of the posterior:

$$\pi(\boldsymbol{\theta}, \boldsymbol{x}) = \pi_m(\boldsymbol{\theta}) r(\boldsymbol{x} \mid \boldsymbol{\theta}),$$



where the marginal posterior density and the corresponding conditional are given as follows:

$$\pi_m(\boldsymbol{\theta}) = \int \pi(\boldsymbol{\theta}, \boldsymbol{x}) \mathrm{d}\boldsymbol{x} \propto \mathrm{pr}(\boldsymbol{\theta}) L(\boldsymbol{\theta})$$

$$r(\boldsymbol{x} \mid \boldsymbol{\theta}) = \frac{p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*)}{L(\boldsymbol{\theta})}$$

with the marginal likelihood L taking the form:

$$L(\boldsymbol{\theta}) = \int p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*) \mathrm{d}\boldsymbol{x}.$$

Separate algorithms for parameters & latents?

- $\boldsymbol{\theta}$ low-dimensional, but π_m often non-standard
  $\implies$ Non-parametric approximation, such as MCMC
- Problems (unless $p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*)$ is of specific form such as Gaussian):
  - $L(\boldsymbol{\theta})$ is intractable.
  - $r(\boldsymbol{x} \mid \boldsymbol{\theta})$ is intractable.
- Two (succesful branches of) solutions:
  - Approximate $L(\boldsymbol{\theta})$ and $r(\boldsymbol{x} \mid \boldsymbol{\theta})$ analytically.
  - Approximate $L(\boldsymbol{\theta})$ and $r(\boldsymbol{x} \mid \boldsymbol{\theta})$ using a specialised Monte Carlo algorithm.

Inference: Integrated nested Laplace approximation (INLA)

(Rue, Martino & Chopin, 2009, *J. R. Stat. Soc. Ser. B. Stat. Methodol.*)

- Suppose $p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}) = f_{\boldsymbol{\theta}}(\boldsymbol{x}) g_{\boldsymbol{\theta}}(\boldsymbol{y})$ where $f_{\boldsymbol{\theta}}(\,\cdot\,)$ is Gaussian (& $g_{\boldsymbol{\theta}}$ of certain form)
- For any given $\boldsymbol{\theta}$, use Gaussian (Laplace) approximation $\hat{p}_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*) \approx p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*)$
  $\implies$ approximate likelihood $L_a(\boldsymbol{\theta}) = \int \hat{p}_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*) \mathrm{d}\boldsymbol{x}$
- Take a finite number of points $(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n)$ and approximate the full posterior as (something like)

$$\hat{\pi}(\mathrm{d}\boldsymbol{\theta}, \mathrm{d}\boldsymbol{x}) = \frac{\sum_{k=1}^n w_k \mathrm{pr}(\boldsymbol{\theta}_k) \hat{p}_{\boldsymbol{\theta}_k}(\boldsymbol{x}, \boldsymbol{y}^*) \delta_{\boldsymbol{\theta}_k}(\mathrm{d}\boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}}{\sum_{j=1}^n w_j \mathrm{pr}(\boldsymbol{\theta}_j) L_a(\boldsymbol{\theta}_j)}.$$

  where the weight w_k depends on the strategy how $(\boldsymbol{\theta}_i)$ chosen...
  - (Further marginal corrections may be applied as well...)
- There is an approximation error (which does not vanish if $n \to \infty$)

(Andrieu, Doucet & Holenstein, 2010, *J. R. Stat. Soc. Ser. B. Stat. Methodol.*)

- For any given $\boldsymbol{\theta}$, it is straightforward to generate random variables $(V^{(i)}, \boldsymbol{X}^{(i)})$, with $V^{(i)} \geq 0$, using particle filter (PF), which satisfy

$$\mathbb{E}\left[\sum_{i=1}^{m} V^{(i)}\right] = L(\boldsymbol{\theta}), \qquad \text{and} \qquad \mathbb{E}\left[\sum_{i=1}^{m} V^{(i)} f(\boldsymbol{X}^{(i)})\right] = \int p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y}^*) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

- The algorithm:
  - Implement Metropolis-Hastings $(\boldsymbol{\Theta}_k)_{k \geq 1}$ targetting π_m, using $\sum_{i=1}^{m} V^{(i)}$ in place of $L(\boldsymbol{\theta})$.
  - Construct an approximation of full $\pi(\boldsymbol{\theta}, \boldsymbol{x})$ using $(V^{(i)}, \boldsymbol{X}^{(i)})$ above.

Particle marginal Metropolis-Hastings (PMMH) algorithm:

- Draw a new proposal $\tilde{\mathbf{\Theta}}_k \sim q(\mathbf{\Theta}_{k-1}, \,\cdot\,)$
- $\boxed{\text{Run PF with } \boldsymbol{\theta} = \tilde{\mathbf{\Theta}}_k} \longrightarrow (\tilde{V}_k^{(i)}, \tilde{\boldsymbol{X}}_k^{(i)})$.
- Accept and set $(\mathbf{\Theta}_k, V_k^{(i)}, \boldsymbol{X}_k^{(i)}) \leftarrow (\tilde{\mathbf{\Theta}}_k, \tilde{V}_k^{(i)}, \tilde{\boldsymbol{X}}_k^{(i)})$ With probability

$$\min\left\{1, \frac{\mathrm{pr}(\tilde{\mathbf{\Theta}}_k)\big(\sum_{i=1}^m \tilde{V}_k^{(i)}\big) q(\tilde{\mathbf{\Theta}}_k, \mathbf{\Theta}_{k-1})}{\mathrm{pr}(\mathbf{\Theta}_{k-1})\big(\sum_{i=1}^m V_{k-1}^{(i)}\big) q(\mathbf{\Theta}_{k-1}, \tilde{\mathbf{\Theta}}_k)}\right\};$$

otherwise reject and set $(\mathbf{\Theta}_k, V_k^{(i)}, \boldsymbol{X}_k^{(i)}) \leftarrow (\mathbf{\Theta}_{k-1}, V_{k-1}^{(i)}, \boldsymbol{X}_{k-1}^{(i)})$

What is nice about this is that:

- This is valid MCMC, in the sense that

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\sum_{i=1}^{m} V_k^{(i)} f(\boldsymbol{\Theta}_k, \boldsymbol{X}_k)}{\sum_{i=1}^{m} V_k^{(i)}} \xrightarrow{n\to\infty} \int f(\boldsymbol{\theta}, \boldsymbol{x})\pi(\boldsymbol{\theta}, \boldsymbol{x})\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{x}, \qquad \text{(a.s.)}$$

  (under a minimal Harris recurrence assumption)
- $\implies$ PMCMC provides (asymptotically) exact inference (as $n \to \infty$).

(NB: There is no asymptotic in 'number of particles' m...
...but m must be 'large enough' to make the MCMC mix sufficiently well...)

Should I use INLA or PMCMC?

- (When applicable) INLA[1] is fast, and often very accurate
  - How accurate? How can you tell?
- PMCMC computationally demanding, but exact (asymptotically)
  - Does not require 'nearly Gaussian' structure of $p_{\boldsymbol{\theta}}(\boldsymbol{x}, \boldsymbol{y})$.
  - Might need large m to work well $\implies$ slow
  - Might still be 'sticky' (slower than geometric if $\sum_{i=1}^{m} V^{(i)}$ are unbounded. . .)
- How about combining ideas both from INLA and PMCMC:

→ Monte Carlo correction/diagnosis for INLA output, or

→ Laplace approximations to speed up PMCMC. . .

[1] The same arguments hold with any other approximate scheme in place of INLA'!

Approximations for speeding up PMMH — delayed acceptance

(Christen & Fox, 2005, *J. Comput. Graph. Statist.*)

Trick to make MCMC faster by using an approximation-based 'screening'.

- Draw a new proposal $\tilde{\mathbf{\Theta}}_k \sim q(\mathbf{\Theta}_{k-1}, \,\cdot\,)$
- With probability

$$\min\left\{1, \frac{\mathrm{pr}(\tilde{\mathbf{\Theta}}_k) L_a(\tilde{\mathbf{\Theta}}_k) q(\tilde{\mathbf{\Theta}}_k, \mathbf{\Theta}_{k-1})}{\mathrm{pr}(\mathbf{\Theta}_{k-1}) L_a(\mathbf{\Theta}_{k-1}) q(\mathbf{\Theta}_{k-1}, \tilde{\mathbf{\Theta}}_k)}\right\}$$

  continue to the next step, otherwise reject.
- $\boxed{\text{Run PF with } \boldsymbol{\theta} = \tilde{\mathbf{\Theta}}_k} \longrightarrow (\tilde{V}_k^{(i)}, \tilde{\boldsymbol{X}}_k^{(i)})$
- With probability

$$\min\left\{1, \frac{\left(\sum_{i=1}^m \tilde{V}_k^{(i)}\right) / L_a(\tilde{\mathbf{\Theta}}_k)}{\left(\sum_{i=1}^m V_{k-1}^{(i)}\right) / L_a(\mathbf{\Theta}_{k-1})}\right\}$$

  accept, otherwise reject.

Importance sampling type estimator based on marginal MCMC

Approximations for speeding up PMMH — importance sampling type

(Review, consistency and CLTs: V, Helske, Franks, arXiv:1609.02541)

Phase 1: MCMC which targets the approximate marginal $\pi_a(\boldsymbol{\theta}) \propto \mathrm{pr}(\boldsymbol{\theta} L_a(\boldsymbol{\theta})$

- Draw a new proposal $\tilde{\mathbf{\Theta}}_k \sim q(\mathbf{\Theta}_{k-1}, \,\cdot\,)$
- With probability

$$\min\left\{1, \frac{\mathrm{pr}(\tilde{\mathbf{\Theta}}_k)L_a(\tilde{\mathbf{\Theta}}_k)q(\tilde{\mathbf{\Theta}}_k, \mathbf{\Theta}_{k-1})}{\mathrm{pr}(\mathbf{\Theta}_{k-1})L_a(\mathbf{\Theta}_{k-1})q(\mathbf{\Theta}_{k-1}, \tilde{\mathbf{\Theta}}_k)}\right\}$$

accept $\mathbf{\Theta}_k = \tilde{\mathbf{\Theta}}_k$; otherwise reject $\mathbf{\Theta}_k = \mathbf{\Theta}_{k-1}$.

Phase 2: For $k = 1, \ldots, n$, $\boxed{\text{run PF with } \boldsymbol{\theta} = \mathbf{\Theta}_k} \longrightarrow (V_k^{(i)}, X_k^{(i)})$ and calculate

$$E_n = \frac{\sum_{k=1}^n \sum_{i=1}^m W_k^{(i)} f(\mathbf{\Theta}_k, \boldsymbol{X}_k^{(i)})}{\sum_{j=1}^n \sum_{\ell=1}^m W_j^{(\ell)}} \qquad \text{where} \qquad W_k^{(i)} = \frac{V_k^{(i)}}{L_a(\mathbf{\Theta}_k)}.$$

Why IS might be better than DA?

- Phase 2 corrections entirely independent ('post-processing') $\Longrightarrow$ parallelisable $\Longrightarrow$ scalable.
- Allows for calculating the correction only for accepted states ('jump chain') $\Longrightarrow$ less expensive than DA
- Allows for (further) *thinning* before (expensive) correction $\Longrightarrow$ further savings
- The approximate marginal MCMC $(\boldsymbol{\Theta}_k)$ need not rely on estimators $\Longrightarrow$ safer & easier to implement efficiently (e.g. adaptive MCMC...)
- The MCMC $(\boldsymbol{\Theta}_k)$ need not be reversible $\Longrightarrow$ new exciting non-reversible samplers readily applicable!
- Non-negativity of the estimator W_k not required $\Longrightarrow$ allows for direct 'debiasing' tricks (or 'randomised multi-level Monte Carlo') (cf. Rhee & Glynn, *Oper. Res.* 2015; V, *Oper. Res.*, 2018)

General setup & assumptions

General setup for IS *type* estimators based on approximate marginal MCMC:

- $\pi(\boldsymbol{\theta}, \boldsymbol{x}) = \pi_m(\boldsymbol{\theta}) r(\boldsymbol{x} \mid \boldsymbol{\theta})$.
- $\pi_m \ll \pi_a$
- $(\boldsymbol{\Theta}_k)_{k \geq 1}$ MCMC Harris ergodic wrt π_a
- $\big(\xi_k\big)_{k \geq 1}$ conditionally independent finite random signed measures given $(\boldsymbol{\Theta}_k)_{k \geq 1}$, which form "proper weighting":

General	The LVM example
$\pi_m(\boldsymbol{\theta})$	$\propto \mathrm{pr}(\boldsymbol{\theta}) L(\boldsymbol{\theta})$
$\pi_a(\boldsymbol{\theta})$	$\propto \mathrm{pr}(\boldsymbol{\theta}) L_a(\boldsymbol{\theta})$
$\xi_k(f)$	$\sum_{i=1}^m W_k^{(i)} f(\boldsymbol{\Theta}_k, \boldsymbol{X}_k^{(i)})$

$$\mathbb{E}[\xi_k(\mathbf{1}) \mid \boldsymbol{\Theta}_k = \boldsymbol{\theta}] = w_u(\boldsymbol{\theta}), \qquad \text{where} \qquad w_u(\boldsymbol{\theta}) = c_w \frac{\pi_m(\boldsymbol{\theta})}{\pi_a(\boldsymbol{\theta})}, \quad c_w > 0$$

$$\mathbb{E}[\xi_k(f) \mid \boldsymbol{\Theta}_k = \boldsymbol{\theta}] = w_u(\boldsymbol{\theta}) \int r(\boldsymbol{x} \mid \boldsymbol{\theta}) f(\boldsymbol{\theta}, \boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

Consistency & CLT

- If $\pi_a(m^{(1)}) < \infty$ where $m^{(1)}(\boldsymbol{\theta}) = \mathbb{E}\big[|\xi_k(1)| + |\xi_k(f)| \;\big|\; \boldsymbol{\Theta}_k = \boldsymbol{\theta}\big]$, then

$$E_n = \frac{\sum_{k=1}^n \xi_k(f)}{\sum_{j=1}^n \xi_j(\mathbf{1})} \xrightarrow[\text{a.s.}]{n\to\infty} \pi(f) = \int f(\boldsymbol{\theta}, \boldsymbol{x})\pi(\boldsymbol{\theta}, \boldsymbol{x})\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{x}.$$

- Suppose further that (for instance):
  - $\pi_a(m^{(2)}) < \infty$ with $m^{(2)} = \mathbb{E}\big[\xi_k(\bar{f})^2 \;\big|\; \boldsymbol{\Theta}_k = \boldsymbol{\theta}\big]$ where $\bar{f}(\boldsymbol{\theta}, \boldsymbol{x}) = f(\boldsymbol{\theta}, \boldsymbol{x}) - \pi(f)$,
  - $(\boldsymbol{\Theta}_k)_{k\geq 1}$ follows P which is aperiodic and reversible, with asymptotic variance $\mathrm{Var}(w_u\bar{f}^*, P) < \infty$, where $\bar{f}^*(\boldsymbol{\theta}, \boldsymbol{x}) = \int \bar{f}(\boldsymbol{\theta}, \boldsymbol{x}')r(\boldsymbol{x}' \mid \boldsymbol{\theta})\mathrm{d}\boldsymbol{x}'$,

Then,

$$\sqrt{n}\big[E_n - \pi(f)\big] \xrightarrow[\text{d}]{n\to\infty} N\Bigg(0, \overbrace{\frac{\mathrm{Var}(w_u\bar{f}^*, P)}{c_w^2}}^{\text{MCMC}} + \overbrace{\frac{\pi_a(v)}{c_w^2}}^{\text{IS corr}}\Bigg),$$

where $v(\boldsymbol{\theta}) = \mathrm{Var}\big(\xi_k(\bar{f}) \;\big|\; \boldsymbol{\Theta}_k = \boldsymbol{\theta}\big)$.

Pseudo-marginal approximate chain

- The IS-type correction may be applied also when $(\mathbf{\Theta}_k, U_k)_{k\geq 1}$ is a pseudo-marginal chain arising from estimators $\tilde{U}_{\boldsymbol{\theta}}$ satisfying $E[\tilde{U}_{\boldsymbol{\theta}}] = L_a(\boldsymbol{\theta})$.

In the pseudo-marginal case, consistency is more delicate:

- If $\tilde{U}_{\boldsymbol{\theta}} > 0$ a.s., then then we may always use $W_k^{(i)} = V_k^{(i)}/U_k$.
- When $\mathbb{P}(\tilde{U}_{\boldsymbol{\theta}} = 0)$ depends on $\boldsymbol{\theta}$, this must be accounted for.
- For instance, $V_k^{(i)}$ is constructed independent of U_k, then we must compensate for an extra factor $p(\boldsymbol{\theta}) = \mathbb{P}(U_{\boldsymbol{\theta}} > 0)\ldots \rightsquigarrow$ lazy ABC

Examples

State space model with linear-Gaussian state dynamics

- State dynamics linear-Gaussian.
- Family of non-linear/non-Gaussian observation models.
- Approximate inference based on Laplace approximation (Durbin & Koopman, *Biometrika*, 1997).
- Proper weighting based on:
  (i) Bootstrap particle filter (BSF).
  (ii) Simple importance sampling & antithetic variables (SPDK, Shephard & Pitt, *Biometrika*, 1997; Durbin & Koopman, *Biometrika*, 1997)
  (iii) ψ-auxiliary particle filter (ψ-APF: bootstrap PF for 'Laplace twisted model;' see Guarniero, Johansen & Lee, *JASA*, 2017)
- Simple IS-corrected estimator (IS1) or estimator based on jump chain (IS2).
- Compare against direct pseudo-marginal (PM) and delayed acceptance (DA).

State space model with linear-Gaussian state dynamics: empirical results

(Stochastic volatility model with $T = 5473$ observations, S&P index data. The numbers are 'inverse relative efficiencies' (avg. time (h) $\times$ MSE) — lower is better.)

	BSF		SPDK				ψ-APF					
	AI	AI[G]	IS2	IS2[8]	PM	DA	IS1	IS2	PM	DA	IS1	IS2
Time	1.3	0.2	25.2	4.6	4.4	1.9	2.8	1.5	2.4	1.4	1.5	1.3
ϕ	0.083	0.062	0.304	0.050	1.015	0.696	0.684	0.483	0.021	0.024	0.009	0.017
σ_η	0.726	0.298	0.483	0.096	3.090	3.307	0.603	0.710	0.044	0.055	0.016	0.028
ν	0.008	0.747	0.287	0.042	1.208	2.544	0.228	0.404	0.026	0.027	0.010	0.020
X_1	0.133	0.035	0.321	0.071	3.054	1.883	0.346	0.373	0.029	0.026	0.007	0.018
X_{5473}	1.887	0.417	0.540	0.112	6.574	1.871	0.444	0.810	0.057	0.064	0.012	0.039

Discretely observed (time-discretised) diffusion

- ‘Ideal’ state dynamics follows a stochastic differential equation (SDE).
    - Cannot simulate exactly from the ideal transition.
    - Easy to simulate from time-discretised model (Euler, Milstein, . . .).
    - The denser discretisation, the more simulation costs.
- Conditionally independent observations at discrete times.
- Approximate inference: particle marginal Metropolis-Hastings (PMMH) with ‘coarse’ (and cheap) time-discretisation.
- Correction with particle filter using ‘fine’ time-discretisation.

Discretely observed (time-discretised) diffusion: empirical results

(Geometric Brownian motion observed at integer times, linear-Gaussian observations of log-state, Milstein discretisation. Parallel implementation with 48 cores, time mins.)

	Mean						IRE				
Init.		Prior mean			Prior sample		Prior mean			Prior sample	
	GT	DA	IS2	IS2$^{\mathrm{t}}$	DA	IS2	DA	IS2	IS2$^{\mathrm{t}}$	DA	IS2
Time	—	12.3	3.4	1.9	14.0	3.3	12.3	3.4	1.9	14.0	3.3
ν	0.053	0.061	0.053	0.053	0.064	0.053	0.069	0.004	0.002	0.135	0.004
σ_x	0.253	0.278	0.253	0.253	0.251	0.252	0.576	0.029	0.019	0.336	0.022
σ_y	1.058	1.054	1.058	1.058	1.083	1.058	0.088	0.020	0.014	1.010	0.022
X_1	1.254	1.273	1.254	1.246	1.243	1.252	0.670	0.109	0.119	0.805	0.103
X_{50}	2.960	2.953	2.966	2.935	20.773	2.971	12.605	1.880	2.074	4×10^6	2.308

Discretely observed (time-discretised) diffusion: empirical results

(Geometric Brownian motion observed at integer times, linear-Gaussian observations of log-state, Milstein discretisation. Parallel implementation with 48 cores, time mins.)

	Mean						IRE				
Init.		Prior mean			Prior sample		Prior mean			Prior sample	
	GT	DA	IS2	IS2^{t}	DA	IS2	DA	IS2	IS2^{t}	DA	IS2
Time	—	12.3	3.4	1.9	14.0	3.3	12.3	3.4	1.9	14.0	3.3
ν	0.053	0.061	0.053	0.053	0.064	0.053	0.069	0.004	0.002	0.135	0.004
σ_x	0.253	0.278	0.253	0.253	0.251	0.252	0.576	0.029	0.019	0.336	0.022
σ_y	1.058	1.054	1.058	1.058	1.083	1.058	0.088	0.020	0.014	1.010	0.022
X_1	1.254	1.273	1.254	1.246	1.243	1.252	0.670	0.109	0.119	0.805	0.103
X_{50}	2.960	2.953	2.966	2.935	20.773	2.971	12.605	1.880	2.074	4×10^6	2.308

Discretely observed diffusion: Ideal diffusion inference with randomised MLMC

- Instead of correcting with 'fine' dynamics, it is possible to do IS correction for ideal SDE dynamics (Franks, Jasra, Law & V, arXiv:1807.10259, 2018).
- The correction is based on
  - Debiasing trick/randomised MLMC (Rhee & Glynn, *Oper. Res.*, 2015) with
  - 'Δ-PF' (Jasra, Kamatani, Law & Zhou. *SIAM J. Sci. Comp.*, 2018).
- Detailed presentation in the closing workshop!

Discussion

Is IS correction always better than DA (with same n)?

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No.

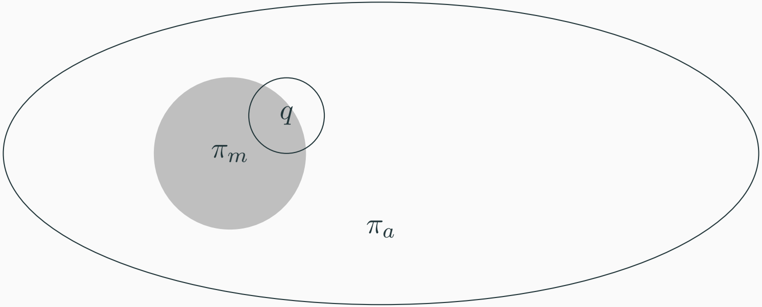
Is IS correction always better than DA (with same n)?

No.

More details Wed 12 Sep at 4pm...

DA can be much better than IS

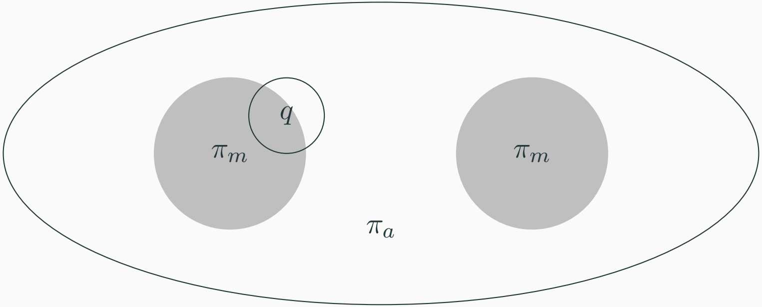
(Franks, V: arXiv:1706.09873)



DA better than IS: π_m and π_a are uniform, q uniform random walk. Approximate chain spends a lot of time outside the support of π_m.

But DA can also be much worse than IS...

(Franks, V: arXiv:1706.09873)



IS better than DA chain, which is reducible (cannot switch mode of π_m).

Can we say something about IS vs DA?

- In practice, we have $\pi_a \approx \pi_m$, which is clearly not the case in the examples above.
- Empirical results suggest that IS often improves on DA slightly

(Franks, V: arXiv:1706.09873):

- If $c_w^{-1} W_k \leq C$ a.s., then

$$\mathrm{Var}(\mathsf{IS}) \leq C\mathrm{Var}(\mathsf{DA}) + \bar{\pi}(\xi^2[C - c_w^{-1}W])$$

  where $\bar{\pi}$ corresponds to the stationary distribution of the DA chain.
  $\rightsquigarrow$ With parallelisation, IS might be a better choice...

- NB: In the LVM setting we may modify the likelihood approximation:
  - $L_a(\boldsymbol{\theta}) \to L_a(\boldsymbol{\theta}) + \epsilon$

  This leads to bounded weights if the likelihood estimators are bounded.

Concluding remarks

- If there is an approximation available, use it!
- IS type correction is a natural way to use the approximation
  - May be a useful alternative to DA pseudo-marginal algorithm (because of the several possible advantages)...
  - ... but not guaranteed to be uniformly better
- Our contributions:
  - arXiv:1609.02541: Review, consistency/CLT results; application in the state-space context, using Laplace approximation and coarse discretisation of diffusion model
  - arXiv:1706.09873: Theoretical bounds relating the efficiencies of IS/DA
  - arXiv:1807.10259: Full inference of SDE driven HMM based on randomised MLMC.
- Ongoing work:
  - Application beyond the state-space context.
  - Some insights for ABC-MCMC...

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Thank you!