

# On Hurwicz–Nash Equilibria of Non–Bayesian Games under Incomplete Information

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# Outline of the Talk

## A Large Non-Atomic Games: A Forty-five Year-old Subject

- The initiation: Classical Papers
- Two Counterexamples and Two Open Questions
- A Resolution of Sorts

## B The announced paper with Patrick Beissner

- A one-theorem paper with an eye to the literature on *imprecise* probabilities

## C Current ongoing brooding:

- Statistics and Game Theory: An interface of two registers
- Countable and Finite Additivity
- Nonstandard Analysis: Two Methods of Pushing-down

## Outline of the Talk: Item A

Game Theory revolves around what the late philosopher Donald Davidson termed *triangulation*: the self, its other and the environment in which the self and its other are placed, and in which they co-operatively and non-cooperatively negotiate

I want to begin by presenting a taxonomy.

- Large non-anonymous games: individualized – microeconomic
- Large anonymous games: distributionalized – macroeconomic
- Large games with bio-social traits: anonymous and non-anonymous
- Finite games with incomplete information: independent and correlated information.

But first, I mention an analytical shift – a linguistic turn, if one prefers – from the cardinality of the space of information to the cardinality of the set of events that can be formed from the set of the sample points of information. This is to say from the cardinality of the set of sample points to their combinability. This emphasis on *combinability* is novel and has taken some time to emerge.

Put differently, this turn has involved the notion of *necessity* as in Keisler-Sun (2002), The necessity of rich probability spaces, *mimeo*  
Keisler-Sun (2009), Why saturated probability spaces are necessary equilibria, *Advances in Mathematics*.

## Outline of the Talk: Item B

- An abstract
- Motivation: Hurwicz and Morris
- Bayesian game with incomplete information
- Non-Bayesian game and Hurwicz-Nash equilibrium
- The result
- Some technicalities.

## An Abstract

- We consider finite-player simultaneous-play games of private information in which a player has no prior belief concerning the information under which the other players take their decisions, and which he or she therefore cannot discern.
- This dissonance leads us to develop the notion of Hurwicz-Nash equilibrium of non-Bayesian games, based on non-expected utility under ambiguity as developed in Gul-Pesendorfer (2014, 2015).
- We present a theorem on the existence of such a pure-strategy equilibrium in a finite-action setting.

## Questions of Morris (1995)

- Why is it that common priors are implicit or explicit in the vast majority of the differential information literature in economics and game theory?
- Why has the economic community been unwilling, in practice, to accept and actually use the idea of truly private probabilities in much the same way that it did accept the idea of private utility functions? After all, in Savage's expected utility theory, both the utilities and probabilities are derived separately for each decision maker.
- Why were the utilities accepted as private, and the probabilities not?

## An Alternative Entry: Hurwicz (1951)

The emphasis is on ... the *technology* of the processes whereby decisions are reached and the choices are made. [W]hen the information processing aspects are taken explicitly into account it is found that the concept of “rational action” is modified. This is true when applied to the action of a single individual, but it becomes particularly interesting when considered in situations involving many persons ... The uncertainty need not be generated by external factors like weather prospects: it may be man-made.



## In Other Words

- The conceptual question is how a player is to move from subjective (private probabilities) beliefs on his subjective (private  $\sigma$ -algebras) information regarding the set of states that he can discern, to objective (public) beliefs on the information available to the others in the game and which he cannot, by definition, discern.
- The answer that we pursue in this paper is simply that he has to move ambiguously.

- This is to say that each player has little option but to extend his private probability on his private information to a possible *set* of probabilities on *all* of the available information in the game; and rather than an expectation taken with respect to a *single* Bayesian prior, he has to modify his objective function in accordance with this extended set of probabilities.
- Thus our question leads us naturally to the literature on ‘ambiguity and the Bayesian paradigm’ authoritatively surveyed in Gilboa-Marinacci (2016).
- But we focus here on the *extension* of individual beliefs as opposed to the *restriction* of an exogenously-assumed universal public belief on the totality of the privately-available, and presumably secret, information.

## Additional Features

- We do not assume a linear structure on the individual action sets, and therefore cannot appeal to any quasi-concavity assumptions on the individual payoffs as in the literature on “ambiguous games” stemming from Marinacci (2000) and his followers.
- Instead, we do assume private information to be “diffused” and “dispersed”, as originally formulated in Radner-Rosenthal (1982), but complemented by an assumption on the existence of independent atomless supplements, original to Aumann (1974).
- Our formulation involves a multi-valued *extension* of an individual’s prior to the join of the finest  $\sigma$ -algebra  $\mathcal{F}$  of the information of the other players, and an absolute-continuity assumption on an individual’s belief with respect to the extended beliefs on  $\mathcal{F}$ .

# Bayesian Primitives

Aumann description

## Definition

A **model of incomplete information** is a specification  $(I, \Omega, (\mathcal{F}_i)_{i \in I}, P)$  where

- $I$  is the set of players.
- $T_i$  denotes the set of types of player  $i$ .  $\Omega = \times_{i \in I} T_i$  consists of the states of the world.
- $\mathcal{F}_i$  is a  $\sigma$ -algebra on  $\Omega$ , the private information of player  $i \in I$ .
- $P$  is a probability measure on  $\mathcal{F}$ , the coarsest  $\sigma$ -algebra containing  $(\mathcal{F}_i)_{i \in I}$ .

# Bayesian Game

## Definition

A **Bayesian game with incomplete information** is a specification  $(I, \Omega, (\mathcal{F}_i)_{i \in I}, P, (A_i)_{i \in I})$  where:

- Each agent  $i$  can take actions from a finite set  $A_i$ .
- Any pure strategy  $s_i : \Omega \rightarrow A_i$  of agent  $i$  is  $\mathcal{F}_i$ -measurable.
- The  $\mathcal{F}_i$ -conditional  $P$ -expected payoffs are computed by a state-dependent utility index  $u_i : \Omega \times A \rightarrow \mathbb{R}$ , where  $A = \times_{i \in I} A_i$  :

$$\mathbb{E}^P [u_i(s_i, s_{-i}) | \mathcal{F}_i] = \int_{\Omega} u_i(\omega, s_i(\omega), s_{-i}(\omega)) dP(\omega | \mathcal{F}_i)$$

# Bayesian–Nash Equilibrium

## Definition

A **Bayesian–Nash equilibrium** for a game with incomplete information is a set of strategies  $s_i^{\text{B}*} : \Omega \rightarrow A_i$ ,  $\mathcal{F}_i$ -measurable for each  $i \in I$ , that satisfies

$$\mathbb{E}^P [u_i(s_i^{\text{B}*}, s_{-i}^{\text{B}*}) | \mathcal{F}_i] \geq \mathbb{E}^P [u_i(s_i, s_{-i}^{\text{B}*}) | \mathcal{F}_i] \quad P\text{-almost surely}$$

for all pure and  $\mathcal{F}_i$ -measurable strategies  $s_i$  of player  $i$ .

# Non-Bayesian Primitives

## Definition

A **model of probabilistically incomplete information** is a specification  $(I, \Omega, (P_i, \mathcal{F}_i)_{i \in I})$  where:

- $I$  is the set of players.
- $T_i$  is the set of types of player  $i$ .  $\Omega = \times_{i \in I} T_i$  consists of the states of the world.
- For any  $i \in I$ ,  $\mathcal{F}_i$  is a  $\sigma$ -algebra of  $\Omega$ .
- $P_i$  is a probability measure **ON**  $(\Omega, \mathcal{F}_i)$  for each  $i \in I$ .

## Non Bayesian Primitives II

To overcome the inability to evaluate expected payoffs depending on the opponents strategies, we define for each agent  $i$  the set

$$\mathcal{P}(\mathcal{F}_i) = \{P \in \Delta(\Omega, \mathcal{F}) : P = P_i \text{ on } \mathcal{F}_i\} \quad (1)$$



# Non-Bayesian Game

## Definition

A **game  $\Gamma$  with probabilistically incomplete information** is a specification  $(I, \Omega, (\mathcal{F}_i, P_i)_{i \in I}, (A_i)_{i \in I})$  where:

- Each agent  $i$  can take actions from a finite set  $A_i$ .
- Any **pure** strategy  $s_i : \Omega \rightarrow A_i$  of player  $i$  is a  $\mathcal{F}_i$ -measurable mapping.
- The  $\tilde{P}_i$ -expected payoff are computed by a state dependent utility index  $u_i : \Omega \times A \rightarrow \mathbb{R}$ , where  $A = \times_{i \in I} A_i$  :

$$\mathbb{E}^{\tilde{P}}[u_i(s_i, s_{-i})] = \int_{\Omega} u_i(\omega, s_i(\omega), s_{-i}(\omega)) d\tilde{P}(\omega)$$

where  $\tilde{P} : \mathcal{F} \rightarrow [0, 1]$  is an **extension** of  $P_i$  and  $(s_i, s_{-i})$  denotes a given profile of pure strategies.

- Any agent has a **preference for ambiguity**, via  $\alpha_i \in [0, 1]$ .

# Hurwicz–Nash Equilibrium

Assume each player applies a **Hurwicz expected payoff**  $W_i$ , based on [Gul and Pesendorfer \(2015\)](#):

$$W_i(s_i, s_{-i}) = \alpha_i \min_{\tilde{P} \in \mathcal{P}(\mathcal{F}_i)} \mathbb{E}^{\tilde{P}}[u_i(s_i, s_{-i})] + (1 - \alpha_i) \max_{\tilde{P} \in \mathcal{P}(\mathcal{F}_i)} \mathbb{E}^{\tilde{P}}[u_i(s_i, s_{-i})].$$

## Definition

A **Hurwicz–Nash equilibrium** for a game with probabilistically incomplete information is a list of  $\mathcal{F}_i$ -measurable strategies  $s_i^{\text{H}^*} : \Omega \rightarrow A_i$  that satisfies

$$W_i(s_i^{\text{H}^*}, s_{-i}^{\text{H}^*}) \geq W_i(s_i, s_{-i}^{\text{H}^*})$$

for all pure  $\mathcal{F}_i$ -measurable strategies  $s_i$  of player  $i \in I$ .

# Existence of Hurwicz–Nash Equilibrium

## Assumption 1

- (i) Players agree on the null set, which is determined by an atomless  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ , where  $\mathcal{F} = \sigma(\mathcal{F}_1, \dots, \mathcal{F}_I)$ . Moreover,  $\mathbb{P} = \prod_{i \in I} \lambda_i$ , where each  $\lambda_i$  is an atomless probability measure on  $(T_i, \mathcal{T}_i)$ .
- (ii) For each player  $i$ , the utility index  $u_i(\cdot, a)$  is  $\mathbb{P}$ -square integrable and continuous for any  $a = (a_1, \dots, a_I) \in A$ .
- (iii) The utility index  $u_i$  only depends on the  $i$ -th component of  $\omega = (t_1, \dots, t_I)$ .
- (iv) For each  $i \in I$ , there exists a  $\sigma_i \in L^2(\Omega, \mathcal{F}, \mathbb{P})$  such that only those extensions  $\tilde{\mathbb{P}}$  are considered that satisfy  $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \leq \sigma_i$ .

$$\mathcal{P}(\mathcal{F}_i) = \left\{ \tilde{P} \in \Delta(\Omega, \mathcal{F}) : \tilde{P} = P_i \text{ on } \mathcal{F}_i \text{ and } \tilde{P} \ll \mathbb{P} \text{ with } \frac{d\tilde{P}}{d\mathbb{P}} \leq \sigma_i \right\}. \quad (2)$$

### Theorem

Under Assumption 1, Hurwicz–Nash equilibria exist.

## Distribution of a Correspondence

Let  $A$  be a finite set,  $Y$  a metric space,  $(T, \mathcal{T}, \lambda)$  an atomless probability space, and  $\Xi : T \times Y \Rightarrow A$  be a correspondence. For each  $y \in Y$ , let  $\Xi(\cdot, y) : T \Rightarrow A$  be  $\mathcal{T}$ -measurable. Define the correspondence from  $Y$  to  $\Delta(A)$  by

$$\mathcal{D}_{\Xi(\cdot, y)} = \left\{ \lambda \circ \phi^{-1} : \phi \in \mathbf{MB}^{sel}(\Xi(\cdot, y)) \right\} \quad \text{for all } y \in Y,$$

where  $\mathbf{MB}^{sel}$  denotes the collection of all measurable selections of a correspondence. Then

- (i)  $\mathcal{D}_{\Xi(\cdot, y)}$  is convex and compact valued;
- (ii) if, in addition, the correspondence  $F(t, \cdot)$  is upper hemi-continuous on  $Y$  for each  $t \in T$ , then  $\mathcal{D}_{\Xi(\cdot, y)}$  is upper hemi-continuous on  $Y$ .

## The Literature: Classical Papers

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