# Efficient Disposal Equilibria of Pseudomarkets

Andy McLennan

Workshop on Game Theory
Institute for Mathematical Sciences
National University of Singapore

June 2018

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.
- They trade on a market.

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.
- They trade on a market.

Several unusual features are natural:

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.
- They trade on a market.

Several unusual features are natural:

- Objects may be indivisible.

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.
- They trade on a market.

Several unusual features are natural:

- Objects may be indivisible.
- Free disposal can be important.

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.
- They trade on a market.

Several unusual features are natural:

- Objects may be indivisible.
- Free disposal can be important.
- Some consumers may be sated.

# Introduction

A *pseudomarket* is a mechanism for allocating a collectively owned endowment in settings where monetary transfers are impossible or inappropriate.

- Agents are endowed with artificial currency or commodity bundles.
- They trade on a market.

Several unusual features are natural:

- Objects may be indivisible.
- Free disposal can be important.
- Some consumers may be sated.

Outcomes may need to be computed.

# Plan of the Talk

- Closely related literature will be reviewed.

# Plan of the Talk

- Closely related literature will be reviewed.
- The main contribution is an existence-of-competitive-equilibrium result.

# Plan of the Talk

- Closely related literature will be reviewed.
- The main contribution is an existence-of-competitive-equilibrium result.
  - This result encompasses existence results in related literature, as well as classical existence theorems.

# Plan of the Talk

- Closely related literature will be reviewed.
- The main contribution is an existence-of-competitive-equilibrium result.
  - This result encompasses existence results in related literature, as well as classical existence theorems.
  - It is more general in ways that are relevant for certain applications.

# Plan of the Talk

- Closely related literature will be reviewed.
- The main contribution is an existence-of-competitive-equilibrium result.
  - This result encompasses existence results in related literature, as well as classical existence theorems.
  - It is more general in ways that are relevant for certain applications.
  - The proof has some interesting and novel features.

# Plan of the Talk

- Closely related literature will be reviewed.
- The main contribution is an existence-of-competitive-equilibrium result.
  - This result encompasses existence results in related literature, as well as classical existence theorems.
  - It is more general in ways that are relevant for certain applications.
  - The proof has some interesting and novel features.
- Two open problems are described.

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

- A finite set of agents.

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

- A finite set of agents.
- A finite set of indivisible objects.

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

- A finite set of agents.
- A finite set of indivisible objects.
    - Each object has an integral capacity.

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

- A finite set of agents.
- A finite set of indivisible objects.
    - Each object has an integral capacity.
- Each agent has a vNM utility over the objects.

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

- A finite set of agents.
- A finite set of indivisible objects.
  - Each object has an integral capacity.
- Each agent has a vNM utility over the objects.

The goal is to find a probability distribution over feasible assignments of an object to each agent that is efficient and fair.

# Hylland and Zeckhauser (1979)

Hylland and Zeckhauser (henceforth HZ) study a setting in which there are:

- A finite set of agents.
- A finite set of indivisible objects.
  - Each object has an integral capacity.
- Each agent has a vNM utility over the objects.

The goal is to find a probability distribution over feasible assignments of an object to each agent that is efficient and fair.

- They propose equilibrium allocations of a market with currency endowments and goods that are probabilities of being assigned to each object.

# Compact Consumption Sets

Bergstrom (1976), Mas-Colell (1992), and Polemarchakis and Siconolfi (1993) study existence of general competitive equilibrium with compact consumption sets.

# Compact Consumption Sets

Bergstrom (1976), Mas-Colell (1992), and Polemarchakis and Siconolfi (1993) study existence of general competitive equilibrium with compact consumption sets.

- This encompasses classical general equilibrium with more-is-always-better consumers.

# Compact Consumption Sets

Bergstrom (1976), Mas-Colell (1992), and Polemarchakis and Siconolfi (1993) study existence of general competitive equilibrium with compact consumption sets.

- This encompasses classical general equilibrium with more-is-always-better consumers.
- Mas-Colell allows redistribution of sated consumers' excess income.

# Compact Consumption Sets

Bergstrom (1976), Mas-Colell (1992), and Polemarchakis and Siconolfi (1993) study existence of general competitive equilibrium with compact consumption sets.

- This encompasses classical general equilibrium with more-is-always-better consumers.
- Mas-Colell allows redistribution of sated consumers' excess income.
- Following Gale and Mas-Colell (1975, 1979), Mas-Colell allows quite general externalities.

# Compact Consumption Sets

Bergstrom (1976), Mas-Colell (1992), and Polemarchakis and Siconolfi (1993) study existence of general competitive equilibrium with compact consumption sets.

- This encompasses classical general equilibrium with more-is-always-better consumers.
- Mas-Colell allows redistribution of sated consumers' excess income.
- Following Gale and Mas-Colell (1975, 1979), Mas-Colell allows quite general externalities.
    - Probably that could be done here, but this has not been pursued.

# Compact Consumption Sets

Bergstrom (1976), Mas-Colell (1992), and Polemarchakis and Siconolfi (1993) study existence of general competitive equilibrium with compact consumption sets.

- This encompasses classical general equilibrium with more-is-always-better consumers.
- Mas-Colell allows redistribution of sated consumers' excess income.
- Following Gale and Mas-Colell (1975, 1979), Mas-Colell allows quite general externalities.
  - Probably that could be done here, but this has not been pursued.
- These papers do not allow free disposal.

# Course Allocation

At some business schools students report preferences and allocations of seats in courses are computed.

# Course Allocation

At some business schools students report preferences and allocations of seats in courses are computed.

- Budish and Cantillon (2012): versions of random priority used at Harvard.

# Course Allocation

At some business schools students report preferences and allocations of seats in courses are computed.

- Budish and Cantillon (2012): versions of random priority used at Harvard.
  Budish and Kessler (2016): auction and market-like mechanisms used at Wharton.

# Course Allocation

At some business schools students report preferences and allocations of seats in courses are computed.

- Budish and Cantillon (2012): versions of random priority used at Harvard.
  Budish and Kessler (2016): auction and market-like mechanisms used at Wharton.
- Budish, Che, Kojima, and Milgrom (2013) (henceforth BCKM) study probabilistic allocations of seats.

# Course Allocation

At some business schools students report preferences and allocations of seats in courses are computed.

- Budish and Cantillon (2012): versions of random priority used at Harvard.
  Budish and Kessler (2016): auction and market-like mechanisms used at Wharton.
- Budish, Che, Kojima, and Milgrom (2013) (henceforth BCKM) study probabilistic allocations of seats.
  - They give conditions under which assignments of probabilities can be realized by distribution over pure assignments.

# Course Allocation

At some business schools students report preferences and allocations of seats in courses are computed.

- Budish and Cantillon (2012): versions of random priority used at Harvard.
  Budish and Kessler (2016): auction and market-like mechanisms used at Wharton.
- Budish, Che, Kojima, and Milgrom (2013) (henceforth BCKM) study probabilistic allocations of seats.
  - They give conditions under which assignments of probabilities can be realized by distribution over pure assignments.
  - They also give a highly restricted existence theorem generalizing HZ.

# Other Pseudomarket Papers

- Eisenberg and Gale (1959) and Eisenberg (1961): equilibrium of a pari-mutuel betting system.

# Other Pseudomarket Papers

- Eisenberg and Gale (1959) and Eisenberg (1961): equilibrium of a pari-mutuel betting system.
- Varian (1974): trade from equal incomes leads to allocations of a commonly owned endowment that are efficient and envy free.

# Other Pseudomarket Papers

- Eisenberg and Gale (1959) and Eisenberg (1961): equilibrium of a pari-mutuel betting system.
- Varian (1974): trade from equal incomes leads to allocations of a commonly owned endowment that are efficient and envy free.
- Bogomolnaia, Moulin, Sandomirsky, and Yanovskaya (2017): division of a commonly owned endowment mixing goods and bads.

# Other Pseudomarket Papers

- Eisenberg and Gale (1959) and Eisenberg (1961): equilibrium of a pari-mutuel betting system.
- Varian (1974): trade from equal incomes leads to allocations of a commonly owned endowment that are efficient and envy free.
- Bogomolnaia, Moulin, Sandomirsky, and Yanovskaya (2017): division of a commonly owned endowment mixing goods and bads.
  - Their existence results are implied by Mas-Colell's.

# Other Pseudomarket Papers

- Eisenberg and Gale (1959) and Eisenberg (1961): equilibrium of a pari-mutuel betting system.
- Varian (1974): trade from equal incomes leads to allocations of a commonly owned endowment that are efficient and envy free.
- Bogomolnaia, Moulin, Sandomirsky, and Yanovskaya (2017): division of a commonly owned endowment mixing goods and bads.
  - Their existence results are implied by Mas-Colell's.

Of course there is also a vast literature on matching and school choice. In such models usually (not always!) both sides of the market are strategic.

# The Model

We work in a general equilibrium setting:

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.
- There are $m$ agents indexed by $i$. For each $i$ there are:

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.
- There are $m$ agents indexed by $i$. For each $i$ there are:
    - a compact convex consumption set $X_i \subset \mathbb{R}^\ell$;

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.
- There are $m$ agents indexed by $i$. For each $i$ there are:
  - a compact convex consumption set $X_i \subset \mathbb{R}^\ell$;
  - a continuous quasiconcave utility function $u_i : X_i \to \mathbb{R}$;

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.
- There are $m$ agents indexed by $i$. For each $i$ there are:
  - a compact convex consumption set $X_i \subset \mathbb{R}^\ell$;
  - a continuous quasiconcave utility function $u_i : X_i \to \mathbb{R}$;
  - an endowment $\omega_i \in \mathbb{R}^\ell$.

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.
- There are $m$ agents indexed by $i$. For each $i$ there are:
    - a compact convex consumption set $X_i \subset \mathbb{R}^\ell$;
    - a continuous quasiconcave utility function $u_i : X_i \to \mathbb{R}$;
    - an endowment $\omega_i \in \mathbb{R}^\ell$.
- There are compact production sets $Y_1, \ldots, Y_n \subset \mathbb{R}^\ell$ that contain the origin.

# The Model

We work in a general equilibrium setting:

- There are $\ell$ goods indexed by $h$.
- There are $m$ agents indexed by $i$. For each $i$ there are:
  - a compact convex consumption set $X_i \subset \mathbb{R}^\ell$;
  - a continuous quasiconcave utility function $u_i : X_i \to \mathbb{R}$;
  - an endowment $\omega_i \in \mathbb{R}^\ell$.
- There are compact production sets $Y_1, \ldots, Y_n \subset \mathbb{R}^\ell$ that contain the origin.
- There is an $m \times n$ matrix $\theta$ of nonnegative ownership shares such that $\sum_i \theta_{ij} = 1$ for all $j$.

- If $u_i(x_i) = \max_{x_i' \in X_i} u_i(x_i')$, then agent $i$ is *sated* at $x_i \in X_i$ and $x_i$ is a *bliss point* for $i$. Otherwise $i$ is *unsated* at $x_i$.

- If $u_i(x_i) = \max_{x'_i \in X_i} u_i(x'_i)$, then agent $i$ is *sated* at $x_i \in X_i$ and $x_i$ is a *bliss point* for $i$. Otherwise $i$ is *unsated* at $x_i$.
- Let $X = \prod_i X_i$ and $Y = \prod_j Y_j$.

- If $u_i(x_i) = \max_{x'_i \in X_i} u_i(x'_i)$, then agent $i$ is *sated* at $x_i \in X_i$ and $x_i$ is a *bliss point* for $i$. Otherwise $i$ is *unsated* at $x_i$.
- Let $X = \prod_i X_i$ and $Y = \prod_j Y_j$.
- For each $j$ and $p \in \mathbb{R}^\ell$ let

$$\pi_j(p) = \max_{y_j \in Y_j} \langle p, y_j \rangle, M_j(p) = \operatorname{argmax}_{y_j \in Y_j} \langle p, y_j \rangle.$$

- If $u_i(x_i) = \max_{x'_i \in X_i} u_i(x'_i)$, then agent $i$ is *sated* at $x_i \in X_i$ and $x_i$ is a *bliss point* for $i$. Otherwise $i$ is *unsated* at $x_i$.
- Let $X = \prod_i X_i$ and $Y = \prod_j Y_j$.
- For each $j$ and $p \in \mathbb{R}^\ell$ let

$$\pi_j(p) = \max_{y_j \in Y_j} \langle p, y_j \rangle, M_j(p) = \operatorname{argmax}_{y_j \in Y_j} \langle p, y_j \rangle.$$

- For each $i$ and $p \in \mathbb{R}^\ell$, $i$'s total income is

$$\mu_i(p) = \langle p, \omega_i \rangle + \sum_j \theta_{ij} \pi_j(p).$$

# Efficient Disposal Equilibrium

A triple $(p, x, y) \in \mathbb{R}^{\ell}_{+} \times X \times Y$ is an *efficient disposal equilibrium* (EDE) if:

# Efficient Disposal Equilibrium

A triple $(p, x, y) \in \mathbb{R}^\ell_+ \times X \times Y$ is an *efficient disposal equilibrium* (EDE) if:

(a) For each $i$ there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle \leq \langle p, x_i \rangle$ and $u_i(x'_i) > u_i(x_i)$, and there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle < \langle p, x_i \rangle$ and $u_i(x'_i) \geq u_i(x_i)$.

# Efficient Disposal Equilibrium

A triple $(p, x, y) \in \mathbb{R}^\ell_+ \times X \times Y$ is an *efficient disposal equilibrium* (EDE) if:

(a) For each $i$ there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle \leq \langle p, x_i \rangle$ and $u_i(x'_i) > u_i(x_i)$, and there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle < \langle p, x_i \rangle$ and $u_i(x'_i) \geq u_i(x_i)$.

(b) For each $i$, if $i$ is unsated at $x_i$, then $\langle p, x_i \rangle \geq \mu_i(p)$.

# Efficient Disposal Equilibrium

A triple $(p, x, y) \in \mathbb{R}^{\ell}_{+} \times X \times Y$ is an *efficient disposal equilibrium* (EDE) if:

(a) For each $i$ there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle \leq \langle p, x_i \rangle$ and $u_i(x'_i) > u_i(x_i)$, and there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle < \langle p, x_i \rangle$ and $u_i(x'_i) \geq u_i(x_i)$.

(b) For each $i$, if $i$ is unsated at $x_i$, then $\langle p, x_i \rangle \geq \mu_i(p)$.

(c) For each $j$, $y_j \in M_j(p)$.

# Efficient Disposal Equilibrium

A triple $(p, x, y) \in \mathbb{R}^\ell_+ \times X \times Y$ is an *efficient disposal equilibrium* (EDE) if:

(a) For each $i$ there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle \leq \langle p, x_i \rangle$ and $u_i(x'_i) > u_i(x_i)$, and there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle < \langle p, x_i \rangle$ and $u_i(x'_i) \geq u_i(x_i)$.

(b) For each $i$, if $i$ is unsated at $x_i$, then $\langle p, x_i \rangle \geq \mu_i(p)$.

(c) For each $j$, $y_j \in M_j(p)$.

(d) $\sum_i x_i \leq \omega + \sum_j y_j$.

# Efficient Disposal Equilibrium

A triple $(p, x, y) \in \mathbb{R}^\ell_+ \times X \times Y$ is an *efficient disposal equilibrium* (EDE) if:

(a) For each $i$ there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle \leq \langle p, x_i \rangle$ and $u_i(x'_i) > u_i(x_i)$, and there is no $x'_i \in X_i$ such that $\langle p, x'_i \rangle < \langle p, x_i \rangle$ and $u_i(x'_i) \geq u_i(x_i)$.

(b) For each $i$, if $i$ is unsated at $x_i$, then $\langle p, x_i \rangle \geq \mu_i(p)$.

(c) For each $j$, $y_j \in M_j(p)$.

(d) $\sum_i x_i \leq \omega + \sum_j y_j$.

(e) For all $h$, if $\sum_i x_{ih} < \omega_h + \sum_j y_{jh}$, then $p_h = 0$.

# The Main Result

**Let** $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^\ell$**,** $V_0 = \{ x \in \mathbb{R}^\ell : \langle \mathbf{e}, x \rangle = 0 \}$**.**

# The Main Result

Let $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^\ell$, $V_0 = \{\, x \in \mathbb{R}^\ell : \langle \mathbf{e}, x \rangle = 0 \,\}$.

**Theorem:** If, for each $i$ there is an $x_i^0 \in X_i$ such that $x_i^0 \leq \omega_i$, $X_i \subset x_i^0 + V_0$, and $x_i^0$ is in the interior (relative to $x_i + V_0$) of $X_i$, then for any $\alpha \in \mathbb{R}^m_{++}$ there is an EDE $(p, x, y)$ such that

$$\langle p, x_i \rangle - \mu_i(p) = \frac{\alpha_i}{\sum_{i' \in U} \alpha_{i'}} \Big( \sum_{i'' \in S} \mu_{i''}(p) - \langle p, x_{i''} \rangle \Big)$$

for all $i \in U$, where $U$ is the set of $i$ that are unsated at $x_i$ and $S = \{1, \ldots, m\} \setminus U$ is the set of $i$ that are sated at $x_i$.

# The Main Result

Let $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^\ell$, $V_0 = \{\, x \in \mathbb{R}^\ell : \langle \mathbf{e}, x \rangle = 0 \,\}$.

**Theorem:** If, for each $i$ there is an $x_i^0 \in X_i$ such that $x_i^0 \le \omega_i$, $X_i \subset x_i^0 + V_0$, and $x_i^0$ is in the interior (relative to $x_i + V_0$) of $X_i$, then for any $\alpha \in \mathbb{R}^m_{++}$ there is an EDE $(p, x, y)$ such that

$$\langle p, x_i \rangle - \mu_i(p) = \frac{\alpha_i}{\sum_{i' \in U} \alpha_{i'}} \Big( \sum_{i'' \in S} \mu_{i''}(p) - \langle p, x_{i''} \rangle \Big)$$

for all $i \in U$, where $U$ is the set of $i$ that are unsated at $x_i$ and $S = \{1, \ldots, m\} \setminus U$ is the set of $i$ that are sated at $x_i$.

This generalizes all prior existence results.

# Approaches to the Proof

The set up:

# Approaches to the Proof

The set up:

- In HZ the consumption set is the simplex over the objects, so it is contained in a translate of $V_0$.

# Approaches to the Proof

The set up:

- In HZ the consumption set is the simplex over the objects, so it is contained in a translate of $V_0$.
- In Mas-Colell's setting we introduce an artificial worthless good to make consumption sets parallel to $V_0$ and production sets contained in $V_0$.

# Approaches to the Proof

The set up:

- In HZ the consumption set is the simplex over the objects, so it is contained in a translate of $V_0$.
- In Mas-Colell's setting we introduce an artificial worthless good to make consumption sets parallel to $V_0$ and production sets contained in $V_0$.

Prices and excess demand:

# Approaches to the Proof

The set up:

- In HZ the consumption set is the simplex over the objects, so it is contained in a translate of $V_0$.
- In Mas-Colell's setting we introduce an artificial worthless good to make consumption sets parallel to $V_0$ and production sets contained in $V_0$.

Prices and excess demand:

- The natural space of prices is $V_0$, which has two problems:

# Approaches to the Proof

The set up:

- In HZ the consumption set is the simplex over the objects, so it is contained in a translate of $V_0$.
- In Mas-Colell's setting we introduce an artificial worthless good to make consumption sets parallel to $V_0$ and production sets contained in $V_0$.

Prices and excess demand:

- The natural space of prices is $V_0$, which has two problems:
  - Budget sets are not lower semicontinuous at $0$.

# Approaches to the Proof

The set up:

- In HZ the consumption set is the simplex over the objects, so it is contained in a translate of $V_0$.
- In Mas-Colell's setting we introduce an artificial worthless good to make consumption sets parallel to $V_0$ and production sets contained in $V_0$.

Prices and excess demand:

- The natural space of prices is $V_0$, which has two problems:
  - Budget sets are not lower semicontinuous at $0$.
  - Aggregate demand may be less valuable than aggregate supply because of satiation.

# Our Methods

There are two main innovations:

# Our Methods

There are two main innovations:

- Each consumer trades in the hyperplane parallel to $V_0$ that contains her endowment, then free disposes to a point in her consumption set.

# Our Methods

There are two main innovations:

- Each consumer trades in the hyperplane parallel to $V_0$ that contains her endowment, then free disposes to a point in her consumption set.
- We introduce a small endowment of an artificial good $0$ that is always desired.

# Our Methods

There are two main innovations:

- Each consumer trades in the hyperplane parallel to $V_0$ that contains her endowment, then free disposes to a point in her consumption set.
- We introduce a small endowment of an artificial good 0 that is always desired.

Let $V = \mathbb{R} \times V_0$.

# Our Methods

There are two main innovations:

- Each consumer trades in the hyperplane parallel to $V_0$ that contains her endowment, then free disposes to a point in her consumption set.
- We introduce a small endowment of an artificial good 0 that is always desired.

Let $V = \mathbb{R} \times V_0$.

- For each $i$ let $\tilde{X}_i = [-1, \tau_i] \times X_i$ for some sufficiently large $\tau_i$.

# Our Methods

There are two main innovations:

- Each consumer trades in the hyperplane parallel to $V_0$ that contains her endowment, then free disposes to a point in her consumption set.
- We introduce a small endowment of an artificial good $0$ that is always desired.

Let $V = \mathbb{R} \times V_0$.

- For each $i$ let $\tilde{X}_i = [-1, \tau_i] \times X_i$ for some sufficiently large $\tau_i$.
  - Let $\tilde{u}_i(\tilde{x}_{i0}, x_i) = \tilde{x}_{i0} + u_i(x_i)$.

# Our Methods

There are two main innovations:

- Each consumer trades in the hyperplane parallel to $V_0$ that contains her endowment, then free disposes to a point in her consumption set.
- We introduce a small endowment of an artificial good 0 that is always desired.

Let $V = \mathbb{R} \times V_0$.

- For each $i$ let $\tilde{X}_i = [-1, \tau_i] \times X_i$ for some sufficiently large $\tau_i$.
  - Let $\tilde{u}_i(\tilde{x}_{i0}, x_i) = \tilde{x}_{i0} + u_i(x_i)$.
- For each $j$ let $\tilde{Y}_j = \{0\} \times Y_j$.

- For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{\, \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \,\}.$$

- For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{\, \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \,\}.$$

- If excess demand $\tilde{Z}(\tilde{p})$ is defined naturally (and a certain additional condition holds) then:

- For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{\, \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \,\}.$$

- If excess demand $\tilde{Z}(\tilde{p})$ is defined naturally (and a certain additional condition holds) then:
  - $\tilde{Z}$ is upper hemicontinuous.

- For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{\, \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \,\}.$$

- If excess demand $\tilde{Z}(\tilde{p})$ is defined naturally (and a certain additional condition holds) then:
  - $\tilde{Z}$ is upper hemicontinuous.
  - $\langle \tilde{p}, \tilde{z} \rangle = 0$ if $\tilde{z} \in \tilde{Z}(\tilde{p})$ (all income is spent).

- For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{\, \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \,\}.$$

- If excess demand $\tilde{Z}(\tilde{p})$ is defined naturally (and a certain additional condition holds) then:
  - $\tilde{Z}$ is upper hemicontinuous.
  - $\langle \tilde{p}, \tilde{z} \rangle = 0$ if $\tilde{z} \in \tilde{Z}(\tilde{p})$ (all income is spent).
  - Of $\tilde{p}_0 = \varepsilon$, then $\tilde{z}_0 > 0$ for all $\tilde{z} \in \tilde{Z}(\tilde{p})$.

- For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{\, \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \,\}.$$

- If excess demand $\tilde{Z}(\tilde{p})$ is defined naturally (and a certain additional condition holds) then:
  - $\tilde{Z}$ is upper hemicontinuous.
  - $\langle \tilde{p}, \tilde{z} \rangle = 0$ if $\tilde{z} \in \tilde{Z}(\tilde{p})$ (all income is spent).
  - Of $\tilde{p}_0 = \varepsilon$, then $\tilde{z}_0 > 0$ for all $\tilde{z} \in \tilde{Z}(\tilde{p})$.
  - Thus $\tilde{Z}$ is an uhc vector field correspondence that is inward pointing on the boundary of $S_\varepsilon$, so the (generalized) Poincaré-Hopf theorem gives a $\tilde{p}^* \in S_\varepsilon$ such that $0 \in \tilde{Z}(\tilde{p}^*)$.

# A Technical Finesse

Recall that a *polyhedron* in $\mathbb{R}^\ell$ is an intersection of finitely many closed half spaces, and a *polytope* is a bounded polyhedron.

# A Technical Finesse

Recall that a *polyhedron* in $\mathbb{R}^\ell$ is an intersection of finitely many closed half spaces, and a *polytope* is a bounded polyhedron.

**Proposition:** If $P_1$ and $P_2$ are polyhedra in $\mathbb{R}^\ell$, $Q = \{\, q \in \mathbb{R}^\ell = (P_1 + q) \cap P_2 \neq \emptyset \,\}$, and $I : Q \to \mathbb{R}^\ell$ is the correspondence $I(q) = (P_1 + q) \cap P_2$, then $I$ is continuous.

# A Technical Finesse

Recall that a *polyhedron* in $\mathbb{R}^\ell$ is an intersection of finitely many closed half spaces, and a *polytope* is a bounded polyhedron.

**Proposition:** If $P_1$ and $P_2$ are polyhedra in $\mathbb{R}^\ell$, $Q = \{ q \in \mathbb{R}^\ell = (P_1 + q) \cap P_2 \neq \emptyset \}$, and $I : Q \to \mathbb{R}^\ell$ is the correspondence $I(q) = (P_1 + q) \cap P_2$, then $I$ is continuous.

- For each $i$ let $\overline{X}_i$ be the set of bliss points in $X_i$.

# A Technical Finesse

Recall that a *polyhedron* in $\mathbb{R}^\ell$ is an intersection of finitely many closed half spaces, and a *polytope* is a bounded polyhedron.

**Proposition:** If $P_1$ and $P_2$ are polyhedra in $\mathbb{R}^\ell$, $Q = \{ q \in \mathbb{R}^\ell = (P_1 + q) \cap P_2 \neq \emptyset \}$, and $I : Q \to \mathbb{R}^\ell$ is the correspondence $I(q) = (P_1 + q) \cap P_2$, then $I$ is continuous.

- For each $i$ let $\overline{X}_i$ be the set of bliss points in $X_i$.

We take a sequence of expanded economies given by a sequence of endowments of the artificial good that go to zero and a sequence of polyhedra $X_i^k \subset X_i$ such that $X_i^k \to X_i$ and $X_i^k \cap \overline{X}_i \to \overline{X}_i$.

# Concluding Remarks

The paper’s final section points to two open problems:

# Concluding Remarks

The paper's final section points to two open problems:

- In the Hylland-Zeckhauser model, is the set of equilibria finite for generic utilities?

# Concluding Remarks

The paper's final section points to two open problems:

- In the Hylland-Zeckhauser model, is the set of equilibria finite for generic utilities?
- Is the problem of computing an equilibrium of the Hylland Zeckhauser model PPAD-complete?

# Concluding Remarks

The paper's final section points to two open problems:

- In the Hylland-Zeckhauser model, is the set of equilibria finite for generic utilities?
- Is the problem of computing an equilibrium of the Hylland Zeckhauser model PPAD-complete?

The traditional concerns of general equilibrium theory are (mostly) meaningful and conceptually pertinent in relation to pseudomarkets, so one can easily produce a host of original and meaningful problems for further research.