A strategic implementation of the Talmud rule based on concede-and-divide algorithm

Min-hung Tsay and Chun-Hsien Yeh

Department of Economics, National Chung Cheng University, Taiwan
Institute of Economics, Academia Sinica, Taiwan

July 2, 2018

Bankruptcy problem

- When a firm goes to bankrupt, how to divide its liquidation value among its creditors?
- This is the so-called bankruptcy problems motivated by the two puzzles in the ancient Jewish document (the Talmud).
- This literature was initiated by O'Neill (1982).
- For an updated survey, see Thomson (2015).

Bankruptcy Problem: initiation

Contest Garment Problem

Worth of the garment	Claimant 1	Claimant 2
	100	200
200	50	150

Research agenda

Estate Division Problem

Estate of the man	Wife 1	Wife 2	Wife 3
	100	200	300
100	$\frac{100}{3}$	$\frac{100}{3}$	$\frac{100}{3}$
200	50	75	75
300	50	100	150

The Model: formal definition

Many mathematicians and economists try to rationalize the numbers in Puzzles I and II. However, almost all of them fail, including O'Neill (1982). Instead of looking at the specific numerical examples, O'Neill (1982) give a general description for this class of resource allocation problems (bankruptcy problems) as follows:

The Model

$$\varphi\left(N \equiv \{1, \cdots, n\}, c \equiv (c_1, \cdots, c_n) \in \mathbb{R}_+^N, E \in \mathbb{R}_+ \text{ with } \sum_{i \in N} c_i \geq E\right)$$

$$= (x_1, \cdots, x_n) \in \mathbb{R}_+^N \text{ s.t. for each } i \in N,\ 0 \leq x_i \leq c_i, \text{ and } \sum_{i \in N} x_i = E.$$

- The first condition, $0 \leq x_i \leq c_i$, says that creditor i should not receive more than his claim (claims boundedness) and a negative award (non-negativity). The condition is called reasonableness.
- The second condition, $\sum_{i \in N} x_i = E$, says that a rule should allocate the entire resource. This condition is called efficiency or feasibility.

Central rules

A number of bankruptcy rules have been proposed. Among them, the following rules are central in the literature as well as in our analysis. They are:

- The Constrained Equal Awards (CEA) rule (egalitarianism form the perspective of gains) assigns equal awards to all creditors subject to no one receiving more than his claim. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.
- The Constrained Equal Losses (CEL) rule (egalitarianism from the perspective of losses) assigns awards such that the loss (the difference between a creditor's claim and award) experienced by each creditor is equal subject to no one receiving a negative award. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEL_i(c, E) \equiv \max\{c_i - \lambda, 0\}$, where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.

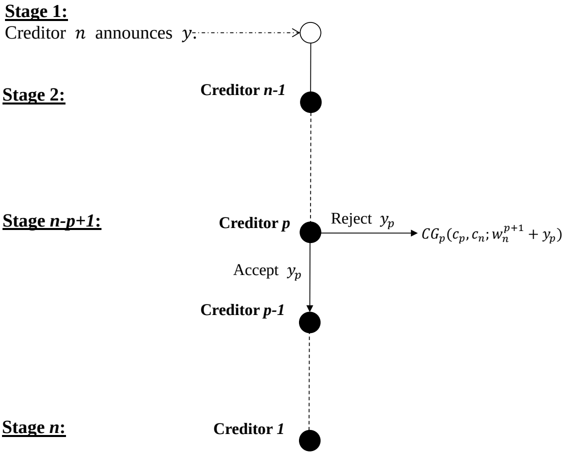
Central rules

- The Talmud (T) rule (Aumann and Maschler, 1985) rationalizes several numerical examples made in the Talmud, and is a "hybrid" of the CEA and CEL rules. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$T_i(c,E) \equiv \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } \sum_{i \in N} \frac{c_i}{2} \geq E; \\ \frac{c_i}{2} + \max\left\{\frac{c_i}{2} - \lambda, 0\right\} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{i \in N} T_i(c,E) = E$.

Serrano (1995)'s game



where $w_n^1 = y_n^1$ and,

for each $p = n-1, \ldots, 1$, $w_n^p = \begin{cases} w_n^{p+1} & \text{if creditor } p \text{ accepts } y_p; \\ CG_n(c_p, c_n; w_n^{p+1} + y_p) & \text{otherwise.} \end{cases}$

Serrano (1995)'s result

He shows that

Theorem: For each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{B}^N$, the unique Subgame Perfect Equilibrium (SPE) outcome of the game $\Gamma^{CG}(c, E)$ is $T(c, E)$.

Serrano (1995)'s result

Remark:

- The exogeneity of the proposer and outcome uniqueness: Creditor n is the only creditor to propose an awards vector. However, if the proposer is not creditor n, the outcome uniqueness of his result does not hold.
- His work relies on contested garment rule (a two-creditor version of the Talmud rule) in solving bilateral negotiations. This leaves a room for improvement since the purpose of the Nash program is to justify cooperative solutions through non-cooperative procedures, ideally no cooperative solution should get involved in the details of non-cooperative procedures. Thus, it would be preferable if bilateral negotiations were resolved by (non-cooperative) procedures.

Our first goal

Our first goal is to strategically justify (or implement) the Talmud rule by introducing a game in which bilateral negotiations are resolved by non-cooperative bilateral bargaining procedures.

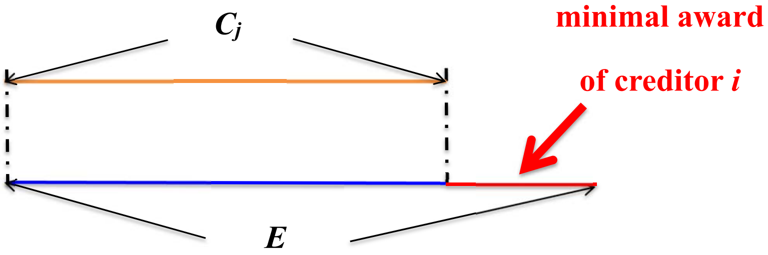
Our second goal

It can be seen that the half-claim vector $\frac{c}{2}$ plays an important role in rationalizing the numerical examples in the Talmud. Aumann and Maschler (1985) justify the vector by invoking legal conventions in the Talmud and psychological presumption (namely, more than half is like the whole and less than half is like nothing). However, there is still no strategic interpretation of the vector. Our second goal is to fill this gap.

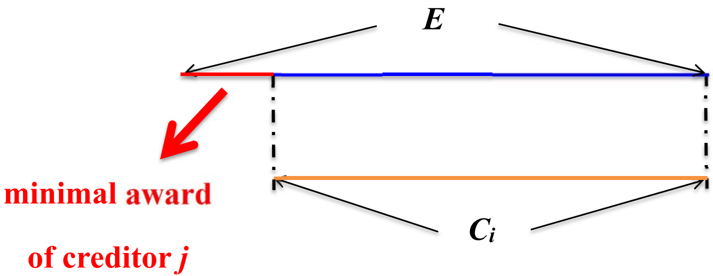
Concede-and-divide algorithm

Aumann and Maschler (1985) mention that for two-creditor problems, the awards vector prescribed by the Talmud rule can be obtained by the following concede-and-divide algorithm. To introduce the algorithm, we define the minimal award of a creditor. The minimal award of a creditor is the remaining endowment after the other has been fully reimbursed if this remaining endowment is positive; it is zero, otherwise. Namely, it is the maximum of the difference between the endowment and the other's claim, and zero. Aumann and Maschler (1985) refer to this minimal award of a creditor as the "minimal right" of a creditor, and consider it as the least concession amount by the other from the perspective of gains. The algorithm says that each creditor first receives her minimal award and then is awarded an equal share of the residual endowment (if any).

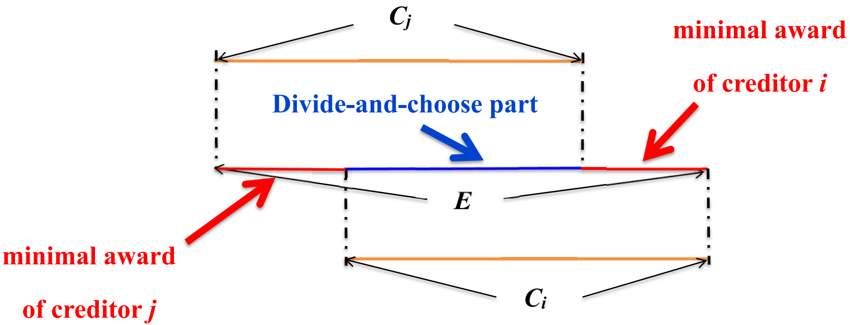
Concede-and-divide algorithm



Concede-and-divide algorithm



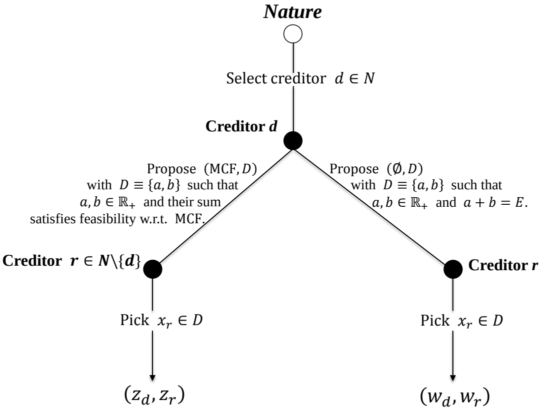
Concede-and-divide algorithm



Concede-and-divide algorithm

This algorithm suggests a two-creditor non-cooperative procedure that involves the following **Minimal concession First (MCF)** process with respect to the minimal awards of creditors. Suppose that the perspective of gains is given. The MCF process suggests first assigning each creditor her minimal award and next dividing the residual endowment. It is a natural and common in the literature on bargaining to allow for a stage of concession during negotiation. For instances, Harsanyi (1956) and Ordover and Rubinstein (1986).

Two-creditor concession game



$$\text{where } z_r = max\{E - c_d, 0\} + x_r \text{ and } z_d = E - z_r;$$

$$w_r = x_r \qquad \text{and } w_d = E - w_r.$$

Inapplicability of the above game

Example. Let $N \equiv \{1,2\}$ and let $\left(\tilde{c} \equiv (\tilde{c}_1, \tilde{c}_2), \tilde{E}\right) = ((3,5),4)$. We exhibit an NE outcome of $\bar{\Omega}'_T(\tilde{c}, \tilde{E})$ that is not the Talmud outcome $\left(\frac{3}{2}, \frac{5}{2}\right)$. Consider the strategy profile $\bar{\sigma}^{T'} \equiv \left(\bar{\sigma}_1^{T'}, \bar{\sigma}_2^{T'}\right)$: each creditor $i \in N$ takes one of the following actions. Let $j \in N \setminus \{i\}$.

• ***i* is the divider:** She proposes

$$(p^{T',i}, D^{T',i}) = \begin{cases} (\emptyset, \{2,2\}) & \text{if } i = 1; \\ \left(\text{MCF}, \left\{\frac{3}{2}, \frac{3}{2}\right\}\right) & \text{if } i = 2. \end{cases}$$

• ***i* is the responder:** Given creditor j's proposal (p, D), creditor i picks $\max D$. Clearly, $\bar{\sigma}^{T'}$ is an SPE of $\bar{\Omega}'_T(\tilde{c}, \tilde{E})$, and is an NE. If *Nature* chooses creditor 1 as the divider, then by following $\bar{\sigma}^{T'}$, the game ends up with outcome $(2,2) \neq \left(\frac{3}{2}, \frac{5}{2}\right)$. However, if *Nature* chooses creditor 2 as the divider, then by following $\bar{\sigma}^{T'}$, the game ends up with the Talmud outcome $\left(\frac{3}{2}, \frac{5}{2}\right)$. □

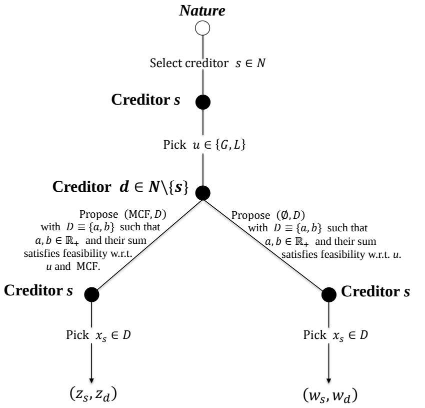
Intuition behind the example

After the decision on whether or not the MCF is conducted, the "divide-and-choose mechanism" is adopted to perform a division of the corresponding endowment. Given this, if creditor 1 (the creditor with the smallest claim) is chosen to be the divider, she will not conduct the MCF process. The game ends up with an outcome that is not the Talmud outcome. If creditor 2 (the creditor with the biggest claim) is the divider, she will conduct the MCF process. The game ends up with the Talmud outcome.

Overcome the inapplicability and treat creditors symmetrically

One way to recover the unique Talmud outcome of the procedure $\bar{\Omega}'_T(c, E)$ is to drop *Nature* and designate creditor 2 as the divider. However, by doing so, the strategy space of creditor 2 is different from that of creditor 1. Thus, the creditors are not treated symmetrically like Serrano (1995) and Dagan et al. (1997). To avoid such an asymmetric treatment and recover the unique Talmud outcome, we introduce the following procedure $\bar{\Omega}_T$ in which the choice between the perspective of gains and the perspective of losses is determined endogenously rather than exogenously.

Our two-creditor Talmud game



where

$$z_s = \begin{cases} max\{E - c_d, 0\} + x_s & if\ u = G; \\ c_s - max\{(c_s + c_d - E) - c_d, 0\} - x_s & if\ u = L, \end{cases} \text{ and } z_d = E - z_s;$$

$$w_s = \begin{cases} x_s & if\ u = G; \\ c_s - x_s & if\ u = L, \end{cases} \text{ and } w_d = E - w_s.$$

Base result

We show that

Proposition: Let $N \in \mathcal{N}$ with $|N| = 2$ and $(c, E) \in \mathcal{B}^N$. The unique NE outcome of $\bar{\Omega}_T(c, E)$ is $T(c, E)$. Moreover, it can be supported by a pure strategy SPE.

A strategic interpretation of the half-claim vector

Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$ with $N \equiv \{1, 2\}$ and $c_1 \leq c_2$. *Nature* chooses one of the two creditors as perspective setter. Given that the divide-and-choose mechanism is adopted to perform a division of the corresponding endowment or deficit, creditor 1 prefers the perspective of gains to the perspective of losses and creditor 2 has reverse preference. Thus, the setter would pick a perspective that is beneficial to her. To balance the advantage given to the setter, the other, called the divider, is allowed to choose either "to conduct", or "not to conduct" the MCF process. It can be seen that in equilibrium, if creditor 2 (creditor 1) is the perspective setter and chooses the perspective of losses (the perspective of gains), then creditor 1 (creditor 2) as the divider conducts the MCF process and proposes a division of the remaining deficit $E' \equiv (c_1 + c_2 - E) - (\xi_1 + \xi_2)$ (the remaining endowment $E'' \equiv E - (\eta_1 + \eta_2)$).

A strategic interpretation of the half-claim vector

Furthermore, in equilibrium, the creditors' awards vector $\left(\eta_1 + \frac{E''}{2}, \eta_2 + \frac{E''}{2}\right)$ (when creditor 1 is the setter) is symmetric to the creditors' losses vector $\left(\xi_1 + \frac{E'}{2}, \xi_2 + \frac{E'}{2}\right)$ (when creditor 2 is the setter) with respect to the half-claim vector. Thus, the vector is a consequence of balancing advantages between the creditors and exploiting the divide-and-choose mechanism.

A strategic implementation of the Talmud rule

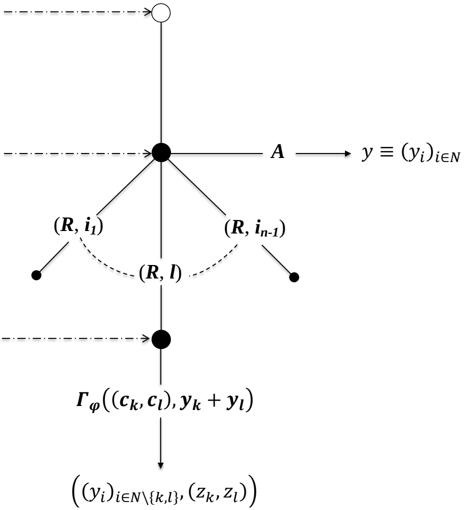
We now extend our base result to more than two creditors. We introduce the following tree-stage extensive form game.

A strategic implementation of the Talmud rule

Stage 1:
Each creditor $i \in N$ announces (y^i, π^i). Let $\pi \equiv \pi^1 \circ \cdots \circ \pi^n$ and $\pi(1) = k$. Let y be the proposal. If for each $i, h \in N \backslash \{k\}$, $y^i = y^h$, then $y = y^i$; otherwise, $y = y^k$.

Stage 2:
Creditor k either takes $\boldsymbol{A}$ (accepts y) or $(\boldsymbol{R}, \boldsymbol{l})$ (rejects y and chooses one creditor from $N \backslash \{k\}$, say creditor l).

Stage 3:
Each creditor $i \in N \backslash \{k, l\}$ receives y_i, and creditors k and l play the two-creditor game $\Gamma_\varphi\big((c_k, c_l), y_k + y_l\big)$. Let (z_k, z_l) be an outcome of $\Gamma_\varphi\big((c_k, c_l), y_k + y_l\big)$.



A strategic implementation of the Talmud rule

Replace φ with T in the above game Ω_φ and call the resulting game Ω_T. We show that

Theorem: Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{B}^N$. The unique NE outcome of $\Omega_T(c, E)$ is $T(c, E)$. Moreover, it can be supported by a pure strategy SPE.

Concluding remarks

Our paper improves the existing implementation of the Talmud rule in Serrano (1995) by opening up the black boxes of bilateral negotiations. Unlike Serrano (1995), our game capture the spirit of the Talmud rule. (Serrano (1995) uses a random dictator game, which captures the spirit of the "random arrival rule".) In addition, *Nature* plays an important role in deriving equilibrium outcome of the game in Serrano (1995). Thus, he obtains an implementation of the Talmud rule in expected term. Our paper obtains exact implementations of the Talmud rule. Our paper is the first one to offer a strategic interpretation of a key element (the half-claim vector) in the Talmud rule.

Thank you!!