A Theory of Stability in Matching with Incomplete Information

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Background

- Two-sided markets:
  - Marriage market
  - Job market
  - College admission market
  - School choice
  - ...

Complete Information Assumption

Assumption: Information is complete (CI), i.e.,

Every agent's characteristics and preferences are common knowledge.

Outline

1. Main consept: Stability.
    - Individual Rationality (IR)
    - No Blocking
    - The fact of IR and no blocking provides no information to agents

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    - Individual Rationality (IR)
    - No Blocking
    - The fact of IR and no blocking provides no information to agents
2. Properties of Blocking and Stability
3. Generality of the Framework

Related Literature

1. One-to-one job market:
   Shapley and Shubik (1971), Crawford and Knoer (1981), Chen et al. (2016), Liu et al. (2014)...

2. Incomplete information:
   Roth (1989), Chakraborty et al. (2010), Yenmez (2013), Pomatto (2015), Bikhchandani (2017), Liu et al. (2014) (LMPS), Chen and Hu (2017), Liu (2017)...

The Model

Agents

- Agents
  - $I \ni i$: a finite set of workers.
  - $J \ni j$: a finite set of firms.
- Types
  - $\mathbf{w} : I \to W$, where W is finite.
  - $\mathbf{f} : J \to F$, where F is finite.
  - $\mathbf{t} = (\mathbf{w}, \mathbf{f})$: a type assignment function.
    $\mathbf{t} \in W^{|I|} \times F^{|J|}$.

Values and Payoffs

- Values for match (w,f)
  - worker premuneration value: $\nu_{wf} \in \mathbb{R}$.
  - firm premuneration value: $\phi_{wf} \in \mathbb{R}$.
  - surplus of the match: $\nu_{wf} + \phi_{wf}$.
- Payoffs $(\mu(i) = j)$:
  - $\nu_{\mathbf{w}(i),\mathbf{f}(j)} + p$ for the worker.
  - $\phi_{\mathbf{w}(i),\mathbf{f}(j)} - p$ for the firm.

Allocation

- matching: $\mu : I \to J \cup \emptyset$, one-to-one on $\mu^{-1}(J)$.
- payment scheme: $\mathbf{p}$ associated with a matching function μ.
  - $\mathbf{p}_{i,\mu(i)} \in \mathbb{R}$ for each $i \in I$.
  - $\mathbf{p}_{\mu^{-1}(j),j} \in \mathbb{R}$ for each $j \in J$.
  - $\mathbf{p}_{\emptyset j} = \mathbf{p}_{i\emptyset} = 0$.
- $(\mu, \mathbf{p})$: an allocation.

  $(\mu, \mathbf{p})$is observable for all agents.

Incomplete Information

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  - $\mathbf{t} \in T \subset W^{|I|} \times F^{|J|}$.

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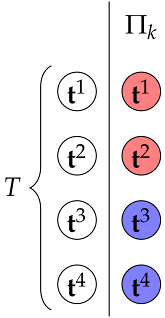
- Π_k: Information Partition of firm $k \in I \cup J$.
  - Π_k is a partition of T.
  - $\mathbf{t}' \in \Pi_k(\mathbf{t})$:
    Agent k thinks $\mathbf{t}'$ is possible when $\mathbf{t}$ is true.



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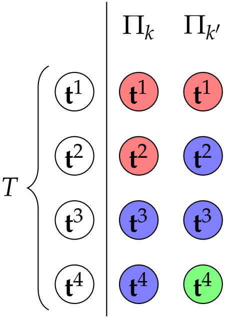
    Agent k thinks $\mathbf{t}'$ is possible when $\mathbf{t}$ is true.
- $\Pi := (\{\Pi_i\}_{i \in I}; \{\Pi_j\}_{j \in J})$: a partition profile.
- Complete info: every partition cell is a singleton.



State of the Market

A state of the matching market, $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$, specifies

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Assumption (LMPS):
Agents can observe the true type of their own partner, if any.

Stability

Individual Rationality

Definition 1
A state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is said to be individually rational if

$$v_{\mathbf{t}(i),\mathbf{t}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \geq 0 \text{ for all } i \in I \text{ and}$$
$$\phi_{\mathbf{t}(\mu^{-1}(j)),\mathbf{t}(j)} - \mathbf{p}_{\mu^{-1}(j),j} \geq 0 \text{ for all } j \in J.$$

Naive Blocking

- Following LMPS: an agent cares about the worst case of a potential partner if she does not know his true type.

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- Given $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$, consider a potential blocking by $(i, j; p)$.

$$v_{\mathbf{t}'(i),\mathbf{t}'(j)} + p > v_{\mathbf{t}'(i),\mathbf{t}'(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ for all } \mathbf{t}' \in \Pi_i(\mathbf{t}) \text{ and}$$

$$\phi_{\mathbf{t}'(i),\mathbf{t}'(j)} - p > \phi_{\mathbf{t}'(\mu^{-1}(j)),\mathbf{t}'(j)} - \mathbf{p}_{\mu^{-1}(j),j} \text{ for all } \mathbf{t}' \in \Pi_j(\mathbf{t}).$$

Naive Blocking is Too Demanding

Market Ingredients:

$I = \{x, y\}$, $J = \{a, b\}$.

$T = \{\mathbf{t}^1, \mathbf{t}^2\}$:

	x	y	a	b
$\mathbf{t}^1$:	2	3	3	2
$\mathbf{t}^2$:	2	1	3	4

$\phi_{wf} = v_{wf} = wf$.

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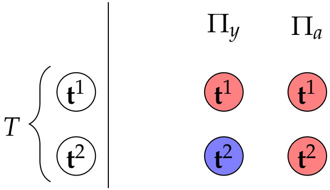
$\mu(x) = a$ and $\mu(y) = b$; $\mathbf{p} = \mathbf{0}$.

$\mathbf{t}^* = \mathbf{t}^1$.



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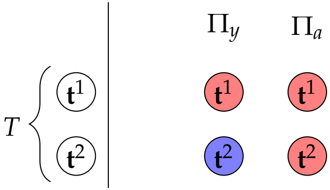

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$(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is blocked by $(y, a; 0)$.

$\mathbf{t}'' \in \Pi_a(\mathbf{t}^1)$ AND $[\nu_{\mathbf{t}''(y),\mathbf{t}''(a)} + 0] - [\nu_{\mathbf{t}''(y),\mathbf{t}''(\mu(y))} + \mathbf{p}_{y,\mu(y)}] > 0$.

Refining Consideration Sets

- Given $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$, consider $(i, j; p)$.

$\Pi_i^{[1]}(\mathbf{t}')$: $\mathbf{t}'' \in \Pi_i(\mathbf{t}')$ and

$$\max_{\tilde{\mathbf{t}} \in \Pi_j(\mathbf{t}'')} [\phi_{\tilde{\mathbf{t}}(i), \tilde{\mathbf{t}}(j)} - p] - [\phi_{\tilde{\mathbf{t}}(\mu^{-1}(j)), \tilde{\mathbf{t}}(j)} - \mathbf{p}_{\mu^{-1}(j), j}] > 0,$$

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$\Pi_i^{[2]}(\mathbf{t}')$: $\mathbf{t}'' \in \Pi_i(\mathbf{t}')$, $\Pi_j^{[1]}(\mathbf{t}'') \neq \emptyset$ and

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Blocking

Definition 2

A state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is said to be blocked, if there exists a worker-firm pair (i, j) and a payment $p \in \mathbb{R}$ such that $\Pi_i^{[l^*]}(\mathbf{t}) \neq \emptyset$, $\Pi_j^{[l^*]}(\mathbf{t}) \neq \emptyset$ and

$$v_{\mathbf{t}'(i),\mathbf{t}'(j)} + p > v_{\mathbf{t}'(i),\mathbf{t}'(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ for all } \mathbf{t}' \in \Pi_i^{[l^*]}(\mathbf{t}) \text{ and}$$

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An Example of Blocking by $(y, a; 0)$

Market Ingredients:

$I = \{x, y\}$, $J = \{a, b\}$.

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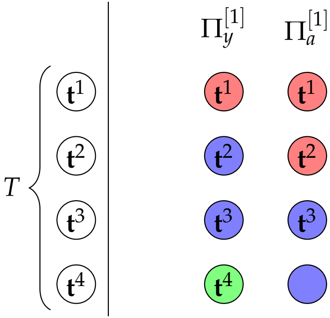

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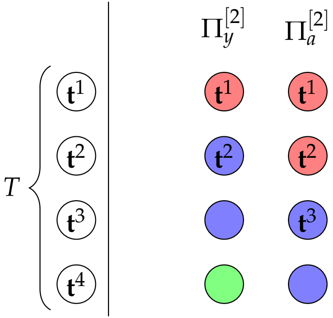

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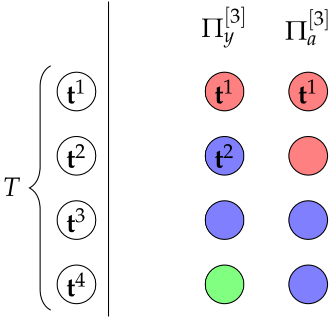

$\Pi_y^{[3]}(\mathbf{t}^1) = \Pi_a^{[3]}(\mathbf{t}^1) = \{\mathbf{t}^1\}$.
$(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is blocked by $(y, a; 0)$.

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1. Given a state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$, let $N^{(\mu,\mathbf{p},\Pi)}$ be a partition of T:

   $N^{(\mu,\mathbf{p},\Pi)}(\mathbf{t}') = N^{(\mu,\mathbf{p},\Pi)}(\mathbf{t}'')$ if and only if either neither $(\mu, \mathbf{p}, \mathbf{t}', \Pi)$ nor $(\mu, \mathbf{p}, \mathbf{t}'', \Pi)$ is blocked or both of them are blocked.

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2. Aggregating two pieces of information $\rightarrow$ Join of two partitions.

- Inferences: $[H_{\mu,\mathbf{p}}(\Pi)]_k := \Pi_k \vee N^{(\mu,\mathbf{p},\Pi)}, \forall k \in I \cup J$, i.e.,

$$[H_{\mu,\mathbf{p}}(\Pi)]_k(\mathbf{t}') := \Pi_k(\mathbf{t}') \cap N^{(\mu,\mathbf{p},\Pi)}(\mathbf{t}'), \forall \mathbf{t}' \in T, \forall k \in I \cup J.$$

Stability

Definition 3

A state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is said to be stable if

1. *it is individually rational,*
2. *it is not blocked, and*
3. *Π is a fixed point of $H_{\mu,\mathbf{p}}$, i.e. $H_{\mu,\mathbf{p}}(\Pi) = \Pi$.*

Convergence: Path to Stability

Proposition 1

Suppose payments permitted in the job market are all integers.
Then the random learning-blocking path starting from an arbitrary state converges with probability one to a stable state.

Properties of Blocking

Properties of Blocking: True-State Improvement

Proposition 2

If $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ *is blocked by* $(i, j; p)$, *then*

$$v_{\mathbf{t}(i),\mathbf{t}(j)} + p > v_{\mathbf{t}(i),\mathbf{t}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ and } \phi_{\mathbf{t}(i),\mathbf{t}(j)} - p > \phi_{\mathbf{t}(\mu^{-1}(j)),\mathbf{t}(j)} - \mathbf{p}_{\mu^{-1}(j),j}.$$

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Proof.

Suppose to the contrary that

$$v_{\mathbf{t}(i),\mathbf{t}(j)} + p \leq v_{\mathbf{t}(i),\mathbf{t}(\mu(i))} + \mathbf{p}_{i,\mu(i)}.$$

Then $\mathbf{t} \notin \Pi_i^{[l^*]}(\mathbf{t})$

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Then $\mathbf{t} \notin \Pi_i^{[l^*]}(\mathbf{t})$, which implies that either $\Pi_j^{[l^*]}(\mathbf{t}) = \emptyset$ or

$$\max_{\mathbf{t}'' \in \Pi_j^{[l^*]}(\mathbf{t})} \left[\phi_{\mathbf{t}''(i),\mathbf{t}''(j)} - p\right] - \left[\phi_{\mathbf{t}''(\mu^{-1}(j)),\mathbf{t}''(j)} - \mathbf{p}_{\mu^{-1}(j),j}\right] \leq 0,$$

a contradiction. $\square$

Properties of Blocking: Naive Blocking

- (One-Dimensional Type) $W \subset \mathbb{R}$ and $F \subset \mathbb{R}$.
- (Non-Transferable Utility) No transfer is permitted in the model.
- (Knowledge within One Side) It is CK that each worker knows the types of all workers and each firm knows the the types of all firms.
- (Increasing and Continuous Utility) The premuneration functions $\nu(w,f)$ and $\phi(w,f)$ are strictly increasing and continuous in w and f.

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Proposition 3

Under these Assumptions, $(\mu, \mathbf{t}, \Pi)$ is blocked if and only if it is naïvely blocked.

Property of Stability (Comparative Statics):

Welfare Effect of Adding One Agent

New Positions and New Workers

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$$\Gamma = (I, J, \mathbf{t}^*, T, \nu, \phi) \quad \xrightarrow{\text{add one agent}} \quad \Gamma' = (I', J', \mathbf{t}^{*\prime}, T', \nu', \phi')$$

Throughout this section, we take Γ' as a one-agent extension of Γ.

Welfare Effect of Adding One Agent

Property. Adding one worker (firm) to a stable market state, the result of any blocking path makes all other workers (firms) weakly worse off and all firms (workers) weakly better off.

- Intuition: expanding one side of the market increases the competition within that side.

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- Intuition: expanding one side of the market increases the competition within that side.
- With incomplete information, **Property** fails because of the correlation of agent types.

A Counterexample where **Property** Fails

Γ:

$I = \{x\}$, $J = \{a\}$.

$T = \{\mathbf{t}^*, \mathbf{t}\}$:

$$\begin{array}{rcc} & x & a \\ \mathbf{t}^* : & 4 & 1 \\ \mathbf{t} : & -4 & 1 \end{array}$$

$\phi_{wf} = wf$, $\nu_{wf} = |wf|$.

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A stable Γ-state:

$$\mu(x) = \emptyset;$$
$$\Pi_x = \{\{\mathbf{t}^*\}, \{\mathbf{t}\}\},$$
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Γ':

$I' = \{x, y\}$, $J' = \{a\}$.

$T' = \{\mathbf{t}^{*\prime}, \mathbf{t}'\}$:

$$\begin{array}{rccc} & x & y & a \\ \mathbf{t}^{*\prime}: & 4 & 2 & 1 \\ \mathbf{t}': & -4 & 3 & 1 \end{array}$$

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	x	a
$\mathbf{t}^*$:	4	1
$\mathbf{t}$:	-4	1

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$\mathbf{t}^{*\prime}$:	4	2	1
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$\phi_{wf} = wf$, $\nu_{wf} = |wf|$.

The unique stable Γ'**-state**:

$$\mu'(x) = a,$$
$$\mu'(y) = \emptyset;$$
$$\Pi'_x = \Pi'_y = \Pi'_a = \left\{\{\mathbf{t}^{*\prime}\}, \{\mathbf{t}'\}\right\}.$$

Restoring Comparative Statics

Strict preferences: $\phi_{\mathbf{t}^{*\prime}(i),\mathbf{t}^{*\prime}(j)} \neq \phi_{\mathbf{t}^{*\prime}(i'),\mathbf{t}^{*\prime}(j)}$ for $i \neq i'$ and
$\nu_{\mathbf{t}^{*\prime}(i),\mathbf{t}^{*\prime}(j)} \neq \nu_{\mathbf{t}^{*\prime}(i),\mathbf{t}^{*\prime}(j')}$ for $j \neq j'$.

Proposition 4

Suppose preferences are strict and no transfer is permitted.

If $(\mu, \mathbf{t}^*, \Pi)$ is a stable Γ-state such that μ is a complete-info. stable allocation,

then for any stable Γ'-state $(\mu', \mathbf{t}^{*\prime}, \Pi')$ produced by Learning-Blocking Paths*,*
when $J \subsetneq J'$ (resp. $I \subsetneq I'$),
the payoffs of all workers (resp. firms) increases
and the payoffs of all existing firms (resp. workers) decreases
compared with the payoffs under $(\mu, \mathbf{t}^*, \Pi)$.

Concluding Remarks

1. Stability with incomplete information.
2. Properties of Blocking.
3. Comparative statics.

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1. Stability with incomplete information.
2. Properties of Blocking.
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- Generality:
  - Observability
  - Correlation (characteristics and preferences)
  - Bayesian Stability

Bayesian Stability

Prior

Common Prior:

$$\text{Assume } \mathbf{t} \sim F.$$

OR

Heterogeneous Prior:

$$\text{Assume } \mathbf{t} \sim F_k \text{ for every } k.$$

Bayesian Blocking: Agents' Willingness

Given $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ and the prior, we consider a potential blocking by $(i, j; p)$.

Indicator correspondence χ over $\Pi_i \vee \Pi_j \ni \pi$, where $\chi(\pi) \subset \{Y, N\}$:

$$\chi_i(\pi) := \begin{cases} \{Y\} & \text{if } \mathbb{E}\left[v_{\tilde{\mathbf{t}}(i), \tilde{\mathbf{t}}(j)} \middle| \pi \right] + p > v_{\mathbf{t}(i), \mathbf{t}(\mu(i))} + \mathbf{p}_{i, \mu(i)} \\ \{N\} & \text{otherwise,} \end{cases}$$

$$\chi_j(\pi) := \begin{cases} \{Y\} & \text{if } \mathbb{E}\left[\phi_{\tilde{\mathbf{t}}(i), \tilde{\mathbf{t}}(j)} \middle| \pi \right] - p > \phi_{\mathbf{t}(\mu^{-1}(j)), \mathbf{t}(j)} - \mathbf{p}_{\mu^{-1}(j), j}, \\ \{N\} & \text{otherwise;} \end{cases}$$

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for each $k = i, j$ and each $\pi \in \Pi_k$,

$$\chi_k(\pi) := \bigcup_{\pi' \in \Pi_i \vee \Pi_j : \pi' \subset \pi} \chi_k(\pi').$$

Refinement of the Consideration Set

Define $\Pi^{[0]} = \Pi$ and recursively for $l = 1, 2, \ldots$ that

$$\Pi_i^{[l]}(\mathbf{t}') := \left\{ \mathbf{t}'' \in \Pi_i(\mathbf{t}') : \Upsilon \in \chi_j(\Pi_j^{[l-1]}(\mathbf{t}'')) \right\}$$

$$\Pi_j^{[l]}(\mathbf{t}') := \left\{ \mathbf{t}'' \in \Pi_j(\mathbf{t}') : \Upsilon \in \chi_i(\Pi_i^{[l-1]}(\mathbf{t}'')) \right\}.$$

Bayesian Blocking

Definition 4

A state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ is said to be Bayesian blocked if there exists $(i, j; p)$ such that $\Pi_i^{[l^*]}(\mathbf{t}) \neq \emptyset$, $\Pi_j^{[l^*]}(\mathbf{t}) \neq \emptyset$ and

$$\mathbb{E}\left[\nu_{\tilde{\mathbf{t}}(i),\tilde{\mathbf{t}}(j)} | \Pi_i^{[l^*]}(\mathbf{t})\right] + p > \nu_{\mathbf{t}(i),\mathbf{t}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \text{ and}$$

$$\mathbb{E}\left[\phi_{\tilde{\mathbf{t}}(i),\tilde{\mathbf{t}}(j)} | \Pi_j^{[l^*]}(\mathbf{t})\right] - p > \phi_{\mathbf{t}(\mu^{-1}(j)),\mathbf{t}(j)} - \mathbf{p}_{\mu^{-1}(j),j}.$$

Bayesian Stability

Definition 5

A state $(\mu, \mathbf{p}, \mathbf{t}, \Pi)$ *is said to be Bayesian stable if*

1. *it is individually rational,*
2. *it is not Bayesian blocked, and*
3. Π *is a fixed point of* $H_{\mu,\mathbf{p}}$*, i.e.* $H_{\mu,\mathbf{p}}(\Pi) = \Pi$.

Bikhchandani, S. (2017). Stability with one-sided incomplete information. *Journal of Economic Theory*, 168:372–399.

Chakraborty, A., Citanna, A., and Ostrovsky, M. (2010). Two-sided matching with interdependent values. *Journal of Economic Theory*, 145(1):85–105.

Chen, B., Fujishige, S., and Yang, Z. (2016). Random decentralized market processes for stable job matchings with competitive salaries. *Journal of Economic Theory*, 165:25–36.

Chen, Y.-C. and Hu, G. (2017). Learning by matching. *Working Paper*.

Crawford, V. P. and Knoer, E. M. (1981). Job matching with heterogeneous firms and workers. *Econometrica*, pages 437–450.

Liu, Q. (2017). Stable belief and stable matching. *Working Paper*.

Liu, Q., Mailath, G. J., Postlewaite, A., and Samuelson, L. (2014). Stable matching with incomplete information. *Econometrica*, 82(2):541–587.

Pomatto, L. (2015). Stable matching under forward-induction reasoning. *Working Paper*.

Roth, A. E. (1989). Two-sided matching with incomplete information about others' preferences. *Games and Economic Behavior*, 1(2):191–209.

Shapley, L. S. and Shubik, M. (1971). The assignment game i: The core. *International Journal of Game Theory*, 1(1):111–130.

Yenmez, M. B. (2013). Incentive-compatible matching mechanisms: consistency with various stability notions. *American Economic Journal: Microeconomics*, 5(4):120–141.