The Weak α-Core of Large Games

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Literature

- Aumann and Peleg (1960) introduce the notions of α and β cores for finite-player games. Aumann (1961) explores the issues further.
- General existence theorems are proved in Scarf (1967, 1971). (The notion of balancedness is important.)
- Notable contributions since have been many; e.g., Shapley (1973), Border (1982), Ichiishi (1982), Kajii (1992).
- Weber (1981): weak-core for games with a continuum of player in a characteristic function form.

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- Notable contributions since have been many; e.g., Shapley (1973), Border (1982), Ichiishi (1982), Kajii (1992).
- Weber (1981): weak-core for games with a continuum of player in a characteristic function form.
- We consider a large (strategic) game over an atomless probability space of players where a player's payoff (continuously) depends on the choice of own action and the societal action distribution.

Literature, cont'd

- Nash equilibrium (NE) in a large game: Existence results
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  - Countable actions: Khan and Sun (1995), Yu and Zhang (2007)

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  - But it may fail for uncountable actions: Rath, Sun and Yamashige (1995), Khan, Rath and Sun (1997)
    Positive results with additional assumptions: Khan and Sun (1999), Keisler and Sun (2009), Khan *et al.* (2013), He, Sun and Sun (2017), He and Sun (2018), *etc.*

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- α-core in a large game:
  - Askoura (2011): The non-emptiness of weak α-core is shown by assuming that a player's (quasi-concave) payoff depends only on the societal distribution but does not depend on her own choice.
  - Askoura(2017), Example 3: Weak α-core is empty for a large game with finite actions if a player's payoff depends on his or her own action.

This Talk

1. We consider
   - the relationship among NE, strong NE and the α-core in a large game.
   - By assuming two conditions in Konishi *et al.* (1997), we can show that the α-core in a large game is non-empty.

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1. We consider
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This Talk

1. We consider
   - the relationship among NE, strong NE and the α-core in a large game.
   - By assuming two conditions in Konishi *et al.* (1997), we can show that the α-core in a large game is non-empty.
2. We also consider
   - a weak α-core in randomized strategies in a large game,
   - We show that under some conditions, the weak α-core in randomized strategies is non-empty.

A large game

- Player space: an atomless probability space $(T, \mathcal{T}, \lambda)$
- Common action set: A compact metric space A.

  Societal summaries: $\mathcal{M}(A)$, the set of probability measures on A endowed with the topology of weak convergence.
- Space of payoff functions: $\mathcal{U}$, the space of all continuous functions on $A \times \mathcal{M}(A)$ with the sup-norm topology.

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- A large game is a measurable function $\mathcal{G} : T \longrightarrow \mathcal{U}$.
- A *(pure strategy) profile* is a measurable function $f : T \longrightarrow A$.

The Notion of α-Core

- A *coalition* is a measurable subset of T with positive measure.
- Given a coalition E, $B(E, A)$ denotes the set of measurable functions from E to A.
- A coalition E *blocks* a strategy profile f if there is a measurable function $h_E \in B(E, A)$, such that for every $h_{E^c} \in B(E^c, A)$ and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \lambda h^{-1}) > u_t(f(t), \lambda f^{-1}) \text{ for almost all } t \in E,$$

  where we abbreviate $\mathcal{G}(t)$ as u_t.
- The *α-core* of the game is the set of profiles that are not blocked by any coalition E.

Nash Equilibrium and Strong Nash Equilibrium

- A strategy profile $f \in B(T, A)$ is a (pure-strategy) *Nash equilibrium (NE)* if

$$u_t(f(t), \lambda f^{-1}) \geq u_t(a, \lambda f^{-1})$$

  for all $a \in A$ and almost all $t \in T$.
- An NE f^s is a *strong NE* if there does not exist any coalition E and $h_E \in B(E, A)$ such that

$$u_t(h(t), \lambda h^{-1}) > u_t(f, \lambda f^{-1})$$

  for almost all $t \in E$ where $h = (h_E, f|_{E^c})$.

Some Observations

In a large game $\mathcal{G}$, it is not hard to show:

Claim

Suppose an NE is not in the α-core. Then it is not a strong NE.

Claim

If f is a strong NE then it is in the α-core.

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Claim

If f is a strong NE then it is in the α-core.

So, once an NE exists in a large game, if we can obtain the existence of strong NE, then we know that α-core is not empty.

A Known Existence Result of NE

Nowhere equivalence (He, Sun and Sun, 2017)

A σ-algebra $\mathcal{T}$ is said to be nowhere equivalent to a sub-σ-algebra $\mathcal{F}$ if for every nonnegligible subset $E \in \mathcal{T}$, there exists an $\mathcal{T}$-measurable subset E_0 of E such that $\lambda(E_0 \triangle E_1) > 0$ for any $E_1 \in \mathcal{F}^E$, where $E_0 \triangle E_1$ is the symmetric difference $(E_0 \setminus E_1) \cup (E_1 \setminus E_0)$.

Proposition 1

There exists an NE in $\mathcal{G}$, provided that

(i) *A is countable, or*

(ii) *$\mathcal{T}$ is nowhere equivalent to $\sigma(\mathcal{G})$.*

Two Assumptions in Konoshi et al. (1997)

Assumption IIC: Independence of Irrelevant Choices

Given any strategy profile $f \in B(T, A)$, for almost all player $t \in T$, if $\tau \in \mathcal{M}(A)$ such that $\tau(f(t)) = \lambda f^{-1}(f(t))$, then $u_t(f(t), \lambda f^{-1}) = u_t(f(t), \tau)$.

IIC says that a player's payoff depends on her own choice and the proportion of others who choose the same choice.

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Assumption PR: Partial Rivalry

Given any strategy profile $f \in B(T, A)$, for almost all player $t \in T$, if $\tau \in \mathcal{M}(A)$ such that $\tau(f(t)) \leq \lambda f^{-1}(f(t))$, then $u_t(f(t), \lambda f^{-1}) \geq u_t(f(t), \tau)$.

PR says that given a choice, a player's payoff is negatively related to the proportion of others who choose the same alternative.

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Examples: Congestion, public goods with negative externalities,

The First Result on α-Core

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Under Assumptions IIC and PR, an NE must be a strong NE in $\mathcal{G}$.

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Theorem 1

Under Assumptions IIC and PR, the α-core of $\mathcal{G}$ is not empty if

(i) *A is countable, or*

(ii) *$\mathcal{T}$ is nowhere equivalent to $\sigma(\mathcal{G})$.*

Randomized Strategies

- A *randomized strategy profile* is a measurable function $g : T \longrightarrow \mathcal{M}(A)$.
- When g is played, the expect payoff of player $t \in T$ is

$$U_t(g) = \int_A u_t(a, \int_{s \in T} g(s) d\lambda(s)) dg(t; da).$$

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- Let $B(T, \mathcal{M}(A))$ (the set of all randomized strategy profiles) be endowed with the weak topology which is defined as the weakest topology for which the functional

$$g \to \int_T \int_A c(t, a) g(t; da) d\lambda(t)$$

  is continuous for every bounded Caratheodory function $c : T \times A \longrightarrow \mathbb{R}$.
- $B(T, \mathcal{M}(A))$ is a compact space under the weak topology.

The Notion of Weak α-Core in Randomized Strategies

- A coalition E *blocks* a randomized strategy profile g if there is a $h_E \in B(E, \mathcal{M}(A))$, such that for every $h_{E^c} \in B(E^c, \mathcal{M}(A))$ and $h = (h_E, h_{E^c})$,

$$U_t(h) > U_t(g) \text{ for almost all } t \in E.$$

- The *α-core* in randomized strategies of the game is the set of randomized profiles that are not blocked by any coalition E.
- A coalition E *strongly blocks* a strategy profile g if there is $\epsilon > 0$ and a $h_E \in B(E, \mathcal{M}(A))$, such that for every $h_{E^c} \in B(E^c, \mathcal{M}(A))$ and $h = (h_E, h_{E^c})$,

$$U_t(h) > U_t(g) + \epsilon \ \text{ for almost all } t \in E.$$

- The *weak α-core* in randomized strategies of $\mathcal{G}$ is the set of profiles that are not strongly blocked by any coalition E.

Assumptions

The following three assumptions are respectively; integrably boundedness, equicontinuity and quasiconcavity.

Assumption 1

The family of functions $\{U_t(g) : g \in B(T, \mathcal{M}(A))\}$ *is integrably bounded.*

Assumption 2

Let $g \in B(T, \mathcal{M}(A))$*. If* $\epsilon > 0$ *then there is an open neighborhood* $V(g, \epsilon)$ *such that* $|U_t(g) - U_t(g')| < \epsilon$ *for all* $g' \in V(g, \epsilon)$ *and* $t \in T$*.*

For a coalition E and $g \in B(T, \mathcal{M}(A))$, let $z(E, g) = \int_E U_t(g)\, \mathrm{d}\lambda$.

Assumption 3

For every coalition E*,* $z(E, \cdot)$ *is quasiconcave.*

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Theorem 2

Under Assumptions 1-3, the weak α-core in randomized strategies of a large game $\mathcal{G}$ is nonempty.

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For a coalition E, let $\mathcal{H}(E) = \{g \in B(T, \mathcal{M}(A)) : g \text{ is not strongly blocked by } E\}$. The proof consists of two lemmas.

Theorem 2

Under Assumptions 1-3, the weak α-core in randomized strategies of a large game $\mathcal{G}$ is nonempty.

For a coalition E, let
$\mathcal{H}(E) = \{g \in B(T, \mathcal{M}(A)) : g$ is not strongly blocked by $E\}$. The proof consists of two lemmas.

Lemma A

For every coalition E, $\mathcal{H}(E)$ is a nonempty, closed (and hence compact) subset of $B(T, \mathcal{M}(A))$.

The Second Main Result

Theorem 2

Under Assumptions 1-3, the weak α-core in randomized strategies of a large game $\mathcal{G}$ is nonempty.

For a coalition E, let $\mathcal{H}(E) = \{g \in B(T, \mathcal{M}(A)) : g$ is not strongly blocked by $E\}$. The proof consists of two lemmas.

Lemma A

For every coalition E, $\mathcal{H}(E)$ is a nonempty, closed (and hence compact) subset of $B(T, \mathcal{M}(A))$.

Lemma B

Let E_i, $i \in I$ be a finite collection of coalitions. Then $\cap_{i \in I} \mathcal{H}(E_i)$ is nonempty.

Proof of Lemma A

$\mathcal{H}(E) = \{g \in B(T, \mathcal{M}(A)) : g$ is not strongly blocked by $E\}$.

- $\mathcal{H}(E) \neq \emptyset$. The function $z(E, \cdot) = \int_E U_t(\cdot) d\lambda(t)$ is continuous. Since $B(T, \mathcal{M}(A))$ is compact, $z(E, \cdot)$ attains its maximum, say at g^*. The coalition E cannot strongly block the strategy profile g^* and $g^* \in \mathcal{H}(E)$.
- If E strongly blocks g then there exist $\epsilon > 0$ and $h_E \in B(E, \mathcal{M}(A))$, such that for every $h_{E^c} \in B(E^c, \mathcal{M}(A))$ and $h = (h_E, h_E^c)$,

$$U_t(h) > U_t(g) + \epsilon \text{ for almost all } t \in E.$$

By Assumption 2, given $\epsilon/2 > 0$, there is an open neighborhood $V(g, \epsilon/2)$ of f such that if $g' \in V(g, \epsilon/2)$ then

$$|U_t(g) - U_t(g')| < \epsilon/2 \text{ for all } t \in T.$$

For almost all $t \in E$,

$$U_t(g') + (\epsilon/2) < U_t(g) + \epsilon < U_t(h).$$

This means the coalition E strongly blocks every profile $g' \in V(g, \epsilon/2)$. Thus, the complement of $\mathcal{H}(E)$ is open and $\mathcal{H}(E)$ is closed. ∎

Outline of Proof of Lemma B

If I is a finite set then $\cap_{i \in I} \mathcal{H}(E_i) \neq \emptyset$.

- Let $\{E_i\}_{i \in I}$ be a finite family of coalitions such that $\cup_{i \in I} E_i = T$.
- Let $\{K_j\}_{j \in J}$ be a finite family of pairwise disjoint elements of $\mathcal{T}$ such that $\mu(K_j) > 0$ for all j and each E_i is a union of some of the K_js.
- For $B \subseteq J$, define $K_B = \cup_{j \in B} K_j$. If $B \subset J$ then K_{B^c} is nonempty and automatically defined as $T \setminus (\cup_{j \in B} K_j)$.
- For $B \subseteq J$, define a subset $V(B)$ of $\mathbb{R}^J$ as follows.

$$V(B) = \{v \in \mathbb{R}^J : \exists\, h_{K_B} \text{ such that } \forall\, h_{K_{B^c}} \text{ and } h = (h_{K_B}, h_{K_{B^c}}), \quad z(K_j, h) \geq v_j,\ \forall\, j \in B\}.$$

  Note that if $j \notin B$ then $v_j \in V(B)$ can be any number in $\mathbb{R}$.
- The following properties hold:
  (1) For every $B \subseteq J$, $V(B)$ is nonempty and closed.
  (2) For every $B \subseteq J$, if $v \in V(B)$ and $v' \leq v$ then $v' \in V(B)$.
  (3) $V(J)$ is bounded from above.
  (4) J is balanced.(By Assumption 3)

Proof of Lemma 2, contd.

- Scarf' theorem: The core of $G = (J, V)$ is nonempty.

  (If v is in the core then v is not in the interior of $V(B)$ for any $B \subseteq J$.)

- If the core of $G = (J, V)$ is not empty, then $\cap_{i \in I} \mathcal{H}(E_i) \neq \emptyset$.
- Let v be in the core of $G = (J, V)$. Let $g : T \longrightarrow \mathcal{M}(A)$ such that $z(K_j, g) \geq v_j$ for all $j \in J$.
- Fix an arbitrary index $i \in I$. E_i is a finite union of some sets K_j, $j \in J$. Let $E_i = \cup_{j \in J_i} K_j$ where $J_i \subseteq J$.
- Since v is not in the interior of $V(J_i)$, for every h_{E_i}, there exists $h_{E_i^c}$ and an index $j \in J_i$ such that for $h = (h_{E_i}, h_{E_i^c})$,

$$z(K_j, h) \leq v_j \leq z(K_j, g).$$

- Thus, for any h_{E_i}, there exists $h_{E_i^c}$ and a subset D_i of E_i of positive measure such that $u_t(h) \leq U_t(g)$ for all $t \in D_i$.
- This shows that $g \in \cap_{i \in I} \mathcal{H}(E_i)$ and completes the proof. ■

Weak α-Core in Pure Strategies?

- We have proved the existence of a randomized strategy profile in the weak α-core. Does the core contain a pure strategy profile?
  - Purification (in progress)
    1. A is countable: Use the DWW theorem.
    2. A is uncountable: assume the no-where equivalence conditions.

Example 1

- The player space is $T = [0,1]$ and λ denotes Lebesgue measure.
- The set of Nash equilibria is a proper subset of the core.
- Let $A = \{a_1, a_2\}$. For any $\eta \in \mathcal{M}(A)$, let

$$u(a_1, \eta) = \frac{1}{2}, \qquad u(a_2, \eta) = 1 - \eta(a_2).$$

  For each $t \in T$, let $u_t = u$.

- f is a Nash equilibrium of this game iff $\lambda \circ f^{-1}(a_2) = 1/2$.
- Since the payoff function is the same for all the players, the weak α-core and the α-core are the same.
- We will show that the α-core of this game is any f such that $\lambda \circ f^{-1}(a_2) \leq 1/2$.

  (Thus, the set of Nash equilibria is contained in the α-core.)

Example 1: Blocked Profiles

- If $\lambda \circ f^{-1}(a_2) > 1/2$ then f is not in the core.
- Let $E \subseteq \{t \in T : f(t) = a_2\}$ such that $\lambda(E) > 0$.
- For any $t \in E$,

$$u_t(f(t), \lambda \circ f^{-1}) = 1 - \lambda \circ f^{-1}(a_2) < \frac{1}{2}.$$

- Let $h_E(t) = a_1$ for any $t \in E$. Then for any h_{E^c} and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \lambda \circ h^{-1}) = \frac{1}{2} \text{ for } t \in E.$$

- So, the coalition E blocks f.

Example 1: Unblocked Profiles

- Now consider any f such that $\lambda \circ f^{-1}(a_2) \leq 1/2$.
  We will show that it is in the core.
- Suppose there is a coalition E which blocks f.
  Let h_E be the function on E such that for any function h_{E^c} on E^c and $h = (h_E, h_{E^c})$,

$$u_t(h(t), \lambda \circ h^{-1}) > u_t(f(t), \lambda \circ f^{-1}).$$

- Consider

$$S_{ij} = \{t \in E : f(t) = a_i \text{ and } h(t) = a_j,\ i, j = 1, 2\}.$$

  - If $t \in S_{11}$ then $u_t(h(t), \lambda \circ h^{-1}) = u_t(f(t), \lambda \circ f^{-1}) = 1/2$, a contradiction. So, $\lambda(S_{11}) = 0$.
  - If $t \in S_{21}$ then $u_t(f(t), \lambda \circ f^{-1}) = 1 - \lambda \circ f^{-1}(a_2) \geq 1/2$ and $u_t(h(t), \lambda \circ h^{-1}) = 1/2$, again a contradiction. So, $\lambda(S_{21}) = 0$.
  - Thus, $E = S_{12} \cup S_{22}$.

Example 1: Unblocked Profiles, contd.

- We have

$$S_{ij} = \{t \in E : f(t) = a_i \text{ and } h(t) = a_j,\ i, j = 1, 2\}, \qquad E = S_{12} \cup S_{22}.$$

  - If $t \in S_{12}$ then $u_t(f(t), \lambda \circ f^{-1}) = 1/2$.
    If $t \in S_{22}$ then $u_t(f(t), \lambda \circ f^{-1}) = 1 - \lambda \circ f^{-1}(a_2) \geq 1/2$.
  - Let $h_{E^c}(t) = a_2$. Then $\lambda \circ h^{-1}(a_2) = 1$.
  - For any $t \in E$, $u_t(h(t), \lambda \circ h^{-1}) = 1 - \lambda \circ h^{-1}(a_2) = 0$.
    This is a contradiction.

- So, no coalition can block f and any f with $\lambda \circ f^{-1}(a_2) \leq 1/2$ is in the α-core.

Example 2

- In this example the weak α-core does not contain any Nash equilibrium.
- Let $A = \{a_1, a_2, a_3\}$, $M_t = \max\{1/10, t\}$ and $m_t = \min\{9/10, t\}$. For $t \in T$ define

$$\begin{aligned} u_t(a_1, \eta) &= 2[1 - \eta(a_2)]M_t \\ u_t(a_2, \eta) &= 1 - \eta(a_2) \\ u_t(a_3, \eta) &= 3[\eta(a_1) - \eta(a_2)](1 - m_t) \end{aligned}$$

- This game has two Nash equilibria f_1 and f_2 where:
  - (1) $f_1(t) = a_1$ if $t > 1/2$ and $f_1(t) = a_2$ if $t \leq 1/2$ and
  - (2) $f_2(t) = a_2$ for all t.
- None of the Nash equilibrium is in the weak α-core.

Example 2: Nash Equilibria

Payoff Functions:

$$u_t(a_1,\eta) = 2[1-\eta(a_2)]M_t$$
$$u_t(a_2,\eta) = 1-\eta(a_2)$$
$$u_t(a_3,\eta) = 3[\eta(a_1)-\eta(a_2)](1-m_t)$$

Nash Equilibria:

(1) $f_1(t) = a_1$ if $t > 1/2$
$f_1(t) = a_2$ if $t \leq 1/2$.

(2) $f_2(t) = a_2$ for all t.

- Observation: If $\eta(a_2) < 1$ then for any $t > 1/2$, $u_t(a_1,\eta) > u_t(a_2,\eta)$ and for $t < 1/2$, $u_t(a_2,\eta) > u_t(a_1,\eta)$.
- (1) If $\eta = \lambda \circ (f_1)^{-1}$ then $\eta(a_1) = \eta(a_2) = 1/2$.
  The payoffs from a_3 is zero and from a_1 and a_2 are positive for all t.
  a_1 is the BR for $t > 1/2$ and a_2 is the BR for $t < 1/2$. So, f_1 is an NE.
- (2) If $f_2(t) = a_2$ and $\eta = \lambda \circ (f_2)^{-1}$ then $\eta(a_2) = 1$.
  For all t, the payoffs from a_1 and a_2 are zero and from a_3 is negative.
  So, a_2 is a BR for $t \in [0,1]$ and f_2 is an NE.
- The arguments to show that these are the only NE are omitted.

Example 2: No Nash Equilibrium in the Weak α-Core

Payoff Functions:

$$u_t(a_1, \eta) = 2[1 - \eta(a_2)]M_t$$
$$u_t(a_2, \eta) = 1 - \eta(a_2)$$
$$u_t(a_3, \eta) = 3[\eta(a_1) - \eta(a_2)](1 - m_t)$$

Nash Equilibria:

(1) $f_1(t) = a_1$ if $t > 1/2$
$f_1(t) = a_2$ if $t \leq 1/2$.

(2) $f_2(t) = a_2$ for all t.

- At f_2 the payoff to each player is zero.
  At f_1, the payoff is t if $t > 1/2$ and the payoff is $1/2$ if $t \leq 1/2$.
  So, $u_t(f_1(t), \lambda \circ (f_1)^{-1}) \geq u_t(f_2(t), \lambda \circ (f_2)^{-1}) + (1/2)$ for all t.
  So, f_2 is not in the weak core.
- At f_1 the payoff is t if $t > 1/2$ and the payoff is $1/2$ if $t \leq 1/2$.
  - Let $h(t) = a_1 = f_1(t)$ if $t > 1/2$ and $h(t) = a_3$ if $t \leq 1/2$.
  - If $\rho = \lambda \circ h^{-1}$ then $\rho(a_1) = 1/2$ and $\rho(a_2) = 0$.
  - The payoff at h is $2t$ if $t > 1/2$ and $(3/2)(1 - t) \geq 3/4$ if $t \leq 1/2$.
  - $u_t(h(t), \lambda \circ h^{-1}) \geq u_t(f_1(t), \lambda \circ (f_1)^{-1}) + (1/4)$ for almost all t.

  So, f_1 is not in the weak α-core.

Example 2: A α-Core Profile

Payoff Functions:

$$u_t(a_1, \eta) = 2[1 - \eta(a_2)]M_t$$
$$u_t(a_2, \eta) = 1 - \eta(a_2)$$
$$u_t(a_3, \eta) = 3[\eta(a_1) - \eta(a_2)](1 - m_t)$$

A *alpha*-Core Profile:

$f(t) = a_1$ if $t > 1/2$
$f(t) = a_3$ if $t \leq 1/2$.

- If $\eta = \lambda \circ f^{-1}$ then $\eta(a_1) = \eta(a_3) = 1/2$ and $\eta(a_2) = 0$.
  $t > 1/2$: $u_t(a_1, \eta) = 2t > 1$. $t \leq 1/2$: $u_t(a_3, \eta) = (3/2)(1 - t) \geq 3/4$.
- f is not an NE because at $t = 1/2$, $u_t(a_3, \eta) = 3/4 < 1 = u_t(a_2, \eta)$.
- Suppose a coalition E blocks f. Let $h = (h_E, h_{E^c})$ and $\rho = \lambda \circ h^{-1}$.
- Let $t > 1/2$. Then $u_t(a_2, \rho) \leq u_t(a_1, \rho) \leq u_t(a_1, \eta)$.
  - If $t \geq 2/3$ then $1 - m_t \leq 1/3$ and $u_t(a_3, \rho) \leq 1$. $\lambda(E \cap [2/3, 1]) = 0$.
  - Let $h(t) = a_2$ on $[2/3,1]$. Then $\rho(a_1) - \rho(a_2) \leq 1/3$ and $u_t(a_3, \rho) \leq 1$ if $t \in (1/2, 2/3)$. $\lambda(E \cap (1/2, 2/3)) = 0$.
- Let $t \leq 1/2$. Assume that $h(t) = a_2$ if $t > 1/2$.
  Then $u_t(a_1, \rho) \leq u_t(a_2, \rho) \leq 1/2$ and $u_t(a_3, \rho) \leq 0$. $\lambda(E \cap [0, 1/2]) = 0$.

Example 3

Payoff Functions:

$$u_t(a_1, \eta) = \eta(a_1) - \eta(a_3)$$
$$u_t(a_2, \eta) = 0$$
$$u_t(a_3, \eta) = -2$$

Nash Equilibria:

(1) $f_1(t) = a_1$ for all t.

(2) $f_2(t) = a_2$ for all t.

f_1 is in the core but not f_2.

- (1) If $\eta = \lambda \circ (f_1)^{-1}$ then $\eta(a_1) = 1$ and $\eta(a_2) = \eta(a_3) = 0$. a_1 is the unique BR for $t \in [0, 1]$. So, f_1 is an NE.
- (2) If $\eta = \lambda \circ (f_2)^{-1}$ then $\eta(a_2) = 1$ and $\eta(a_1) = \eta(a_3) = 0$. So, a_2 is a best response for $t \in [0, 1]$ and f_2 is an NE.
- Conversely suppose that f is an NE and $\eta = \lambda \circ (f_1)^{-1}$.
  - If $\eta(a_1) > \eta(a_3)$ then $u_t(a_1, \eta) > u_t(a_i, \eta)$ for $i = 2, 3$. So, $f = f_1$.
  - If $\eta(a_1) \leq \eta(a_3)$ then $u_t(a_2, \eta) = u_t(a_1, \eta) > u_t(a_3, \eta)$. So, $\eta(a_3) = 0$ which implies that $\eta(a_1) = 0$. Thus, $f = f_2$.
- The payoff to every player from f_1 is 1, which is the highest payoff in the game. So, no coalition can block it and f_1 is in the core.
- The payoff is zero to every player from f_2. So, the all member coalition can strongly block f_2 (via f_1) and f_2 is not in the weak core.

Example 4

- The core is a proper subset of the set of NE.
- Let $A = \{a_1, a_2\}$ and $u(a_i, \eta) = \eta(a_1)$ for $i = 1, 2$.
  For all $t \in [0, 1]$, let $u_t = u$.
- Each player has the same payoff function and the payoff depends only on the measure.
  So, every measure (or the corresponding strategy profile) is an NE.
- We will show that $f(t) = a_1$ for all t is the only core profile.
- Let $\eta = \lambda \circ f^{-1}$. Then $\eta(a_1) = 1$ and the payoff is 1 to each. This is the highest payoff in the game. So, no coalition can block it and f_1 is in the core.
- Let h be any strategy profile, $\rho = \lambda \circ h^{-1}$ and $\rho(a_1) < 1$. Then the payoff to each player is $\rho(a_1) < 1$. The all member coalition strongly blocks h.
- So, f is the unique core allocation and the core is a proper subset of the set of NE.

Example 5

- The core and set of NE are identical.
- Let $A = \{a_1, a_2\}$ and $u_t(a_1, \eta) = \eta(a_1)$, $u_t(a_2, \eta) = \eta(a_1) - 1$.
- Let $f^*(t) = a_1$ for each t and $\eta^* = \lambda \circ (f^*)^{-1}$. Then $\eta^*(a_1) = 1$ and $\eta^*(a_2) = 0$. $u_t(a_1, \eta^*) = 1$ and $u_t(a_2, \eta^*) = 0$. So, f^* is an NE.
- Conversely, suppose that f is an NE. Then

  $$u_t(a_1, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1), \qquad u_t(a_2, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1) - 1.$$

  So, $f(t) = a_1$ for almost all t. Thus f^* is the unique NE.
- f^* is in the core. The payoff to t at f^* is 1 and a player never gets more than 1. So, no coalition can block f^*.
- Let f be any profile such that $\lambda \circ f^{-1}(a_2) > 0$. The payoffs are:

  $$u_t(a_1, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1) < 1,$$
  $$u_t(a_2, \lambda \circ f^{-1}) = \lambda \circ f^{-1}(a_1) - 1 < 0.$$

  The all member coalition strongly blocks f (via f^*).
- This shows that the unique NE f^* is in the unique element of the core.