Incentivizing Team Production by Indivisible Prizes: Electoral Competition in Proportional Representation

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June 8, 2018

Prepared for Workshop on Game Theory at the Institute for Mathematical Sciences (NUS)

Proportional Representation in Parliamental Election

- Voters vote for parties (not for candidates)
- Based on the numbers of votes parties collected, the parliament seats are allocated to parties
  - party candidates exert efforts to increase the number of votes
  - candidates may or may not get a parliament seat (probabilistic outcome)
- (Closed) List rule (Argentina, Iceland, Israel, Spain, etc.)
  - Is this a good rule?
  - What is the optimal list if candidates differ in their abilities?
- Optimal rule?

Contest, Team Production, Incentives

Contest by multiple teams (Tullock type contest model)

- How to split a prize into public and private goods? (group-size paradox?)
  - Nitzan (1991), Esteban Ray (2001), Nitzan Ueda (2011)
  - homogeneous player
  - observable and contractable effort
- How to allocate indivisible multiple prizes? (List rule or egalitarian rule?)
  - Crutzen Flamand Sahuguet (2017)
  - homogeneous player
  - unobservable effort (free-riding incentives)
- This paper: Crutzen Flamand Sahuguet + heterogenous abilities

Preview of the Results

- the degrees of complementarity in team production and convexity of cost functions matter
- if team production is not too complementary, and if cost function is not too convex, then a list rule is the optimal *monotonic* rule
- a list rule is the optimal deterministic monotonic rule
- but the highest ability candidate will not be listed at the top—in the middle of the list
  - it is to let her extert the most effort by making her vulnerable
- characterization of the optimal rule without monotonicity requirement

The Model

- there are n seats in the parliament
- J parties: $j = 1, ..., J$
- party j has $i = 1, ..., n$ candidates
- the number of party j's winning seats is a random variable that is a function of j's effort share

$$p_j = \frac{E_j}{E_1 + ... + E_J}$$

where E_j is the aggregate effort of party j (Tullock)

CES Team Production

- party j has $i = 1, ..., n$ candidates
  - ability $a_{ij} \geq 0$ a parameter—observable
  - effort $e_{ij} \geq 0$
  - payoff from getting a seat in the parliament $V > 0$ (common)
- CES team production function ($\sigma \in (0, 1)$)

$$E_j = \left[\sum_{i=1}^{n} a_{ij} e_{ij}^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

as in Ray Baland Dangnelie (2007) + heterogeneous abilities

- candidate i's cost function (common)

$$C_{ij}(e_{ij}) = \frac{1}{\beta} e_{ij}^{\beta}$$

Probability of getting k seats in the parliament

- if p_j is the probability of winning a seat with $i.i.d.$,

$$P_j^k(p_j) = C(n,k)p_j^k(1-p_j)^{n-k}$$

- in general, the probability of party j's getting k seats: $(P_j^k(p_j))_{k=0}^n$ continuous function of effort share p_j
  - $\sum_{k=0}^n P_j^k(p_j) = 1$ for each p_j

Probability of getting k seats in the parliament

- **first-order stochastic dominance (FOSD)**: $\sum_{k=m}^{n} \frac{dP_j^k}{dp_j} \geq 0$ for all $m = 1, ..., n$
  - this condition must be satisfied.

- **single-crossingness in winning probabilities**: there exists $k^*(p_j)$ such that
  1. $\frac{dP_j^k}{dp_j} \leq 0$ for all $k < k^*(p_j)$
  2. $\frac{dP_j^k}{dp_j} > 0$ for all $k \geq k^*(p_j)$
- when $i.i.d.$, single-crossingness is satisfied with $k^*(p_j) = \lfloor np_j \rfloor + 1$:

$$\frac{dP_j^k}{dp_j} = C(n,k)p_j^{k-1}(1-p_j)^{n-k-1}(k - np_j)$$

Assignment Rules

List rules

- if k seats are won, then the top k candidates on the list go to the parliament

General assignment rule

- Let $\mathcal{S}(k, N_j) = \{S_j \subseteq N_j : |S| = k\}$ be the set of subsets of cardinality k.
- Let $q_j^k : \mathcal{S}(k, N_j) \longrightarrow [0, 1]$ with $\sum_{S \in \mathcal{S}(k,N_j)} q_j^k(S) = 1$ be party j's assignment function: which k candidates go to the parliament when k seats are won.
- An **assignment rule** is a list of functions q_j^ks: $q_j = (q_j^1, ..., q_j^k, ..., q_j^n)$.
- The optimal assignment rule is the rule that maximizes p_j (to be justified).

Effort Optimization

Candidate i in party j has the following payoff

$$B_{ij} - C_{ij} = V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) P_j^k(p_j) - \frac{1}{\beta} e_{ij}^{\beta},$$

where $\mathcal{S}_i(k) = \{S \in \mathcal{S}(k, N_j) : i \in S\}$.

- candidate i goes to the parliament with probability $\sum_{S \in \mathcal{S}_i(k)} q_j^k(S)$ when k seats are won by party j

Effort Optimization

Candidate i chooses e_{ij} given E_{-j} and e_{-ij}:

$$
\begin{aligned}
&\frac{\partial B_{ij}}{\partial e_{ij}} - \frac{\partial C_{ij}}{\partial e_{ij}} \\
= \quad & V \sum_{k=1}^{n} \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \frac{dP_j^k}{dp_j} \frac{E_{-j}}{(E_{-j}+E_j)^2} \frac{\partial E_j}{\partial e_{ij}} - e_{ij}^{\beta-1} \\
= \quad & \frac{a_{ij} V}{e_{ij}} \left(\frac{e_{ij}}{E_j} \right)^{1-\sigma} \left[\sum_{k=1}^{n} \left(\sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \right) \frac{dP_j^k}{dp_j} (1-p_j)\, p_j \right] - e_{ij}^{\beta-1}
\end{aligned}
$$

- $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S)$: probability of i gets a seat when k seats are won by party j
- Note that the above has an interior solution iff the contents of the bracket is positive (otherwise, $e_{ij} = 0$)

Equilibrium Effort under an Assignment Rule

The optimal effort under assignment rule $q_j = (q_j^k)_{k=1}^n$ and p_j is:

$$e_{ij} = \left[a_{ij} V \left(\frac{1}{E_j}\right)^{1-\sigma} \max\left\{ \sum_{k=1}^{n} \left(\sum_{\mathcal{S}_i(k)} q_j^k(S) \right) \frac{dP_j^k}{dp_j} (1-p_j)\, p_j, 0 \right\} \right]^{\frac{1}{\sigma+\beta-1}}$$

Equilibrium aggregate party effort under $q_j = (q_j^k)_{k=1}^n$ and p_j is,

$$E_j = \left\{ V \left(\sum_{i=1}^{n} \alpha_{ij} \left[\max\left\{ \sum_{k=1}^{n} r_i^k \frac{dP_j^k}{dp_j} (1-p_j)\, p_j, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right\}^{\frac{1}{\beta}}$$

where $\alpha_{ij} = a_{ij}^{\frac{\beta}{\sigma+\beta-1}}$ and $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0,1]$.

- Note E_j is a function of p_j only. (This is the key.)

Existence of Equilibrium

Theorem 1. *For any party profile* $((a_{ij})_{i=1}^{n}, \sigma_j, \beta_j)_{j=1}^{J}$, *and any assignment functions* $(q_j^k)_{k=1}^{n}$ *for any party* j, *there exists an equilibrium.*

- $E_j(p_j)$ is continuous in p_j
- Let $\varphi : \Delta^J \rightarrow \Delta^J$ be such that $\varphi_j(p) = \frac{E_j(p_j)}{\sum_{\ell=1}^{J} E_\ell(p_\ell)}$ for all $j = 1, ..., J$.
- There is $\varphi(p^*) = p^*$ (Brouwer's fixed point theorem)

System of Equations

An equilibrium is described by the following system of equations:

$$\begin{pmatrix} p_1 \\ \vdots \\ p_j \\ \vdots \\ p_J \end{pmatrix} = \begin{pmatrix} \frac{E_1(p_1)}{E_1(p_1)+E_{-1}(p_{-1})} \\ \vdots \\ \frac{E_j(p_j)}{E_j(p_j)+E_{-j}(p_{-j})} \\ \vdots \\ \frac{E_J(p_J)}{E_J(p_J)+E_{-J}(p_{-J})} \end{pmatrix}$$

Comparative statics: impact on p_j by an increase in a_{ij} (ability of candidate i in party j)

- total differentiation of the system of equations

Comparative Statics

$$\begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} =$$

Comparative Statics

$$
\begin{pmatrix}
\frac{\frac{\partial E_1}{\partial p_1}}{E} - \frac{E_1 \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & -\frac{E_1 \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & -\frac{E_1 \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \\
\vdots & \ddots & \vdots & & \vdots \\
-\frac{E_j \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & \frac{\frac{\partial E_j}{\partial p_j}}{E} - \frac{E_j \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & -\frac{E_j \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2} \\
\vdots & & \vdots & \ddots & \vdots \\
-\frac{E_{J-1} \frac{\partial E_1}{\partial p_1}}{E^2} & \cdots & -\frac{E_{J-1} \frac{\partial E_j}{\partial p_j}}{E^2} & \cdots & \frac{\frac{\partial E_{J-1}}{\partial p_{J-1}}}{E} - \frac{E_{J-1} \frac{\partial E_{J-1}}{\partial p_{J-1}}}{E^2}
\end{pmatrix}
$$

$$
\times \begin{pmatrix} dp_1 \\ \vdots \\ dp_j \\ \vdots \\ dp_{J-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial E_j}{\partial \alpha_{ij}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\alpha_{ij}
$$

Stability of Equilibrium

- Let $\eta_j(p_j) = \frac{p_j}{E_j}\frac{\partial E_j}{\partial p_j}$ be party j's winning-prob elasticity of aggregate effort.

Lemma 2. *Suppose that candidate i's ability is increased slightly. Then, we have*

$$\frac{dp_j}{d\alpha_{ij}} = \left[\left(1-\eta_j\right) + \frac{(1-p_j)\,\eta_j}{\sum_{i=1,\ i\neq j}^{J-1}\left(\frac{(1-p_i)\eta_i}{1-\eta_i}\right)}\right]^{-1}\frac{\partial E_j}{\partial \alpha_{ij}}$$

where $\frac{\partial E_j}{\partial \alpha_{ij}} = A\left(\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\}\right)^{\frac{1-\sigma}{\sigma+\beta-1}} \geq 0.$

- When $\eta_j(p_j) < 1$ for all j (individually stable), $\frac{dp_j}{d\alpha_{ij}} \geq 0$ holds (with equality when $\sum_{k=1}^{n} r_i^k \mu^k(p_j) \leq 0$: no incentive to make effort) — the system is well-behaved.
- *From now on, we assume that each party j chooses to maximize E_j.*

Effort-Maximizing Party

Party j chooses assignment rule $(q_j^k)_{k=1}^{n-1}$ to maximize $E_j =$

$$\left\{ V \left(\sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} \left(\sum_{\mathcal{S}_i(k)} q_j^k(S) \right) \frac{dP_j^k}{dp_j} (1 - p_j) p_j, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}} \right)^{\frac{\sigma+\beta-1}{1-\sigma}} \right.$$

This can be written as

$$\max_{(r_i^k)} \sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu_j^k, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}}$$

where $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0, 1]$ and $\mu_j^k = \frac{dP_j^k}{dp_j} (1 - p_j) p_j$.

- weights (incentives to make effort) μs satisfy $\mu_j^1, ..., \mu_j^{k^*-1} \leq 0$, and $0 < \mu_j^{k^*}, ..., \mu_j^n$
- $a_{1j} \geq a_{2j} \geq ... \geq a_{ij} \geq ... \geq a_{nj}$ (or $\alpha_{1j} \geq ... \geq \alpha_{nj}$)

Two Issues in Optimization

$$\max_{(r_i^k)} \sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} \boxed{r_i^k} \mu_j^k, 0 \right\} \right]^{\boxed{\frac{1-\sigma}{\sigma+\beta-1}}}$$

where $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0,1]$ and $\mu_j^k = \frac{dP_j^k}{dp_j}(1-p_j)\, p_j$.

1. $\frac{1-\sigma}{\sigma+\beta-1} \gtreqless 1$: If this power is higher than 1 $(\beta < 2(1-\sigma))$, the objective function is convex, so highest possible $\sum_{k=1}^{n} r_i^k \mu_j^k$ should be given to high ability is. (*we will focus on this case—deterministic case*)
2. Can we choose r_i^ks freely as long as (i) $r_i^k \in [0,1]$ for all i and k, and (ii) $\sum_{i=1}^{n} r_i^k = k$ for all k?

Assignment Matrix

$n \times n$ matrix (**row** candidates; **column** when k seats are won)

$$R = \begin{pmatrix} r_1^1 & \cdots & r_1^k & \cdots & r_1^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ r_i^1 & \cdots & r_i^k & \cdots & r_i^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ r_n^1 & \cdots & r_n^k & \cdots & r_n^n \end{pmatrix}$$

- $\sum_{i=1}^{n} r_i^k = k$ for all $k = 1, ..., n$
- $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0, 1]$

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- $\sum_{i=1}^n r_i^k = k$ for all $k = 1, ..., n$
- $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0, 1]$

Q. *For any R with (i) $r_i^k \in [0,1]$ and (ii) $\sum_{i=1}^n r_i^k = k$ for all k, can we find allocation rule $q_j = (q_j^k)_{k=1}^n$?*

Assignment Matrix

$n \times n$ matrix (**row** candidates; **column** when k seats are won)

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- $\sum_{i=1}^n r_i^k = k$ for all $k = 1, ..., n$
- $r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0, 1]$

Q. *For any R with (i) $r_i^k \in [0, 1]$ and (ii) $\sum_{i=1}^n r_i^k = k$ for all k, can we find allocation rule $q_j = (q_j^k)_{k=1}^n$?*
A. *Yes!* (by induction)

Freedom of Choosing Assignment Matrix (1)

Q. *For any R with (i) $r_i^k \in [0,1]$ and (ii) $\sum_{i=1}^n r_i^k = k$ for all k, can we find allocation rule $q_j = (q_j^k)_{k=1}^n$?*

When $n = 3$:

$$R = \begin{pmatrix} r_1^1 & r_1^2 & r_1^3 \\ r_2^1 & r_2^2 & r_2^3 \\ r_3^1 & r_3^2 & r_3^3 \end{pmatrix}$$

- $r_1^1 + r_2^1 + r_3^1 = 1$ (one goes to the parliament $\iff$ choosing one candidate)
- $r_1^2 + r_2^2 + r_3^2 = 2$ (two go to the parliament $\iff$ excluding one candidate)
- $r_1^3 + r_2^3 + r_3^3 = 3$ (three go to the parliament $\iff$ $r_1^3 = r_2^3 = r_3^3 = 1$)

A. *Yes!*

Freedom of Choosing Assignment Matrix (2)

Q. *For any R with (i) $r_i^k \in [0,1]$ and (ii) $\sum_{i=1}^n r_i^k = k$ for all k, can we find allocation rule $q_j = (q_j^k)_{k=1}^n$?*

Supposing that it works for $n = m \geq 3$, when $n = m + 1$,

$$R = \begin{pmatrix} r_1^1 & \cdots & r_1^k & \cdots & r_1^m & r^{m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ r_i^1 & \cdots & r_i^k & \cdots & r_i^m & r_i^{m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ r_m^1 & \cdots & r_m^k & \cdots & r_m^m & r_m^{m+1} \\ r_{m+1}^1 & \cdots & r_{m+1}^k & \cdots & r_{m+1}^m & r_{m+1}^{m+1} \end{pmatrix}$$

Is there an allocation rule $(q_j^k)_{k=1}^{m+1}$ such that

$$r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0,1]$$

for all $i = 1, ..., m+1$ and $k = 1, ..., m+1$?

Freedom of Choosing Assignment Matrix (3)

Q. *Is there an allocation rule* $(q_j^k)_{k=1}^{m+1}$ *such that*

$$r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0,1]$$

for all $i = 1, ..., m+1$ *and* $k = 1, ..., m+1$*?*

$$R = \begin{pmatrix} \mathbf{r}_1^1 & \cdots & r_1^k & \cdots & \mathbf{r}_1^m & \mathbf{r}_1^{m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{r}_i^1 & \cdots & r_i^k & \cdots & \mathbf{r}_i^m & \mathbf{r}_i^{m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{r}_m^1 & \cdots & r_m^k & \cdots & \mathbf{r}_m^m & \mathbf{r}_m^{m+1} \\ \mathbf{r}_{m+1}^1 & \cdots & r_{m+1}^k & \cdots & \mathbf{r}_{m+1}^m & \mathbf{r}_{m+1}^{m+1} \end{pmatrix}$$

By the same argument as $n = 3$ case, by choosing 1, excluding 1, and assigning $r_i^{m+1} = 1$, we can achieve these three columns.

Freedom of Choosing Assignment Matrix (4)

Q. *Is there an allocation rule q_j^k such that*

$$r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0,1]$$

for all $i = 1, ..., m+1$ and all $k = 2, ..., m-1$?

$$R = \begin{pmatrix} r_1^1 & \cdots & \mathbf{r}_1^k & \cdots & r_1^m & r_1^{m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ r_i^1 & \cdots & \mathbf{r}_i^k & \cdots & r_i^m & r_i^{m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ r_m^1 & \cdots & \mathbf{r}_m^k & \cdots & r_m^m & r_m^{m+1} \\ r_{m+1}^1 & \cdots & \mathbf{r}_{m+1}^k & \cdots & r_{m+1}^m & r_{m+1}^{m+1} \end{pmatrix}$$

- *Let's pick an arbitrary $(r_1^k, ..., r_i^k, ..., r_m^k, r_{m+1}^k)$ with $\sum_{i=1}^{m+1} r_i^k = k$ and $r_i^k \in [0,1]$ for all i.*

Freedom of Choosing Assignment Matrix (5)

Q. *For an arbitrary* $(r_1^k, ..., r_i^k, ..., r_m^k, r_{m+1}^k)$ *with* $\sum_{i=1}^{m+1} r_i^k = k$ *and* $r_i^k \in [0,1]$ *for all* i, *is there an allocation rule* q_j^k *such that*

$$r_i^k = \sum_{S \in \mathcal{S}_i(k)} q_j^k(S) \in [0,1]$$

for all $i = 1, ..., m+1$ *and* $k = 2, ..., m-1$*?*

A. *Yes, we can.*

1. Let $r_{m+1}^k \geq r_i^k$ (we can reshuffle). Drop her.
2. Let $\tilde{r}^k = (\tilde{r}_1^k, ..., \tilde{r}_i^k, ..., \tilde{r}_m^k) \propto (r_1^k, ..., r_i^k, ..., r_m^k)$ with $\sum_{i=1}^m \tilde{r}_i^k = k$. $(\tilde{r}_1^k, ..., \tilde{r}_i^k, ..., \tilde{r}_m^k, 0)$ is supportable.
3. Let $\tilde{r}^{k-1} = (\tilde{r}_1^{k-1}, ..., \tilde{r}_i^{k-1}, ..., \tilde{r}_m^{k-1}) \propto (r_1^k, ..., r_i^k, ..., r_m^k)$ with $\sum_{i=1}^m \tilde{r}_i^{k-1} = k-1$. $(\tilde{r}_1^{k-1}, ..., \tilde{r}_i^{k-1}, ..., \tilde{r}_m^{k-1}, 1)$ is supportable.
4. A convex combination of 2 and 3 supports $(r_1^k, ..., r_i^k, ..., r_m^k, r_{m+1}^k)$.

Optimal Assignment Matrix (convex case) (1)

When $\frac{1-\sigma}{\sigma+\beta-1} > 1$ (convex), or when we focus on deterministic rule, find

$$R = \begin{pmatrix} r_1^1 & \cdots & r_1^k & \cdots & r_1^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ r_i^1 & \cdots & r_i^k & \cdots & r_i^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ r_n^1 & \cdots & r_n^k & \cdots & r_n^n \end{pmatrix}$$

that maximizes

$$\sum_{i=1}^{n} \alpha_{ij} \left[\max \left\{ \sum_{k=1}^{n} r_i^k \mu_j^k, 0 \right\} \right]^{\frac{1-\sigma}{\sigma+\beta-1}}$$

- weights μs satisfy $\mu_j^1, \ldots, \mu_j^{k^*-1} \leq 0$, and $0 < \mu_j^{k^*}, \ldots, \mu_j^n$
- $\alpha_{ij} = a_{ij}^{\frac{\beta}{\sigma+\beta-1}}$: $\alpha_{1j} \geq \alpha_{2j} \geq \ldots \geq \alpha_{ij} \geq \ldots \geq \alpha_{nj}$

Optimal Assignment Matrix (convex case) (2)

- starting from $i = 1$, we assign $(r_i^k)_{k=1}^n$ sequentially ($r_i^k \in \{0, 1\}$)
- $\zeta(i) \subseteq \{1, ...k, ..., n\}$: $k \in \zeta(i) \iff r_i^k = 1$
- counter $\kappa(i) = (\kappa_1(i), ..., \kappa_k(i), ..., \kappa_n(i))$ is the number of seats still available for each k when i is to be assigned
  - $\kappa(1) = (\kappa_1(1), ..., \kappa_k(1), ..., \kappa_n(1)) = (1, ..., k, ..., n)$
  - $\kappa_k(i+1) = \kappa_k(i) - 1$ if $k \in \zeta(i)$; $\kappa_k(i+1) = \kappa_k(i)$ otherwise
- $\mathcal{M}(i) \equiv \{k : \kappa_k(i) > 0\}$: the set of ks that party j can assign to candidate i
- $\mathcal{L}(i) \equiv \{k : \kappa_k(i) = n - i + 1\}$: the set of ks that party j must assign to candidate i
  - $n \in \mathcal{L}(i)$ for all i (to be feasible)
- choose $\zeta(i)$ from $\mathcal{L}(i) \subseteq \zeta(i) \subseteq \mathcal{M}(i)$

Optimal Assignment Matrix (convex case) (Example)

$n = 7$ and $k^* = 3$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\kappa(1) = (1,2,3,4,5,6,7)$, $\mathcal{M}(1) = \{1, ..., 7\}$, $\mathcal{L}(1) = \{7\}$
$\kappa(2) = (1,2,2,3,4,5,6)$, $\mathcal{M}(2) = \{1, ..., 7\}$, $\mathcal{L}(2) = \{7\}$
$\kappa(3) = (1,2,1,2,3,4,5)$, $\mathcal{M}(3) = \{1, ..., 7\}$, $\mathcal{L}(3) = \{7\}$
$\kappa(4) = (1,2,0,1,2,3,4)$, $\mathcal{M}(4) = \{1,2,4, ..., 7\}$, $\mathcal{L}(4) = \{7\}$
$\kappa(5) = (1,2,0,0,1,2,3)$, $\mathcal{M}(5) = \{1,2,5,6,7\}$, $\mathcal{L}(5) = \{7\}$
$\kappa(6) = (1,2,0,0,0,1,2)$, $\mathcal{M}(6) = \{1,2,6,7\}$, $\mathcal{L}(6) = \{2,7\}$
$\kappa(7) = (1,1,0,0,0,0,1)$, $\mathcal{M}(7) = \{1,2,7\}$, $\mathcal{L}(7) = \{1,2,7\}$

Optimal Monotonic Rule (convex case)

Monotonic Rule. *For all* $i = 1, .., n$, *and all* $k = 1, .., n-1$, $r_i^k \leq r_i^{k+1}$.

Proposition 4. *Under any monotonic rule, every candidate exerts effort.*

Proof. Rewriting candidate i's incentive term, we have

$$\sum_{k=1}^{n} r_i^k \mu^k(p_j) = r_i^1 \sum_{k=1}^{n} \mu^k(p_j) + \left(r_i^2 - r_i^1\right) \sum_{k=2}^{n} \mu^k(p_j) + ... + \left(r_i^n - r_i^{n-1}\right) \mu$$

By monotonicity, $r_i^k - r_i^{k-1} \geq 0$ for all $k = 1, ..., n$ ($r_0^k = 0$).
Thus, the FOSD, $\sum_{k=m}^{n} \mu^k(p_j) > 0$ for all $m = 1, ..., n$, implies:

$$\max\left\{\sum_{k=1}^{n} r_i^k \mu^k(p_j), 0\right\} = \sum_{k=1}^{n} r_i^k \mu^k(p_j) > 0$$

Optimal Monotonic Rule (convex case)

- If $k^* = 1$, the optimal rule is monotonic rule—*the list rule with the highest ability to the lowest.*
- If not, the optimal rule is not monotonic.
- If we confine our attention to monotonic rules, one of the list rules is the optimal.
  - if a rule is monotonic, $e_{ij} > 0$ for all $i = 1, ..., n$ (the first-order stochastic dominance).
  - the highest ability candidates gets $\zeta(1) = \{k^*, ..., n\}$
  - the second is either $\{k^* + 1, ..., n\}$ or $\{k^* - 1, ..., n\}$, and so on (single-peaked at k^*)

Optimal Monotonic Rule (convex case)

Proposition 6. Suppose $\beta < 2\,(1-\sigma)$. Then, the optimal monotonic rule is a list rule.

Example: $n = 7$ and $k^* = 3$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

optimal list: $\{6, 3, 1, 2, 4, 5, 7\}$

- If we consider only deterministic rules, then the optimal deterministic monotonic rule is a list rule (no condition needed).

Concave Case?

When $\beta > 2(1-\sigma)$, E_j is a concave function of candidates' efforts e_{ij}s.

- This is a textbook exercise—proportional allocation.
- But depending on parameters, proportional allocation may not be feasible.
  - When $\beta - 2(1-\sigma)$ is small, with substantial ability difference, there is no way to have proportional allocation.

Proposition 7. *Suppose $\beta > 2(1-\sigma)$. Then, whenever feasible, the optimal assignment rule tries to allocate the chances of candidates to get a seat in the parliament proportionally to $a_{ij}^{\frac{\beta}{\beta-2(1-\sigma)}}$.*

Summary

- A tractable model of team production with indivisible prizes.
- A list rule is the optimal monotonic rule, if
  1. complementarity is not too strong and cost function is not too convex, or
  2. we confine our attention to deterministic rules only
- However, the optimal list is not in order of ability
  - the highest ability candidate will be listed in the middle
  - the highest ability candidate needs to make a lot of effort
- optimal (nonmonotonic) rule is characterized