Equal-quantiles rules in resource allocation with uncertain needs

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NUS Game Theory Workshop

Motivation

- Pre-committed division with uncertain needs
- Examples: allocation of public service; division of rescue forces/medical supplies; capacity allocation in a network
- Ex post reallocation may not be possible
- Departure from the literature: waste vs deficit

The model

- $\mathcal{N}$: the set of all finite subsets of $\mathbb{N}$
- $I \in \mathcal{N}$: a finite population of agents
- F_i: a probability measure on $\mathbb{R}_+$ with convex and compact support
- $T \in \mathbb{R}_+$: total endowment
- A problem: $(F, T) \in \mathcal{P}^I$, $I \in \mathcal{N}$
- An allocation: $t \in \mathbb{R}^I_+$ s.t. $\sum t_i \leq T$ and for each i, $t_i \in [0, \max \operatorname{supp} F_i]$
- A rule $r : \bigcup_I P^I \to \bigcup_I \mathbb{R}^I_+$: $r(F, T) = t$

Cost of an assignment to a single agent i

Suppose that $u_i > u_0 > 0$. Agent 0 generates deterministic welfare and is outside the model.

Utility maximization $\iff$ cost minimization

$$\int_{x_i<t_i} u_i x_i + u_0(T - x_i) - [u_i x_i + u_0(T - t_i)] dF_i(x_i)$$

$$+ \int_{x_i>t_i} u_i x_i + u_0(T - x_i) - [u_i t_i + u_0(T - t_i)] dF_i(x_i)$$

$$= \int_{x_i<t_i} u_0(t_i - x_i) dF_i(x_i)$$

$$+ \int_{x_i>t_i} (u_i - u_0)(x_i - t_i) dF_i(x_i)$$

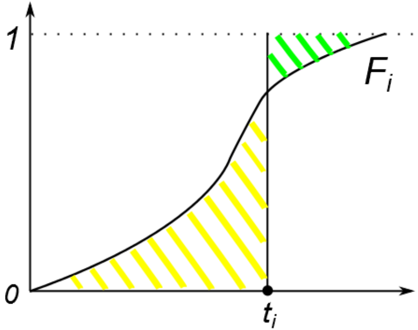
$$= c^w \cdot ew(F_i, t_i) + c_i^d \cdot ed(F_i, t_i)$$

Optimal assignment to a single agent i

$$\min_{t_i} c^w \cdot ew(F_i, t_i) + c_i^d \cdot ed(F_i, t_i)$$

Unconstrained solution: $t_i = F_i^{-1}(\frac{c_i^d}{c^w + c_i^d})$

Constrained solution: $t_i = T$



Discussion 1: Resource may not be exhausted.

Given $i \in \mathbb{N}$ with F_i, it could be that

$$t^*_{F_i} := \sup_{T \in \mathbb{R}_+} r_i(F_i, T) < \max \operatorname{supp} F_i.$$

Given $I \in \mathcal{N}$ with F, it could be that for each $i \in I$,

$$\sup_{T \in \mathbb{R}_+} r_i(F, T) < \max \operatorname{supp} F_i.$$

Recall: an allocation $t \in \mathbb{R}^I_+$ is s.t. $\sum t_i \leq T$ and for each i, $t_i \in [0, \max \operatorname{supp} F_i]$.

Discussion 1: Maximum assignment

Our contribution: We find that some existing axioms, when extended from deterministic to the uncertain context, imply that for each $I \in \mathcal{N}$ and each $(F, T) \in \mathcal{P}^I$,

$$T \leq \sum_{i \in I} t^*_{F_i} \Rightarrow \sum_{i \in I} r_i(F, T) = T;$$

$$T > \sum_{i \in I} t^*_{F_i} \Rightarrow \text{for each } i, r_i(F, T) = t^*_{F_i}.$$

Note: The maximum assignment of an agent does not depend on the number of other agents and their claims.

Discussion 2: Newsvendor problem — Similarity

Each unit of a perishable product can be purchased at price c and sold at price p where $p > c > 0$.

$$\min_{t_i} c^w \cdot ew(F_i, t_i) + c_i^d \cdot ed(F_i, t_i)$$
$$= \min_{t_i} c \cdot ew(F_i, t_i) + (p - c) \cdot ed(F_i, t_i)$$

Unconstrained solution:

$$t_i = F_i^{-1}\left(\frac{c_i^d}{c^w + c_i^d}\right) = F_i^{-1}\left(\frac{p-c}{p}\right)$$

Critical fractile formula (operations management)
Littlewood's rule (revenue management)

Discussion 2: Newsvendor problem — Difference

Unlimited resource vs limited resource (multiple agents)

Maximize a utility function vs social choice function

Profit vs social welfare (single agent/multiple agents)

Our contribution: Axiomatize a family of division rules selecting allocations according to

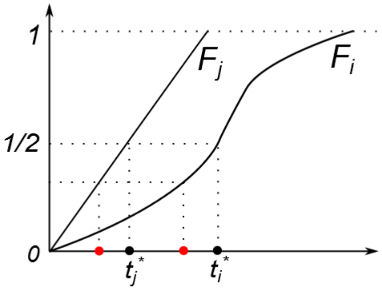
$$\min_t \sum_{i \in I} [c^w \cdot ew(F_i, t_i) + c^d \cdot ed(F_i, t_i)].$$

Equal-quantile rules

An equal-quantile rule associated with $\lambda \in (0, 1]$ selects for each $I \in \mathcal{N}$ and each $(F, T) \in \mathcal{P}^I$ the allocation that solves

$$\min_t \sum_{i \in I} [c^w \cdot ew(F_i, t_i) + c^d \cdot ed(F_i, t_i)],$$

where $c^w, c^d > 0$ are such that $\lambda = \frac{c^d}{c^w + c^d}$.



Axioms: *Continuity*

For each $I \in \mathcal{N}$, r is continuous on P^I.

Axioms: *Strict Ranking*

For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $i, j \in I$, if F_i strictly first-order stochastically dominates F_j, then $r_i(F, T) > r_j(F, T)$.

Axioms: *Ranking*

For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $i, j \in I$, if F_i first-order stochastically dominates F_j, then $r_i(F, T) \geq r_j(F, T)$.

Strict ranking and *continuity* imply *ranking*.

Ranking implies *equal treatment of equals*: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $i, j \in I$, if $F_i = F_j$, then $r_i(F, T) = r_j(F, T)$.

Axioms: *Consistency*

For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, each $J \subseteq I$, and each $i \in J$,

$$r_i(F, T) = r_i(F_J, \sum_{j \in J} r_j(F, T)) \text{ and}$$

$$r_i(F, T) = r_i(F_J, T - \sum_{j \in I \setminus J} r_j(F, T)),$$

where F_J is the restriction of F onto J.

Maximum assignment result

For each $i \in \mathbb{N}$ with F_i, recall $t^*_{F_i} := \sup_{T \in \mathbb{R}_+} r_i(F_i, T)$.

Theorem

If a rule r satisfies consistency and continuity, then for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, $r_i(F, T) \leq t^*_{F_i}$, and if $T \leq \sum t^*_{F_i}$, $\sum r_i(F, T) = T$, and thus for each $i \in I$, $r_i(F, \sum t^*_{F_i}) = t^*_{F_i}$.

If a rule r satisfies consistency, continuity, and equal treatment of equals, then for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, if $T > \sum t^*_{F_i}$, then $r_i(F, T) = t^*_{F_i}$.

Axioms: *Ordinality*

Independence of a (non-linear) transformation of the problem due to a common shock.

For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, each $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that is strictly increasing and continuous, and each $i \in I$, $r_i(F^\phi, \sum \phi(r_j(F, T))) = \phi(r_i(F, T))$, where for each $j \in I$ and each $x_j \in \mathbb{R}_+$, $F_j^\phi(\phi(x_j)) = F_j(x_j)$.

D'Aspremont and Gevers (1977), Sprumont (1998), Chambers (2007)

Implication 1: *Scale invariance*

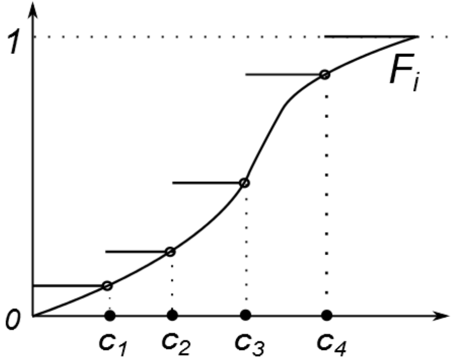
Independence of a uniform rescaling of the problem

Let $k > 0$ and for each $x_j \in \mathbb{R}_+$, $\phi^k(x_j) := kx_j$.

Implication 2: *Coarse ETE*

Agents with "coarsely" equal needs receive "coarsely" equal awards.

For each coarse transformation $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $i, j \in I$, $F_i^\varphi = F_j^\varphi \Rightarrow \varphi(r_i(F, T)) = \varphi(r_j(F, T))$.



Characterization Result

Theorem

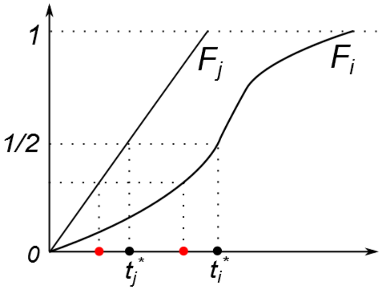
A rule satisfies continuity, strict ranking, consistency, and ordinality if and only if it is an equal-quantile rule.

Equal-quantile rules

An equal-quantile rule associated with $\lambda \in (0,1]$ selects for each $I \in \mathcal{N}$ and each $(F,T) \in \mathcal{P}^I$ the allocation that solves

$$\min_t \sum_{i \in I} [c^w \cdot ew(F_i, t_i) + c^d \cdot ed(F_i, t_i)],$$

where $c^w, c^d > 0$ are such that $\lambda = \frac{c^d}{c^w + c^d}$.



Literature

Deterministic fair division: Moulin (2002), Thomson (2003, 2015)

Operations research: Rawls and Turnquist (2010), Wex, Schryen, Feuerriegel, Neumann (2014), etc.

Fair division under uncertainty: Ertemel and Kumar (2018), Xue (2018), Hougaard and Moulin (2018)

Literature: Fair allocation and welfare economics of risk

Fair allocation: axiomatize division rules (may or may not be rationalizable by some underlying social welfare function)

Welfare economics of risk: axiomatize social welfare functions under risk (Harsanyi (1955), Diamond (1967))

Open question: build a connection between them.